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On monotonic solutions of a recurrence relation

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Let $\{\lambda_n\}$ and $\{F_n\}$ be given sequences of real numbers such that $\lambda_n \neq 0$ for every n and the limit

$$\lim_{n\to\infty} \lambda_n = \lambda \tag{1}$$

exists and is different from zero. (This assumption is valid throughout the paper and will not be repeated.) We shall study the recurrence relation

$$x_{n+1} = \lambda_n x_n + F_n\,, \tag{2}$$

where $\{x_n\}$ is an unknown sequence.

The subject of the present paper is the existence and uniqueness of monotonic, or ultimately monotonic (i.e. monotonic for sufficiently large n) sequences fulfilling (2). Our results are a direct generalization of the results of D. Brydak and J. Kordylewski [1], who have dealt with the case $\lambda_n = \text{const}$. (The cases where $\lambda_n = +1$ or $\lambda_n = -1$ are related to the problems treated in [2], [3], [4], [5].) Also the methods of proofs does not differ from that employed in [1].

In § 3 the results obtained are applied to the problem of monotonic solutions of a linear functional equation.

In the sequence we shall use a shorter notation

$$\Lambda_q^p = \prod_{i=q}^{p} \lambda_i\,.$$

§ 1. THEOREM 1. *If* $\lambda < 0$ *and*

$$\lim_{n\to\infty} \frac{F_n}{\Lambda_0^{n-1}} = 0\,, \tag{3}$$

then exists at most one ultimately monotonic sequence $\{x_n\}$, *satisfying relation* (2). *If it actually does exist, then the series*

$$\sum_{k=0}^{\infty} \frac{F_{n+k}}{\Lambda_n^{n-1}} \tag{4}$$

converges for every n and the sequence $\{x_n\}$ is given by the formula

$$x_n = -\sum_{k=0}^{\infty} \frac{F_{n+k}}{\Lambda_n^{k+n}}. \tag{5}$$

Proof. From relation (2) the formula

$$x_{n+m} = x_n \Lambda_n^{n+m-1} + \sum_{i=n}^{n+m-2} F_i \Lambda_{i+1}^{n+m-1} + F_{n+m-1} \tag{6}$$

for $n = 0, 1, 2, \ldots, m = 1, 2, \ldots$ follows by induction. Since $\lambda < 0$, there exists n_0 such that $\lambda_n < 0$ for every $n \geqslant n_0$. Let us suppose that there exists a sequence $\{x_n\}$ increasing for $n \geqslant n_1 \geqslant n_0$ and satisfying (2). Thus we have for $n \geqslant n_1$ and arbitrary $p \geqslant 1$

$$\lambda_{n+2p} x_{n+2p} + F_{n+2p} = x_{n+2p+1} \geqslant x_{n+2p},$$

whence, since $\lambda_{n+2p} < 0$,

$$x_{n+2p} \leqslant -\frac{F_{n+2p}}{\lambda_{n+2p} - 1}.$$

Now we make use of (6), setting $m = 2p$. Thus

$$x_{n+2p} = x_n \Lambda_n^{n+2p-1} + \sum_{i=n}^{n+2p-2} F_i \Lambda_{i+1}^{n+2p-1} + F_{n+2p-1} \leqslant \frac{F_{n+2p}}{1 - \lambda_{n+2p}},$$

whence

$$x_n \leqslant -\sum_{i=n}^{n+2p-1} \frac{F_i}{\Lambda_n^i} + \frac{F_{n+2p}}{(1-\lambda_{n+2p})\Lambda_n^{n+2p-1}} \quad \text{for every } n \geqslant n_1. \tag{7}$$

Similary, starting from the relation

$$\lambda_{n+2p+1} x_{n+2p+1} + F_{n+2p+1} = x_{n+2p+2} \geqslant x_{n+2p+1} \quad \text{for every } n \geqslant n_1,$$

we get the inequality

$$x_n \geqslant -\sum_{i=n}^{n+2p} \frac{F_i}{\Lambda_n^i} + \frac{F_{n+2p+1}}{(1-\lambda_{n+2p+1})\Lambda_n^{n+2p}} \quad \text{for every } n \geqslant n_1. \tag{8}$$

