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Approximation of exterior conformal mappings*

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Abstract. We denote by D (or G) a bounded domain in E^2 which contains the origin and whose boundary $\dot{D}$ is a closed, rectifiable, Jordan curve. Let D^e denote the exterior of D and let $w = F(z)$ denote the unique function which is holomorphic in D^e, one-to-one and continuous on $\overline{D^e}$, maps $\overline{D^e}$ onto $\{|w| > \nu\}$, and satisfies

$$F(\infty) = \infty, \quad \lim_{z\to\infty} \frac{F(z)}{z} = 1.$$

The quantity ν is called the *outer conformal radius of* D^e. In the following we consider the problem of approximating $F(z)$, $F'(z)$ and ν. These approximations are useful for finding approximate solutions to a number of physical problems (see e.g. [4]). In particular, we find upper and lower bounds for ν, hence also for the transfinite diameter and capacity of $\bar{D}$, since these quantities are equal to ν [1], Chapter 2.

1. Preliminaries. The material of this section is a slight modification of results reported in Goluzin [3], Chapters IX and X (see also [2], Kapitel III).

Let $w = f(z)$ denote the unique function which is holomorphic in G, one-to-one and continuous on $\bar{G}$, maps $\bar{G}$ onto $\{|w| \leqslant R\}$, and satisfies

$$f(0) = 0, \quad f'(0) = 1.$$

R is called the *conformal radius of* G. Let $z = g(w)$ denote the inverse of $f(z)$. Because $\check{G}$ is a closed, rectifiable, Jordan curve, it follows that $g(w)$ is continuous for $|w| \leqslant R$ and is absolutely continuous on the circle $|w| = R$. Furthermore,

$$\sup_{0<r<R} \int_0^{2\pi} |g'(re^{i\theta})|\, d\theta < \infty. \tag{1.1}$$

Now (1.1) implies that there exists $h \in L_1([0, 2\pi))$ such that

$$\lim_{w\to Re^{i\theta}} g'(w) = h(\theta)$$

for almost all θ, where w approaches $Re^{i\theta}$ along any non-tangential path.

* This paper was written while the author was a student at the University of Delaware, Newark, Delaware.

We will denote $h(\theta)$ by $g'(Re^{i\theta})$ and call this function the *non-tangential boundary values of* $g'(w)$. It also follows that

$$\frac{dg}{d\theta}(Re^{i\theta}) = Rie^{i\theta} g'(Re^{i\theta})$$

for almost all θ, and that the length $s(\theta', \theta'')$ of the arc $z = g(Re^{i\theta})$, $\theta' \leqslant \theta \leqslant \theta''$, is given by

$$s(\theta', \theta'') = \int_{\theta'}^{\theta''} |g'(Re^{i\theta})| \, d\theta .$$

Now let

$$s(\theta) = \int_0^{\theta} |g'(Re^{i\theta})| \, d\theta .$$

Then the length L of $\dot{G}$ is given by

$$\int_0^{2\pi} |g'(Re^{i\theta})| \, d\theta .$$

Let $z = z(s)$ denote the parameterization of $\dot{G}$ in terms of arc length. Furthermore, let $\mathfrak{s}$ denote Lebesgue measure on $[0, L)$.

If $H(z)$ is holomorphic in G, the integral

$$\int_{C_r'} |H(z)|^2 ds = r \int_0^{2\pi} \left| H\left(g(re^{i\theta})\right) \sqrt{g'(re^{i\theta})} \right|^2 d\theta ,$$

where $C_r' = g\ (|w| = r)$, is a monotone increasing function of r. We say that $H(z)$ is in class $L_2(G)$ if this integral is bounded for $0 \leqslant r < R$. Now $\dot{G}$ has a tangent for almost all $0 \leqslant s < L$ and if $H(z)$ is in class $L_2(\dot{G})$, there exists $\tilde{h} \in L_2\left([0, L)\right)$ such that

$$\lim_{z \to z(s) \in \dot{G}} H(z) = \tilde{h}(s)$$

for almost all s, where z approaches $z(s) \in \dot{G}$ along any non-tangential path. We will denote $\tilde{h}(s)$ by $H\left(z(s)\right)$ and call this function the non-tangential boundary values of $H(z)$.

