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On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem under the conditions of the Krasnosielski and Krein type

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In this paper we want to show that the conditions of the (K-K) type, which were quoted in note [5] for the Darboux problem, guarantee not only the uniqueness of solutions of this problem but also the convergence of successive approximations, which has been observed successively by J. Luxemburg in his paper [4] and F. Brauer in papers [1] and [2] with respect to equations and system of ordinary differential equations under (K-K) conditions as well as under more general conditions. It is also worth mentioning that in the case of the Darboux problem, in which we are interested, with conditions of the Nagumo type—a theorem on the convergence of successive approximations has been included in J. P. Shanahan's paper [6].

1. Let D denote the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. We denote here by $C^*(D)$ a set of functions $v(x, y)$ defined on D and continuous with v_x, v_y and v_{xy}. Moreover, we assume the functions $\sigma(x)$ and $\tau(y)$ of class C^1 satisfying the condition $\sigma(0) = \tau(0)$ to be respectively defined on $\langle 0, a\rangle$ and $\langle 0, b\rangle$. Let the function $f(x, y, u)$ be defined on the set $E = D \times \times \{-\infty < u < +\infty\}$. We shall be interested in the Darboux problem and so we are looking for the function $u(x, y)$ in class $C^*(D)$ satisfying on D the equation

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u) \tag{1}$$

and the conditions

$$u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y) \tag{2}$$

respectively for $0 \leqslant x \leqslant a$, and $0 \leqslant y \leqslant b$.

The Darboux problem is equivalent to solving the integral equation

$$u(x, y) = \varphi_0(x, y) + \int_0^x \int_0^y f\big(t, \tau, u(t, \tau)\big)\, dt\, d\tau, \tag{3}$$

where $\varphi_0(x, y) = \sigma(x) + \tau(y) - \sigma(0)$.

Proceeding by the usual method we shall define a sequence of successive approximations for equation (3)

$$u_{n+1}(x, y) = \varphi_0(x, y) + \int_0^x\int_0^y f\big(t, \tau, u_n(t, \tau)\big)\,dt\,d\tau \tag{4}$$

for $n = 0, 1, 2, \ldots$, where $u_0(x, y) \in C(D)$.

It will be our aim now to prove that the uniqueness conditions of the (K-K) type for equation (1) are also satisfactory for the convergence of the sequence of successive approximations (4).

Before we proceed to do that we shall prove the following:

LEMMA 1. *If the function $v(x, y) \geqslant 0$ continuous on D fulfil there the system of the integral inequalities*

$$\begin{aligned} v(x, y) &\leqslant \int_0^x\int_0^y Cv^a(t, \tau)\,dt\,d\tau, \\ v(x, y) &\leqslant \int_0^x\int_0^y \frac{k}{t\tau} v(t, \tau)\,dt\,d\tau, \end{aligned} \tag{5}$$

where C, k, a are positive constants and

$$0 < a < 1, \quad k(1-a)^2 < 1, \tag{6}$$

then $v(x, y) \equiv 0$ on D.

Proof (cf. [3]). Let the function $v(x, y) \geqslant 0$ satisfy system (5) and let $M = \sup\limits_D v(x, y)$. As in paper [5], we shall prove that the function $v(x, y)$ must then satisfy the estimation

$$0 \leqslant v(x, y) \leqslant C^{\frac{1}{1-a}} \cdot (xy)^{\frac{1}{1-a}} \quad \text{for} \quad (x, y) \in D. \tag{7}$$

Let us consider now the function $Q(s) = Q(x, y)$ defined as follows:

$$Q(s) = \begin{cases} (xy)^{-\sqrt{k}} v(x, y) & \text{for} \quad (x, y) \in D/\Gamma, \\ 0 & \text{for} \quad (x, y) \in \Gamma, \end{cases}$$

where Γ is a broken line consisting of intervals $\langle 0, a\rangle$ on the x-axis and of intervals $\langle 0, b\rangle$ on the y-axis.