It follows from (7) and (8) that

$$\frac{F_{n+2p+1}}{(1-\lambda_{n+2p+1})\Lambda_n^{n+2p}} - \frac{F_{n+2p}}{\Lambda_n^{n+2p}} \leqslant x_n + \sum_{i=n}^{n+2p-1} \frac{F_i}{\Lambda_n^i} \leqslant \frac{F_{n+2p}}{(1-\lambda_{n+2p})\Lambda_n^{n+2p-1}} \tag{9}$$

$$\text{for every } n \geqslant n_1.$$

From (3) we obtain

$$\lim_{p\to\infty} \frac{F_{n+2p}}{(1-\lambda_{n+2p})\Lambda_n^{n+2p}} = \frac{1}{1-\lambda} \Lambda_0^{n-1} \lim_{p\to\infty} \frac{F_{n+2p}}{\Lambda_0^{n+2p-1}} = 0 , \tag{10}$$

$$\lim_{p\to\infty} \frac{F_{n+2p}}{\Lambda_n^{n+2p}} = \frac{1}{\lambda} \Lambda_0^{n-1} \lim_{p\to\infty} \frac{F_{n+2p}}{\Lambda_0^{n+2p-1}} = 0 , \tag{11}$$

and

$$\lim_{p\to\infty} \frac{F_{n+2p+1}}{(1-\lambda_{n+2p+1})\Lambda_n^{n+2p}} = \frac{1}{1-\lambda} \Lambda_0^{n-1} \lim_{n\to\infty} \frac{F_{n+2p+1}}{\Lambda_0^{n+2p}} = 0 . \tag{12}$$

In virtue of (9), (10), (11) and (12)

$$\lim_{p\to\infty} \left(x_n + \sum_{i=n}^{n+2p-1} \frac{F_i}{\Lambda_n^i}\right) = 0 ,$$

whence

$$x_n = -\sum_{i=n}^{\infty} \frac{F_i}{\Lambda_n^i} ,$$

which proves formula (5). The uniqueness of the sequence $\{x_n\}$ results hence immediately.

THEOREM 2. *If $|\lambda| > 1$, then there may exist at most one sequence $\{x_n\}$, satisfying relation* (2) *and fulfilling the condition*

$$\limsup_{n\to\infty} |x_{n+1} - x_n| < \infty . \tag{13}$$

Proof. Let us take two sequences $\{x_n'\}$ and $\{x_n''\}$ fulfilling (13) and (2). Then

$$\limsup_{n\to\infty} |(x_{n+1}' - x_{n+1}'') - (x_n' - x_n'')| \leqslant \limsup_{n\to\infty} |x_{n+1}' - x_n'| + \limsup_{n\to\infty} |x_{n+1}'' - x_n''| < \infty .$$

Consequently the sequence $\{x_n\}$ such that $x_n = x_n' - x_n''$ fulfils (13) and moreover

$$x_{n+1} = \lambda_n x_n . \tag{14}$$

Formula (6) has now the form $x_{n+m} = x_n \Lambda_n^{n+m-1}$. We suppose that there exists such an index N that $x_N \neq 0$. Then we have

$$\limsup_{n\to\infty} |x_{n+1} - x_n| = \limsup_{n\to\infty} |\lambda_n - 1|\, |x_n| = \limsup_{n\to\infty} |\lambda_n - 1|\, |x_N|\, |\Lambda_N^{n-1}|$$
$$= \limsup_{n\to\infty} |\lambda_n - 1|\, |x_N|\, |\Lambda_N^{n-1}| = +\infty ,$$

which contradicts (13). Thus $x_n = 0$, $n = 0, 1, \ldots$ Whence $x_n' = x_n''$, $n = 0, 1, 2, \ldots$, which was to be proved.