If we introduce the inner product

$$(\tilde{H}, H)_{L_2(\dot{G})} = \lim_{r \to R^-} \int_{C_r'} \tilde{H}(z) \overline{H(z)} \, ds = \int_0^L \tilde{H}\left(z(s)\right) \overline{H\left(z(s)\right)} \, d\mathfrak{s} ,$$

then $L_2(\dot{G})$ becomes a Hilbert space. For simplicity of notation we write

$$\int_0^L \tilde{H}(z(s))\overline{H(z(s))}\,ds = \int_{\dot{G}} \tilde{H}(z)\overline{H(z)}\,ds.$$

Throughout Section 1, $(\cdot,\cdot) = (\cdot,\cdot)_{L_2(\dot{G})}$ and $\|\cdot\| = (\cdot,\cdot)^{1/2}_{L_2(\dot{G})}$.

It is useful to have a condition on $\dot{G}$ which will ensure that the set of polynomials is dense in $L_2(\dot{G})$. This is the subject of the following theorem:

THEOREM 1.1. *The set of polynomials is dense in $L_2(\dot{G})$ if and only if the function $\log|g'(w)|$, which is harmonic in $|w| < R$, can be represented by its Poisson integral, i.e.,*

$$\log|g'(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \log|g'(Re^{i\varphi})|\,\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)}\,d\varphi. \tag{1.2}$$

If (1.2) *holds, $\dot{G}$ is said to satisfy condition* (S), *after Smirnov.*

There exists a domain $\dot{G}$ with $\dot{G}$ a closed, rectifiable, Jordan curve such that $\dot{G}$ does not satisfy condition (S). If G is a domain with $\dot{G}$ a closed, rectifiable, Jordan curve, then any one of the following is sufficient for $\dot{G}$ to satisfy condition (S):

(a) G is convex, or starlike with respect to a point of G,

(b) $\dot{G}$ is piecewise smooth, and the smooth arcs are joined with interior angles $\neq 0$ (i.e., no cusps are allowed),

(c) there exists $M > 0$ such that for all $0 \leqslant s_1 < s_2 < L$

$$\frac{s_2-s_1}{|z(s_2)-z(s_1)|} < M.$$

Now let $\mathfrak{L} = \{F \in L_2(\dot{G}) \colon F(0) = 1\}$. Then we have the following extremal characterization for $\sqrt{f'(z)}$.

THEOREM 1.2. (Julia). *The function $H_0(z) = \sqrt{f'(z)}$ and only this function minimizes the integral*

$$\int_{\dot{G}} |H(z)|^2\,ds$$

over the class $\mathfrak{L}$. This minimum is equal to $2\pi R$.

The function $H_0(z)$ is also completely characterized by the following property.

LEMMA 1.1. *The function $H_0(z)$ is orthogonal to each function $K(z)$ in class $L_2(\dot{G})$ with $K(0) = 0$, i.e.,*

$$(1.3) \qquad (H_0, K) = \int_{\dot{G}} H_0(z)\overline{K(z)}\,ds = 0.$$

Next, we present the Ritz method for approximation of $H_0(z)$. Let $\mathfrak{L}_m$ denote the set of polynomials $P(z)$ of degree less than or equal to m with $P(0) = 1$.

PROPOSITION 1.1. *For each $m \geqslant 1$, there exists a polynomial $P_m(z)$ in class $\mathfrak{L}_m$ which uniquely minimizes*

$$\int_{\dot{G}} |P(z)|^2 ds$$

over $\mathfrak{L}_m$. Furthermore, $P_m(z)$ is orthogonal to each polynomial $Q(z)$ of degree less than or equal to m with $Q(0) = 0$, i.e.,

$$(1.4) \qquad (P_m, Q) = \int_{\dot{G}} P_m(z)\overline{Q(z)}\,ds = 0,$$

and $P_m(z)$ is completely characterized by property (1.4).