From (7) it follows that

$$0 \leqslant Q(s) \leqslant C^{\frac{1}{1-a}} \cdot (xy)^{\frac{1-\sqrt{k}(1-a)}{1-a}}$$

with respect to condition (6), whence $1-\sqrt{k}\cdot(1-a) > 0$, gives $\lim\limits_{D/\Gamma \ni S \to S_0 \in \Gamma} Q(s) = 0$. Thus the function $Q(s)$ defined above is continuous on D. We state that $Q(s) \equiv 0$ on D. If it were not so, there would exist a point $\bar{s} = (\xi, \eta) \in D/\Gamma$ where $Q(\bar{s}) > 0$ and $Q(\bar{s}) = \sup Q(s)$.

It follows from (5) that

$$Q(\bar{s}) = (\xi\eta)^{-\sqrt{k}} v(\xi, \eta) \leqslant (\xi\eta)^{-\sqrt{k}} \int_0^\xi \int_0^\eta \frac{k}{t\tau} v(t, \tau)\, dt\, d\tau$$

$$= (\xi\eta)^{-\sqrt{k}} \int_0^\xi \int_0^\eta \left(\sqrt{k}\, t^{\sqrt{k}-1}\right)\left(\sqrt{k}\, \tau^{\sqrt{k}-1}\right) Q(t, \tau)\, dt\, d\tau$$

$$< Q(\bar{s})(\xi\eta)^{-\sqrt{k}} \int_0^\xi \sqrt{k}\, t^{\sqrt{k}-1} dt \int_0^\eta \sqrt{k}\, \tau^{\sqrt{k}-1} d\tau = Q(\bar{s}),$$

which is impossible. This ends the proof of the lemma.

Note 1. It follows from lemma 1 that if the function $v(x, y) \geqslant 0$ continuous on D satisfies the integral inequality of the form

$$(8) \qquad v(x, y) \leqslant \int_0^x \int_0^y h\big(t, \tau, v(t, \tau)\big)\, dt\, d\tau, \qquad (x, y) \in D,$$

where

$$(8') \qquad h(x, y, u) = \min\left[Cu^a, \frac{k}{xy} u\right], \qquad C, a, k \text{ — as in lemma 1},$$

then $v(x, y) \equiv 0$ on D.

Note 2. With respect to note 1 we immediately obtain the following theorem on the uniqueness of solutions for the Darboux problem for equation (1) (cf. [5]).

THEOREM 1. *If the function $f(x, y, u)$ defined and continuous on the set E satisfies the condition*

$$(9) \qquad |f(x, y, u) - f(x, y, \bar{u})| \leqslant h(x, y, |u - \bar{u}|)$$

for $(x, y) \in D/\Gamma$, $u, \bar{u} \in (-\infty, +\infty)$, where $h(x, y, u)$ is defined by formula (8′), then the Darboux problem has no more than one solution in class $C^*(D)$.

And so if $u(x, y)$ and $\bar{u}(x, y)$ are two solutions of equation (1) satisfying conditions (2), then putting $v(x, y) = |u(x, y) - \bar{u}(x, y)|$ and taking into consideration (3) and (9) and note 1 we obtain $v(x, y) \equiv 0$.

Note 3. Referring to the above theorem on uniqueness we can observe on a simple example that in (6) condition: $k(1-a)^2 < 1$ cannot be weakened. The following example will show that with $k(1-a)^2 \geqslant 1$ uniqueness does not take place.

Thus, assuming (cf. [3])

$$f^*(x, y, u) = \begin{cases} 0 & \text{for} \quad -\infty < u \leqslant 0\,, \\ \dfrac{k}{xy}u & \text{for} \quad 0 < u \leqslant \left(\dfrac{xy}{k}\right)^{\frac{1}{1-a}}, \\ \left(\dfrac{xy}{k}\right)^{\frac{a}{1-a}} & \text{for} \quad u > \left(\dfrac{xy}{k}\right)^{\frac{1}{1-a}}, \quad k > 0\,, \quad 0 < a < 1\,, \end{cases}$$