In the case where $|\lambda| < 1$ every sequence of the form

$$x_n = x_0 \Lambda_0^{n-1}$$

fulfils relations (14) and (13), since

$$\begin{aligned}\limsup_{n\to\infty}|x_{n+1}-x_n| &= \limsup_{n\to\infty}|x_0|\,|\Lambda_0^{n-1}|\,|\lambda_n-1|\\ &= \limsup_{n\to\infty}|\lambda_{n-1}-1|\,|x_0|\,|\Lambda_0^{n_2-1}|\,|\Lambda_{n_2}^{n-1}|\\ &\leqslant \limsup_{n\to\infty}|\lambda_n-1|\,|x_0|\varepsilon^{n-n_2}|\Lambda_0^{n_2-1}| = 0\,.\end{aligned}$$

Here ε is chosen so that $|\lambda| < \varepsilon < 1$, and n_2 is such that $|\lambda_n| < \varepsilon$ for every $n \geqslant n_2$. Thus condition (13) does not guarantee the uniqueness of $\{x_n\}$ in the case $|\lambda| < 1$.

§ 2. THEOREM 3. *Suppose that*

$$|\lambda| > \limsup_{n\to\infty} \sqrt[n]{|F_n|}$$

and one of the following four conditions is fulfilled:

(a) $\lambda > 0$, *the sequence* $\{F_n\}$ *is ultimately increasing (decreasing) and non-negative and the sequence* $\{\lambda_n\}$ *is ultimately decreasing (increasing)*;

(b) $\lambda > 0$, *the sequence* $\{F_n\}$ *is ultimately increasing (decreasing) and non-positive and the sequence* $\{\lambda_n\}$ *is ultimately increasing (decreasing)*;

(c) $\lambda < 0$, *the sequence* $\{u_n\}$ *is ultimately increasing (decreasing) and non-negative and the sequence* $\{\lambda_n\}$ *is ultimately decreasing (increasing)*;

(d) $\lambda < 0$, *the sequence* $\{u_n\}$ *is ultimately increasing (decreasing) and non-positive and the sequence* $\{\lambda_n\}$ *is ultimately decreasing (increasing)*; *where*

$$u_n \stackrel{\mathrm{df}}{=} F_{n+1} + \lambda_{n+1} F_n\,.$$

Then formula (5) *defines an ultimately decreasing (increasing) sequence satisfying relation* (2). *In cases* (c), (d) *it is the unique ultimately monotonic sequence satisfying* (2).

Proof. Since

$$\lim_{k\to\infty} \sqrt[k]{|\Lambda_n^{n+k}|} = \lim_{k\to\infty}|\lambda_{n+k}| = |\lambda|\,,$$

we have

$$\begin{aligned}\limsup_{k\to\infty}\sqrt[k]{\left|\frac{F_{n+k}}{\Lambda_n^{n+k}}\right|} &= \limsup_{k\to\infty}\frac{\sqrt[k]{|F_{n+k}|}}{\sqrt[k]{|\Lambda_n^{n+k}|}}\\ &= \frac{1}{|\lambda|}\limsup_{k\to\infty}\sqrt[k]{|F_{n+k}|} = \frac{1}{|\lambda|}\limsup_{k\to\infty}\sqrt[k]{|F_k|}\,.\end{aligned}$$

Hence and from the assumption we have

$$\limsup_{k\to\infty} \sqrt[k]{\left|\frac{F_{n+k}}{\Lambda_n^{n+k}}\right|} < 1 ,$$

whence in virtue of the Cauchy criterion it follows that series (4) converges. Therefore formula (5) defines a sequence which evidently satisfies relation (2). In cases (a), (b) the sequence $\{x_n\}$ is the sum of a sequence of sequences such that every next is subsequence of previous. These sequences are ultimately decreasing (increasing), hence $\{x_n\}$ is ultimately decreasing (increasing). When $\lambda < 0$ for $n \geqslant n_0$, $\lambda_n < 0$