The coefficients of $P_m(z) = 1 + a_1 z + \ldots + a_m z^m$ are characterized by the fact that

$$\int_{\dot{G}} \Big(\sum_{i=0}^{m} a_i z^i\Big)\overline{z^k}\,ds = 0, \qquad a_0 = 1,\ k = 1, \ldots, m.$$

Let

$$(1.5) \qquad A_{ik}(\dot{G}) = \int_{\dot{G}} z^i \overline{z^k}\,ds, \qquad i, k = 0, 1, \ldots, m.$$

Then the coefficients of $P_m(z)$ must satisfy the linear system

$$(1.6) \qquad \sum_{i=1}^{m} A_{ik}(\dot{G})\, a_i = -A_{0k}(\dot{G}), \qquad k = 1, \ldots, m.$$

Hence (1.6) must have a unique solution.

It follows from Lemma 1.1 that if $P(z)$ is in class $\mathfrak{L}$, then $\|H_0 - P\|^2 = \|P\|^2 - \|H_0\|^2$, so that $P_m(z)$ has the additional minimum property that

$$\int_{\dot{G}} |H_0(z) - P(z)|^2 ds$$

is minimal in $\mathfrak{L}_m$ for $P(z) = P_m(z)$.

Since $\mathfrak{L}_m$ is a proper subset of $\mathfrak{L}_{m+1}$, it follows that $\|H_0^- P_m\|$ is decreasing. If $\dot{G}$ satisfies condition (S), then it follows from Theorem 1.1 that $\|H_0 - P_m\| \to 0$ as $m \to \infty$.

Now let $\{Q_k(z)\}_{k=0}^{\infty}$ denote the unique orthonormal sequence of polynomials with positive leading coefficient obtained by applying the Gram–Schmidt orthogonalization procedure to $\{z^k\}_{k=0}^{\infty}$ with respect to the inner product of $L_2(\dot{G})$. Then it follows that

$$P_m(z) = \frac{\sum_{k=0}^{m} \overline{Q_k(0)} Q_k(z)}{\sum_{k=0}^{m} |Q_k(0)|^2} \tag{1.7}$$

and

$$\|P_m\|^2 = \Big(\sum_{k=0}^{m} |Q_k(0)|^2\Big)^{-1}.$$

The $Q_k(z)$ have the advantage that they are determined algebraically in terms of the $A_{ik}(\dot{G})$ defined in (1.5). Hence no linear system need be solved to determine them [2], p. 132–133.

Next, we consider the approximation of $f(z)$. First, we need

LEMMA 1.2. *If $H(z)$ is in class $L_2(\dot{G})$, then for $z \in G$,*

$$\left| \int_0^z H(\xi)\, d\xi \right| \leqslant \tfrac{1}{2} \int_{\dot{G}} |H(z)|\, ds.$$

Using Lemma 1.2, it can be shown that

THEOREM 1.3. *If $P(z)$ is in class $\mathfrak{L}$ and*

$$p(z) = \int_0^z P^2(\xi)\, d\xi,$$

then

$$\max_{\bar{G}} |f(z) - P(z)| \leqslant \sqrt{2\pi R}\, \|F_0 - P\| + \tfrac{1}{2} \|F_0 - P\|^2$$
$$= 2\pi R [R^{-1} \|P\|^2 / 2\pi - 1]^{1/2} + \pi R [R^{-1} \|P\|^2 / 2\pi - 1]. \tag{1.8}$$

2. The class $L_2(\dot{D}^e)$. Let $z = G(W)$ denote the inverse of $F(z)$ and let $C_r = G\ (|w| = r)$. If $\mathfrak{F}(z)$ is holomorphic in D^e and has an expansion at ∞ of the form

$$\mathfrak{F}(z) = a_0 + a_{-1} z^{-1} + a_{-2} z^{-2} + \ldots,$$

then the integral

$$\frac{1}{r} \int_{C_r} |\mathfrak{F}(z)|^2\, ds = \int_0^{2\pi} |\mathfrak{F}(G(re^{i\theta}))\, \sqrt{G'(re^{i\theta})}|^2\, d\theta$$