we obtain a continuous and bounded function on E, where $D = \langle 0, 1\rangle \times \langle 0, 1\rangle$. It can easily be checked that a function so defined satisfies the conditions

$$|f^*(x, y, u) - f^*(x, y, \bar{u})| \leqslant \frac{k}{xy}|u - \bar{u}|$$

and

$$|f^*(x, y, u) - f^*(x, y, \bar{u})| \leqslant |u - \bar{u}|^a\,.$$

And thus, if $k(1-a)^2 < 1$ then the Darboux problem for equation (1) with $f = f^*$ and conditions (2) has no more than one solution; if conditions (2) are null, then $u(x, y) \equiv 0$ is the only solution. It is also obvious that if $k(1-a)^2 \geqslant 1$ then the function

$$u_1(x, y) = k^{\frac{1}{1-a}} \cdot (xy)^{\sqrt{k}}$$

also satisfies equation (1) with $f = f^*$ and null conditions (2). In such a case we have at least two solutions, $u(x, y) \equiv 0$ and $u_1(x, y)$.

2. Consequently we can now prove a theorem on the convergence of successive approximations defined by formula (4).

THEOREM 2. *If the function $f(x, y, u)$ is defined, continuous and bounded on E and satisfies condition* (9) *then the sequence of successive approximations* (4) *is convergent only to one solution of the Darboux problem for equation* (1).

Proof. Note that with conditions imposed on function $f(x, y, u)$ when

$$M = \sup_E |f(x, y, u)|$$

we obtain for the function

$$v_u(x, y) = \psi(x, y) + \int_0^x \int_0^y f\big(t, \tau, u(t, \tau)\big)\, dt\, d\tau\,, \qquad \psi, u \in C(D)$$

the estimation

$$|v_u(\bar{x}, \bar{y}) - v_u(x, y)| \leqslant \omega_\psi(|\bar{x} - x| + |\bar{y} - y|) + M(a|\bar{y} - y| + b|\bar{x} - x|)$$

where

$$\omega_\psi(t) = \sup_{\substack{(x,y),(\bar{x},\bar{y}) \in D \\ |\bar{x}-x|+|\bar{y}-y| \leqslant t}} |\psi(\bar{x}, \bar{y}) - \psi(x, y)|$$

for the arbitrary function $u(x, y) \in C(D)$.

Besides we have

$$|v_u(x, y)| \leqslant \max_D |\psi(x, y)| + Mab = l .$$

From the above it follows that the sequence of functions $\{u_n(x, y)\}$ defined by formula (4) is a sequence of equicontinuous and uniformly bounded functions. We shall prove that this sequence is uinformly convergent.

For that reason let us assume

$$(10) \qquad \delta_n(x, y) = \sup_{1 \leqslant m < \infty} |u_{n+m}(x, y) - u_n(x, y)| \qquad (n = 0, 1, 2, \ldots) ,$$

$$(11) \qquad \delta(x, y) = \limsup_{n \to \infty} \delta_n(x, y) .$$

It can easily be checked that the sequence of functions $\{\delta_n(x, y)\}$ forms a sequence of equicontinuous and uniformly bounded functions, and thus $\delta(x, y)$ is a continuous function on D. On the basis of (4), (9), (10) and (11) we obtain

$$\begin{aligned} |u_{n+m+1}(x, y) - u_{n+1}(x, y)| &\leqslant \Big| \int_0^x \int_0^y \big[f\big(t, \tau, u_{n+m}(t, \tau)\big) - f\big(t, \tau, u_n(t, \tau)\big) \big] dt d\tau \Big| \\ &\leqslant \int_0^x \int_0^y \big| f\big(t, \tau, u_{n+m}(t, \tau)\big) - f\big(t, \tau, u_n(t, \tau)\big) \big| dt d\tau \\ &\leqslant \int_0^x \int_0^y h\big(t, \tau, |u_{n+m}(t, \tau) - u_n(t, \tau)|\big) dt d\tau \\ &\leqslant \int_0^x \int_0^y h\big(t, \tau, \delta_n(t, \tau)\big) dt d\tau . \end{aligned}$$