$$x_n = -\sum_{k=0}^{\infty} \frac{F_{n+k}}{\Lambda_n^{n+k}} = -\sum_{k=0}^{\infty} \frac{F_{n+2j}}{\Lambda_n^{n+2j}} - \sum_{j=0}^{\infty} \frac{F_{n+2j+1}}{\Lambda_n^{n+2j+1}}$$

$$= \sum_{j=0}^{\infty} \frac{F_{n+2j+1} + \lambda_{n+2j+1} F_{n+2j}}{\Lambda_n^{n+2j+1}} = -\sum_{j=0}^{\infty} \frac{u_{n+2j}}{\Lambda_n^{n+2j+1}} .$$

Similarly, in virtue of the latter equalities in cases (c), (d) $\{x_n\}$ is ultimately decreasing (increasing). Convergence of series (4) implies condition (3). In cases (c), (d) in virtue of Theorem 1 the sequence defined by (5) is the unique ultimately monotonic sequence fulfilling (2). This completes the proof.

COROLLARY. *If $\lambda < 0$ and series* (4) *diverges, then there may exist ultimately monotonic sequences satisfying* (2) *only if*

$$\frac{F_n}{\Lambda_0^{n-1}} \to 0 .$$

Now we shall find a necessary and sufficient condition of the existence of a monotonic sequence satisfying (2) in the case $\lambda < 0$. To this purpose we write

$$S_p \overset{\mathrm{df}}{=} -\sum_{k=0}^{p} \frac{F_k}{\Lambda_0^k} - \frac{F_{p+1}}{(\lambda_{p+1}-1)\Lambda_0^p}$$

and we put further

$$\bar{\sigma}_e \overset{\mathrm{df}}{=} \sup_{p=1,2,\ldots} S_{2p} , \qquad \underline{\sigma}_e \overset{\mathrm{df}}{=} \inf_{p=1,2,\ldots} S_{2p} ,$$

$$\bar{\sigma}_0 \overset{\mathrm{df}}{=} \sup_{p=1,2,\ldots} S_{2p-1} , \qquad \underline{\sigma}_0 \overset{\mathrm{df}}{=} \inf_{p=1,2,\ldots} S_{2p-1} .$$

THEOREM 4. *Let us assume that $\lambda_n < 0$ for $n = 1, 2, \ldots$ A necessary and sufficient condition that the sequence $\{x_n\}$ defined by the recurrence relation* (2) *be increasing is that*

$$\bar{\sigma}_e \leqslant x_0 \leqslant \underline{\sigma}_0 . \tag{15}$$

Similarly, a necessary and sufficient condition that the sequence $\{x_n\}$ *defined by the recurrence relation* (2) *be decreasing is that*

$$\bar{\sigma}_0 \leqslant x_0 \leqslant \underline{\sigma}_e \,. \tag{16}$$

Proof. We shall prove only the first part of the Theorem, the proof for decreasing sequences is quite analogical. If $\{x_n\}$ is an increasing sequence satisfying (2), then we have by (7) and (8) ($n = 0$)

$$S_{2p} \leqslant x_0 \leqslant S_{2p-1} \quad \text{for} \quad p = 1, 2, \ldots, \tag{17}$$

whence (15) follows immediately. On the other hand let $\{x_n\}$ be a sequence satisfying (2), and let us assume that inequalities (15) hold. Then inequalities (17) hold too, and we have by (6) ($n = 0$)