is a monotone decreasing function of r. We say that $\mathfrak{F}(z)$ is in class $L_2(\dot{D}^e)$ if this integral is bounded for $\infty < r < \nu$. Now let $\varrho = I(z) = z^{-1}$ and let $D^* = I(D^e)$. Then it can be shown that $\mathfrak{F}(z)$ is in class $L_2(\dot{D}^e)$ if and only if $\mathfrak{G}(\varrho) = \mathfrak{F}(\varrho^{-1})$ is in class $L_2(\dot{D}^*)$. It follows from this that $\mathfrak{F}(z)$ has non-tangential boundary values $\mathfrak{F}(z(s))$ for almost all $0 \leqslant s < L$, that $\mathfrak{F}(z(s))$ is in class $L_2([0, L))$, and that

$$\lim_{r \to \nu +} \int_{C_r} |\mathfrak{F}(z)|^2 ds = \int_0^L |\mathfrak{F}(z(s))|^2 ds.$$

If we introduce the inner product

$$(\mathfrak{F}, \mathfrak{G}) = \lim_{r \to \nu +} \int_{C_r} \mathfrak{F}(z)\overline{\mathfrak{G}(z)}\, ds = \int_0^L \mathfrak{F}(z(s))\overline{\mathfrak{G}(z(s))}\, ds,$$

then $L_2(\dot{D}^e)$ becomes a Hilbert space. Moreover, it follows from Theorem 1.1 that the class of polynomials in z^{-1} are dense in $L_2(\dot{D}^e)$ if and only if $\dot{D}^*$ satisfies condition (S). Again, we will use the notation

$$\int_0^L \mathfrak{F}(z(s))\overline{\mathfrak{G}(z(s))}\, ds = \int_{\dot{D}} \mathfrak{F}(z)\overline{\mathfrak{G}(z)}\, ds.$$

Throughout Sections 3 and 4 $(\cdot, \cdot) = (\cdot, \cdot)_{L_2(\dot{D}^e)}$ and $\|\cdot\| = (\cdot, \cdot)^{1/2}_{L_2(\dot{D}^e)}$.

3. The Ritz method of approximating $\sqrt{F'(z)}$. The proofs of the results of this section are similar to the proofs of the results of Section 1 which can be found in [2], Chapters IX and X.

Let $\mathfrak{M}$ denote the class of functions $\mathfrak{F}(z)$ in $L_2(\dot{D}^e)$ which have an expansion at ∞ of the form

$$\mathfrak{F}(z) = 1 + a_{-2}z^{-2} + a_{-3}z^{-3} + \ldots$$

Then $\sqrt{F'(z)}$ has the following extremal characterization.

THEOREM 3.1. *The function $\mathfrak{F}_0(z) = \sqrt{F'(z)}$ and only this function minimizes the integral*

$$\int_{\dot{D}} |\mathfrak{F}(z)|^2 ds$$

over the class $\mathfrak{M}$.

The function $\mathfrak{F}_0(z)$ is also completely characterized by the following property.

PROPOSITION 3.1. *The function $\mathfrak{F}_0(z)$ is orthogonal to every function $\mathfrak{G}(z)$ in $L_2(\dot{D}^e)$ which has an expansion at ∞ of the form*

$$\mathfrak{G}(z) = a_{-2}z^{-2} + a_{-3}z^{-3} + \ldots,$$

i.e.,

$$(\mathfrak{F}_0, \mathfrak{G}) = \int_{\dot{D}} \mathfrak{F}_0(z)\overline{\mathfrak{G}(z)}\,ds = 0. \tag{3.1}$$

Now, let $\mathfrak{M}_n$ be the class of elements of $\mathfrak{M}$ of the form

$$1 + a_{-2}z^{-2} + a_{-3}z^{-3} + \ldots + a_{-n}z^{-n}.$$

THEOREM 3.2. *For each $n \geqslant 2$, there exists a function $\mathfrak{P}_n(z)$ in class $\mathfrak{M}_n$ which uniquely minimizes*

$$\int_{\dot{D}} |\mathfrak{P}(z)|^2\,ds$$

over $\mathfrak{M}_n$. Moreover, $\mathfrak{P}_n(z)$ is orthogonal to each function $\mathfrak{Q}(z)$ of the form

$$\mathfrak{Q}(z) = a_{-2}z^{-2} + \ldots + a_{-n}z^{-n},$$

i.e.,

$$(\mathfrak{P}_n, \mathfrak{Q}) = \int_{\dot{D}} \mathfrak{P}_n(z)\overline{\mathfrak{Q}(z)}\,ds = 0, \tag{3.3}$$

and $\mathfrak{P}_n$ is completely characterized by this property.