Hence it follows that

$$\delta_{n+1}(x, y) \leqslant \int_0^x \int_0^y h\big(t, \tau, \delta_n(t, \tau)\big) dt d\tau \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

and

$$\begin{aligned} \delta(x, y) = \limsup_{n \to \infty} \delta_{n+1}(x, y) &\leqslant \int_0^x \int_0^y h\big(t, \tau, \limsup_{n \to \infty} \delta_n(t, \tau)\big) dt d\tau \\ &= \int_0^x \int_0^y h\big(t, \tau, \delta(t, \tau)\big) dt d\tau . \end{aligned}$$

Consequently, the function $\delta(x, y) \geqslant 0$, being continuous, satisfies the integral inequality (8), which with respect to note 1 implies the relation $\delta(x, y) \equiv 0$. The last fact with respect to (11) and the equicontinuity of the sequence of functions $\{\delta_n(x, y)\}$ enables us to make the statement that sequence $\{\delta_n(x, y)\}$ is uniformly convergent to zero. As follows from (10) we have the right to state that the sequence $\{u_n(x, y)\}$ is also uniformly convergent and, putting $u(x, y) = \lim\limits_{n\to\infty} u_n(x, y)$, we easily conclude that (4) implies

$$u(x, y) = \varphi_0(x, y) + \int_0^x\int_0^y f\big(t, \tau, u(t, \tau)\big)\,dt\,d\tau\,,$$

by which we end the proof of theorem 2.

Note 4. We shall show on an example that also in case of theorem 2 the condition $\gamma = k(1-a)^2 < 1$ cannot be weakened, even if with $\gamma \geqslant 1$ there exists a solution of the Darboux problem.

In fact (cf. also the example in [4]) putting for $(x, y) \in D$ and $u \in (-\infty, +\infty)$

$$f^{**}(x, y, u) = \begin{cases} \left(\dfrac{xy}{k}\right)^{\frac{a}{1-a}} & \text{for} \quad -\infty < u \leqslant 0\,, \\ \left(\dfrac{xy}{k}\right)^{\frac{a}{1-a}} - \dfrac{k}{xy}u & \text{for} \quad 0 < u \leqslant \left(\dfrac{xy}{k}\right)^{\frac{1}{1-a}}\,, \\ 0 & \text{for} \quad u > \left(\dfrac{xy}{k}\right)^{\frac{1}{1-a}}\,,\ k > 0\,,\ 0 < a < 1 \end{cases}$$

we may easily conclude that the function f^{**} defined in this way is continuous, bounded and satisfies the conditions

$$|f^{**}(x, y, u) - f^{**}(x, y, \bar{u})| \leqslant \frac{k}{xy}|u - \bar{u}|\,,$$

$$|f^{**}(x, y, u) - f^{**}(x, y, \bar{u})| \leqslant |u - \bar{u}|^a\,, \qquad k > 0\,,\ 0 < a < 1\,.$$

Let us now consider for equation (1) with $f = f^{**}$ and null conditions (2) the successive approximations (4) where $u_0(x, y) \equiv 0$ on D. Simple calculations lead to the following results:

1° if $0 < \gamma < 1$, then

$$u_n^{(1)}(x, y) = \left(\frac{xy}{k}\right)^{\frac{1}{1-a}} \sum_{i=1}^{n} (-1)^{i-1}\gamma^i \qquad \text{for} \qquad n = 1, 2, \ldots$$

and

$$u^{(1)}(x, y) = \lim_{n\to\infty} u_n^{(1)}(x, y) = \gamma(1+\gamma)^{-1}\left(\frac{xy}{k}\right)^{\frac{1}{1-a}}\,;$$

$2°$ if $\gamma \geqslant 1$, then

$$u_n^{(2)}(x, y) = \frac{1-(-1)^n}{2}\gamma\left(\frac{xy}{k}\right)^{\frac{1}{1-a}} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

We consequently prove by direct calculation that

$$u_{xy}^{(1)} = f^{**}\big(x, y, u^{(1)}(x, y)\big) \quad \text{with each} \quad \gamma > 0\,,$$

while, for $\gamma \geqslant 1$, $u_n^{(2)}(x, y)$ does not satisfy equation (1) with $f = f^{**}$ for any n; moreover, the sequence of successive approximations $\{u_n^{(2)}(x, y)\}$ is non-convergent.