$$\begin{aligned}
x_{2p+1} - x_{2p} &= x_0 \Lambda_0^{2p} + \sum_{i=0}^{2p-1} F_i \Lambda_{i+1}^{2p} + F_{2p} - x_0 \Lambda_0^{2p-1} - \sum_{i=0}^{2p-2} F_i \Lambda_{i+1}^{2p-1} - F_{2p-1} \\
&= x_0(\lambda_{2p}-1)\Lambda_0^{2p-1} + \sum_{i=0}^{2p-2} F_i(\lambda_{2p}-1)\Lambda_{i+1}^{2p-1} + F_{2p} - F_{2p-1} + F_{2p-1}\lambda_{2p} \\
&= (\lambda_{2p}-1)\Lambda_0^{2p-1}\left[x_0 + \sum_{i=0}^{2p-2} \frac{F_i}{\Lambda_0^i} + \frac{F_{2p-1}}{\Lambda_0^{2p-1}} + \frac{F_{2p}}{(\lambda_{2p}-1)\Lambda_0^{2p-1}}\right] \\
&= (\lambda_{2p-1}-1)\Lambda_0^{2p-1}(x_0 - S_{2p-1}) \geqslant 0 \,,
\end{aligned}$$

and similarly

$$x_{2p+2} - x_{2p+1} = (\lambda_{2p+1}-1)\Lambda_0^{2p}(x_0 - S_{2p}) \geqslant 0 \,.$$

This means that the sequence $\{x_n\}$ is increasing, which was to be proved.

§ 3. The results obtained can be applied to establish some conditions of the uniqueness and existence of solution of the functional equation

$$\varphi[f(t)] = g(t)\varphi(t) + F(t) \,. \tag{18}$$

Here $\varphi(t)$ is the unknown function, $f(t)$, $g(t)$ and $F(t)$ are given functions defined in an interval (a, b) and fulfilling the conditions: $g(t) \neq 0$ for $t \in (a, b)$, the limit $\lim\limits_{t \to a+} g(t) = \lambda$ exists and $\lambda \neq 0$, $a < f(t) < t$ for $t \in (a, b)$ and the function $f(t)$ is strictly increasing in (a, b).

Definition (cf. [2]). A function $h(t)$ defined in the interval (a, b) is *semiincreasing* $\{f\}$ in this interval if $h[f(t)] \geqslant h(t)$ for $t \in (a, b)$. A function $h(t)$ defined in the interval (a, b) is *semidecreasing* $\{f\}$ in this interval if $h[f(t)] \leqslant h(t)$ for $t \in (a, b)$. A function that is semiincreasing $\{f\}$ or semidecreasing $\{f\}$ in (a, b) is called *semimonotonic* $\{f\}$ in (a, b).

By our assumptions about the function f, an increasing function is semidecreasing $\{f\}$, and a decreasing function is semiincreasing $\{f\}$.

Let us add to the previous assumptions for f, F, g yet one: the function g is semimonotonic $\{f\}$ in (a, d), $a < d \leqslant b$.

Let us put

$$x_n \overset{\mathrm{df}}{=} \varphi[f^n(t)], \qquad \lambda_n \overset{\mathrm{df}}{=} g[f^n(t)], \qquad F_n \overset{\mathrm{df}}{=} F[f^n(t)].$$

As an immediate consequence of Theorems 1, 2, 3, 4 we obtain the following

THEOREM 1'. *If* $\lambda < 0$ *and*

$$\lim_{n\to\infty} \frac{F[f^n(t)]}{\prod\limits_{i=0}^{n-1} g[f^i(t)]} = 0 \qquad \textit{for} \qquad t \in (a, b),$$

then there exists at most one solution of equation (18) *in* (a, b) *semimonotonic* $\{f\}$ *in* (a, c), $a < c \leqslant b$. *If such solution exists, then it is given by the formula*

$$\varphi(t) = -\sum_{k=0}^{\infty} \frac{F[f^k(t)]}{\prod\limits_{i=0}^{k} g[f^i(t)]} \qquad \textit{for} \qquad t \in (a, b) \tag{19}$$

and series on the right-hand side of (19) *converges.*

THEOREM 2'. *If* $|\lambda| > 1$, *then equation* (18) *may have at most one solution fulfilling the condition*

$$\limsup_{n\to\infty} |\varphi[f^{n+1}(t)] - \varphi[f^n(t)]| < \infty \qquad \textit{for} \qquad t \in (a, b). \tag{20}$$