The coefficients of $\mathfrak{P}_n(z) = 1 + a_{-2}z^{-2} + \ldots + a_{-n}z^{-n}$ are characterized by the fact that

$$\int_{\dot{D}} \sum_{i=0}^{n} a_i z^{-i}\overline{z^{-k}}\,ds = 0, \quad a_0 = 1,\ a_{-1} = 0,\ k = 2, \ldots, n.$$

Let

$$A_{ik}(\dot{D}^e) = \int_{\dot{D}} z^{-i}\overline{z^{-k}}\,ds, \quad i, k = 0, 1, \ldots, n. \tag{3.4}$$

Then the coefficients of $\mathfrak{P}_n(z)$ must satisfy the linear system

$$\sum_{i=2}^{n} A_{ik}(\dot{D}^e)\,a_{-i} = -A_{0k}(\dot{D}^e), \quad k = 2, \ldots, n, \tag{3.5}$$

so that (3.5) must have a unique solution.

Because of Proposition 3.1, for an arbitrary function $\mathfrak{P}(z)$ in class $\mathfrak{M}$, we have $(\mathfrak{F}_0, \mathfrak{P}) = (\mathfrak{F}_0, \mathfrak{P} - \mathfrak{F}_0) + (\mathfrak{F}_0, \mathfrak{F}_0) = (\mathfrak{F}_0, \mathfrak{F}_0)$, and hence $\|\mathfrak{F}_0 - \mathfrak{P}\|^2 = \|\mathfrak{P}\|^2 - \|\mathfrak{F}_0\|^2$. Thus, $\mathfrak{P}_n(z)$ has the additional extremal property that

$$\int_{\dot{D}} |\mathfrak{F}_0(z) - \mathfrak{P}(z)|^2\,ds$$

is minimized over $\mathfrak{M}_n$ for $\mathfrak{P}(z) = \mathfrak{P}_n(z)$.

Since $\mathfrak{M}_n$ is a proper subset of $\mathfrak{M}_{n+1}$, $\|\mathfrak{F}_0'-\mathfrak{P}_n\|$ is decreasing. If $\dot{D}^*$ satisfies condition (S), then the polynomials in z^{-1} are dense in $L_2(\dot{D}^e)$ and it follows that $\|\mathfrak{F}_0-\mathfrak{P}_n\| \to 0$ as $n \to \infty$ as we now show.

Since $\mathfrak{F}_0(\varrho^{-1})$ is in class $L_2(\dot{D}^*)$ so is the function $[\mathfrak{F}_0(\varrho^{-1})-1]/\varrho^2$. Since $0 \in D$, there exists $r_0 > 0$ such that $\{|z| \leqslant r_0\} \subseteq D$. Let $\varepsilon > 0$ be given. Then according to Theorem 1.1, there exists a polynomial $P(\varrho)$ such that

$$\int\limits_{\dot{D}^*} \left| \frac{\mathfrak{F}_0(\varrho^{-1})-1}{\varrho^2} - P(\varrho) \right|^2 ds < \varepsilon^2 r_0^2.$$

Hence

$$\int\limits_{\dot{D}^*} \frac{1}{|\varrho|^4} \left|\mathfrak{F}_0(\varrho^{-1}) - \big(1+\varrho^2 P(\varrho)\big)\right|^2 ds < \varepsilon^2 r_0^2.$$

Setting $\mathfrak{P}(z) = 1+z^{-2}P(z^{-1})$, we have

$$r_0^2 \int\limits_{\dot{D}} |\mathfrak{F}_0(z)-\mathfrak{P}(z)|^2 ds \leqslant \int\limits_{\dot{D}} |z|^2 |\mathfrak{F}_0(z)-\mathfrak{P}(z)|^2 ds < \varepsilon^2 r_0^2.$$