Referring to theorem 2 it seems worth noticing that the above mentioned results may be transfered to the case of a more general equation of the same type (1).

Let us consider the equation

$$\text{(1')} \qquad \frac{\partial^{m+2} U}{\partial x_0 \partial x_1 \ldots \partial x_{m+1}} = F(s, U)\,,$$

where $F = (f_1, \ldots, f_r)$, $U = (u_1, \ldots, u_r)$, $s = (x_0, \ldots, x_{m+1})$, $m \geqslant 0$, while the functions $f_i(x_0, \ldots, x_{m+1}, u_1, u_2, \ldots, u_r) = f_i(s, U)$ of $m+2+r$ variables are defined on the set $G = D_m \times \{-\infty < u_1, \ldots, u_r < +\infty\}$, $D_m = \{s$: $0 \leqslant x_i \leqslant a_i, a_i > 0\,, i = 0, \ldots, m+1\}$.

Besides, let us consider the sets $D_m^{(k)} = D_m \cap H_k$, where $H_k = \{s$: $x_k = 0\}$ and $B_m = \bigcup_{k=0}^{m+1} D_m^{(k)}$, and let us assume that on B_m the vector function $\Psi(s) = \big(\Psi_1(s), \ldots, \Psi_r(s)\big)$, sufficiently regular on each $D_m^{(k)}$ $(k = 0, 1, \ldots \ldots, m+1)$, has been defined.

We are now able to formulate the Darboux problem for equation (1′).

We shall try to find a vector function $U(s)$ regular on D_m, satisfying equation (1′) and the condition

$$\text{(2')} \qquad U(s) = \Psi(s) \quad \text{for} \quad s \in B_m\,.$$

It can easily be proved that such a problem for equation (1′) with condition (2′) is equivalent to solving the integral equation

$$\text{(3')} \qquad U(s) = \overline{\Psi}(s) + \int_0^{x_0} \ldots \int_0^{x_{m+1}} F\big(t, U(t)\big)\,dt\,,$$

where

$$\begin{aligned}\overline{\Psi}(s) = {} & \Psi(0, x_1, \ldots, x_{m+1}) + \ldots + \Psi(x_0, \ldots, x_m, 0) - \\ & -\Psi(0, 0, x_2, \ldots, x_{m+1}) - \ldots - \Psi(x_0, \ldots, x_{m+1}, 0, 0) + \\ & + \ldots + (-1)^{m+1}\Psi(0, \ldots, 0)\,.\end{aligned}$$

Consequently we denote through $\|V\|$ an arbitrary homogenuous norm for the vector $V = \{v_1, \ldots, v_r\}$.

Now we shall quote

THEOREM 3. *If a vector function $F(s, U)$ which is defined, continuous and bounded on G also satisfies the conditoon*

$$\|F(s, U)-F(s, \overline{U})\| \leqslant h(s, \|U-\overline{U}\|), \tag{9'}$$

where

$$h(s, u) = \min\left[Cu^a, \frac{k}{x_0 \dots x_{m+1}} \cdot u\right] \quad \textit{with} \quad C, k > 0, \ 0 < a < 1$$

and

$$k(1-a)^{m+2} < 1, \tag{13}$$

then the Darboux problem for equation (1') *has a unique regular solution which may be obtain by the method of successive approximations*:

$$U_{n+1}(s) = \overline{\Psi}(s) + \int_0^{x_0} \dots \int_0^{x_{m+1}} F\big(t, U_n(t)\big)\, dt \quad (n = 0, 1, 2, \dots),$$

where $U_0(s)$ is an arbitrary vector function, continuous on D_m.

We omit the proof of this theorem because it proceeds similarly to the proof of theorem 2.

References

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