THEOREM 3'. *Suppose that*

$$|\lambda| > \limsup_{n\to\infty} \sqrt[n]{|F[f^n(t)]|}$$

for every $t \in (a, b)$ *and that one of the following four conditions is fulfilled:*

(i) $\lambda > 0$ *and there exists* b_1, $a < b_1 \leqslant b$ *such that the function* $F(t)$ *is semiincreasing* $\{f\}$ *(semidecreasing* $\{f\}$*)*, $F(t) \geqslant 0$ *and* $g(t)$ *is semidecreasing* $\{f\}$ *(semiincreasing* $\{f\}$*) in* (a, b_1); *or*

(ii) $\lambda > 0$ *and there exists* b_2, $a < b_2 \leqslant b$ *such that the function* $F(t)$ *is semiincreasing* $\{f\}$ *(semidecreasing* $\{f\}$*)*, $F(t) \leqslant 0$ *and* $g(t)$ *is semiincreasing* $\{f\}$ *(semidecreasing* $\{f\}$*) in* (a, b_2); *or*

(iii) $\lambda < 0$ *and there exists* b_3, $a < b_3 \leqslant b$ *such that the function* $u(t)$ *is semiincreasing* $\{f\}$ *(semidecreasing* $\{f\}$*)*, $u(t) > 0$ *and* $g(t)$ *is semidecreasing* $\{f\}$ *(semiincreasing* $\{f\}$*) in* (a, b_3); *or*

(iv) $\lambda < 0$ *and there exists* b_4, $a < b_4 \leqslant b$ *such that the function* $u(t)$ *is semiincreasing* $\{f\}$ *(semidecreasing* $\{f\}$*)*, $u(t) \leqslant 0$ *and* $g(t)$ *is semidecreasing* $\{f\}$ *(semiincreasing* $\{f\}$*) in* (a, b_4);

where

$$u(t) = F[f(t)] + g(t)F(t),$$

then the formula (19) *define a function fulfilling of the equation* (18) *in* (a, b) *semidecreasing* $\{f\}$ (*semiincreasing* $\{f\}$) *in* (a, c), $a < c \leqslant b$.

In cases (iii) and (iv) it is the unique solution semimonotonic $\{f\}$ of (18).

Let $g(t) < 0$. We put

$$S_p(t) \overset{\text{df}}{=} -\sum_{k=0}^{p} \frac{F[f^k(t)]}{\prod_{i=0}^{k} g[f^i(t)]} - \frac{F[f^{p+1}(t)]}{(g[f^{p+1}(t)]-1)\prod_{i=0}^{p} g[f^i(t)]},$$

$$\bar{\sigma}_e(t) \overset{\text{df}}{=} \sup_{p=1,2,\ldots} S_{2p}(t), \qquad \underline{\sigma}_e(t) \overset{\text{df}}{=} \inf_{p=1,2,\ldots} S_{2p}(t),$$

$$\bar{\sigma}_0(t) \overset{\text{df}}{=} \sup_{p=1,2,\ldots} S_{2p-1}(t), \qquad \underline{\sigma}_0(t) \overset{\text{df}}{=} \inf_{p=1,2,\ldots} S_{2p-1}(t).$$

We obtain following

THEOREM 4'. *Let* $g(t) < 0$ *in* (a, b) *and* $f(b) < b$. *A necessary and sufficient condition that the function* $\varphi(t)$ *fulfilling equation* (18) *be semiincreasing* $\{f\}$ *is*

$$\bar{\sigma}_e(t) \leqslant \varphi(t) \leqslant \underline{\sigma}_0(t) \quad \text{in} \quad (f(b), b\rangle.$$

Similarly, a necessary and sufficient condition that the function $\varphi(t)$ *fulfilling equation* (18) *be semidecreasing* $\{f\}$ *in* (a, b) *is that*

$$\bar{\sigma}_0(t) \leqslant \varphi(t) \leqslant \underline{\sigma}_e(t) \quad \text{in} \quad (f(b), b\rangle.$$

References

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