Let $N = N(\varepsilon)$ be the degree of $1+\varrho^2 P(\varrho)$. Then for all $n \geqslant N(\varepsilon)$,

$$\|\mathfrak{F}_0-\mathfrak{P}_n\| \leqslant \|\mathfrak{F}_0-\mathfrak{P}_{N(\varepsilon)}\| \leqslant \|\mathfrak{F}_0-\mathfrak{P}\| < \varepsilon.$$

Next, let $\{\mathfrak{Q}_k(z)\}_{k=0, k\neq 2}^{\infty}$ denote the orthonormal sequence of functions of the form

$$\mathfrak{Q}_k(z) = a_0 + a_{-2}z^{-2} + \ldots + a_{-k}z^{-k}, \qquad a_{-k} > 0,$$

obtained by applying the Gram–Schmidt orthogonalization procedure to $1, z^{-2}, z^{-3}, \ldots$ with respect to the inner product of $L_2(\dot{D}^e)$. Then, it follows that

$$\mathfrak{P}_n(z) = \frac{\sum\limits_{\substack{k=0\\k\neq 2}}^{n} \overline{\mathfrak{Q}_k(\infty)}\mathfrak{Q}_k(z)}{\sum\limits_{\substack{k=0\\k\neq 2}}^{n} |\mathfrak{Q}_k(\infty)|^2} \tag{3.6}$$

and

$$\|\mathfrak{P}_n\|^2 = \Big(\sum_{\substack{k=0\\k\neq 2}}^{n} |\mathfrak{Q}_k(\infty)|^2\Big)^{-1}.$$

The $\mathfrak{Q}_k(z)$ have the advantage that they are determined algebraically in terms of the $A_{ik}(\dot{D}^e)$ defined by (3.4). Hence, no linear system need be solved to determine them.

4. Approximation of $F(z)$. The function $F(z)$ has an expansion at ∞ of the form

$$F(z) = z + c_0 + c_{-1}z^{-1} + c_{-2}z^{-2} + \ldots$$

It follows from Lemma 1.2 that

THEOREM 4.1. *If* $\mathfrak{P}(z)$ *is in class* $\mathfrak{M}$ *and*

$$\mathfrak{p}(z) = z - \int_0^{z^{-1}} [\mathfrak{P}^2(\zeta^{-1})-1]/\zeta^2 d\zeta,$$

then

$$\max_{\overline{D^e}} |F(z) - c_0 - \mathfrak{p}(z)| \leqslant \sqrt{2\pi\nu}\|\mathfrak{F}_0 - \mathfrak{P}\| + \tfrac{1}{2}\|\mathfrak{F}_0-\mathfrak{P}\|^2.$$

Let us consider the problem of approximating c_0. Let $\xi = \psi(\varrho)$ denote the unique function which is holomorphic in D^*, one-to-one and continuous on $\overline{D^*}$, maps $\overline{D^*}$ onto $\{|\xi| \leqslant \nu^{-1}\}$, and satisfies

$$\psi(0) = 0, \quad \psi'(0) = 1.$$

Then

$$F(z) = \frac{1}{\psi(z^{-1})}.$$

Furthermore, $\psi(\varrho)$ has an expansion in a neighborhood of the origin of the form

$$\psi(\varrho) = \varrho - c_0\varrho^2 + b_3\varrho^3 + b_4\varrho^4 + \dots$$

By setting $G = D^*$, the results of Section 1 can be used to approximate c_0, and to find upper bounds for ν^{-1}.

Because $\sqrt{\psi'(\varrho)}$ has the expansion $\sqrt{\psi'(\varrho)} = 1 - c_0\varrho + \dots$ in a neighborhood of the origin, for $P(\varrho) = 1 + a_1\varrho + \dots$ in $L_2(\dot{D}^*)$, we have

$$-c_0 - a_1 = \frac{1}{2\pi i} \int_{|\xi|=\frac{1}{4\nu^2}} \left[\sqrt{\psi'(\xi)} - P(\xi)\right]/\xi^2 d\xi,$$

where we have used the fact that D^* must contain $\{|\varrho| \leqslant 1/4\nu^2\}$, which follows from the proof of the 1/4 theorem. Hence

$$\begin{aligned}
(4.2) \qquad |a_1 - (-c_0)| &\leqslant \frac{16\nu^4}{2\pi} \int_{|\xi|=\frac{1}{4\nu^2}} \left|\sqrt{\psi'(\xi)} - P(\xi)\right| |d\xi| \\
&\leqslant \frac{16\nu^4}{2\pi}\left[\int_{|\xi|=\frac{1}{4\nu^2}} \left|\sqrt{\psi'(\xi)} - P(\xi)\right|^2\right]^{1/2} |d\xi| \cdot \frac{\sqrt{2\pi}}{2\nu} \\
&\leqslant \frac{8\nu^3}{\sqrt{2\pi}} \left\|\sqrt{\psi'} - P\right\|_{L_2(\dot{D}^*)} \\
&= \frac{8\nu^3}{\sqrt{2\pi}} [\|P\|^2_{L_2(\dot{D}^*)} - 2\pi\nu^{-1}]^{1/2}.
\end{aligned}$$

Putting (4.1) and (4.2) together and setting $\mathfrak{P}(z) = \mathfrak{P}_n(z)$ and $P(z) = P_m(z)$ we obtain

$$(4.3)\qquad \max_{\overline{D^e}} \left|F(z) - \left(-a_1 + \mathfrak{p}_n(z)\right)\right| \leqslant 2\pi\nu[\nu^{-1}\|\mathfrak{P}_n\|^2/2\pi - 1]^{1/2} + \\ + \pi\nu[\nu^{-1}\|\mathfrak{P}_n\|^2/2\pi - 1] + \frac{8\nu^{5/2}}{\sqrt{2\pi}}[\nu\|P_m\|^2_{L_2(\dot{D}^*)}/2\pi - 1]^{1/2}.$$

Suppose now that $\dot{D} = \{z = \lambda(t)\colon\ 0 \leqslant t \leqslant 1\}$. Then

$$A_{ik}(\dot{D}^e) = \int_0^1 \lambda^{-i}(t)\overline{\lambda^{-k}(t)}\,|\lambda'(t)|\,dt$$

and

$$A_{ik}(\dot{D}^*) = \int_0^1 \lambda(t)^{-i-1}\,\overline{\lambda(t)^{-k-1}}\,|\lambda'(t)|\,dt = A_{i+1,k+1}(\dot{D}^e).$$

We must find solutions to the systems

$$(4.4)\qquad \sum_{i=2}^{n} A_{ik}(\dot{D}^e)\,a_{-1} = -A_{0k}(\dot{D}^e), \qquad k = 2, \ldots, n,$$

and

$$(4.5)\qquad \sum_{i=1}^{m} A_{i+1,k+1}(\dot{D}^e)\,a_i = -A_{1,k+1}(\dot{D}^e), \quad k = 1, 2, \ldots, m,$$

or use (1.7) and (3.6). Note that if we choose $m = n-1$, then (4.5) can be rewritten as

$$(4.6)\qquad \sum_{j=2}^{n} A_{jl}(\dot{D}^e)\,a_{j-1} = -A_{1l}(\dot{D}^e), \qquad l = 2, \ldots, n,$$

and (4.4) and (4.6) have the same coefficient matrices.

Remark. In practical applications of inequality (1.8), upper bounds for R^{-1} are required. These can be obtained by applying the results of Sections 2 and 3 with D replaced by G^*.

5. An example. Let G be the square $\{z = x + iy\colon\ |x| < 1,\ |y| < 1\}$. Because of symmetry, $A_{jk}(\dot{G}) = A_{kj}(\dot{G})$ and

$$A_{jk}(\dot{G}) = 2(1 + i^{j-k})\,Re\int_{-1}^{1}(x+i)^j(x-i)^k\,dx,$$

from which it follows that

$$(5.1)\qquad A_{jk}(\dot{G}) = \begin{cases} 0, & k - j \neq 4l, \\ 8\displaystyle\int_0^1 (x^2+1)^k Re(x+i)^{j-k}\,dx, & l = 0, \pm 1, \ldots \end{cases}$$

Now let

$$a_{Kl} = \int_0^{\pi/4} \cos^K\theta \cos l\theta \, d\theta .$$

Noting that $A_{00}(\dot{G}) = 8$, a change of variable in the integral appearing in (5.1) yields

$$\frac{A_{j,j+4l}(\dot{G})}{A_{00}(\dot{G})} = a_{-2j-2-4l,4l}.$$

The following recurrence formulas are helpful in evaluating a_{Kl}:

$$(5.2) \qquad a_{Kl} = \frac{K-l+2}{K+1} a_{K+1,l-1} + \frac{(\sqrt{2})^{-K-1}}{K+1} \sin(l-1)^{\pi/4}, \qquad K \neq -1,$$

$$(5.3) \qquad a_{K-1,l} + \frac{(\sqrt{2})^{-K-1}}{K}\left[\cos\frac{l\pi}{4} - \frac{l}{K+1}\sin\frac{l\pi}{4}\right] = \frac{(K+1)^2-l^2}{K(K+1)} a_{K+1,l}, \qquad K \neq 0, -1.$$

Applying Section 1 to $\dot{G}$ we find that $P_{4l+r}(z) = P_{4l}(z)$, $l = 0, 1, 2, \ldots$, $r = 1, 2, 3$ and for $m = 8$ we have $(A_{jk}(\dot{G}) = A_{jk})$

$$a_4 = \frac{-A_{04}A_{88} + A_{08}A_{48}}{A_{44}A_{08} - A_{48}^2} = .1803870919,$$

$$a_8 = \frac{-A_{44}A_{08} + A_{04}A_{48}}{A_{44}A_{88} - A_{48}^2} = -.0043608125;$$

and the corresponding upper bound for R is given by

$$\frac{\|P_8(z)\|}{2\pi} L_2(\dot{G}) = \frac{A_{00}}{2\pi}\left[1 + \frac{2a_4A_{04}}{A_{00}} + \frac{2a_8A_{08}}{A_{00}} + \frac{2a_4a_8A_{48}}{A_{00}} + \frac{a_4^2A_{44}}{A_{00}} + \frac{a_8^2A_{88}}{A_{00}}\right] = 1.079627901.$$

Applying Sections 2 and 3 to G^* we again find that $\mathfrak{P}_{4l+r}(z) = \mathfrak{P}_{4l}(z)$, $l = 0, 1, 2, \ldots, r = 1, 2, 3$ and for $n = 8$ we have $(A_{jk}(\dot{G}^{*e}) = A_{j-1,k-1}(\dot{G}) = A_{j-1,k-1})$

$$a_{-4} = \frac{-A_{3,-1}A_{77} + A_{7,-1}A_{37}}{A_{33}A_{77} - A_{37}^2} = .1077016034,$$

$$a_{-8} = \frac{-A_{33}A_{7,-1} + A_{3,-1}A_{37}}{A_{33}A_{77} - A_{37}^2} = -.0174660527;$$

and the corresponding uper bound for R^{-1} is given by

$$\frac{\|\mathfrak{P}_8(z)\|}{2\pi} L_2(G^{*e}) = \frac{A_{-1,-1}}{2\pi}\left[1 + \frac{2\alpha_{-4}A_{3,-1}}{A_{-1,-1}} + \right.$$
$$\left. + \frac{2\alpha_{-8}A_{7,-1}}{A_{-1,-1}} + \frac{2\alpha_{-4}\alpha_{-8}A_{3,7}}{A_{-1,-1}} + \frac{\alpha_{-4}^2 A_{33}}{A_{-1,-1}} + \frac{\alpha_{-8}^2 A_{77}}{A_{-1,-1}}\right] = .9279807598.$$

This yields a lower bound for R of 1.07760855. Finally, setting $P(z) = P_8(z)$ in Theorem 1.3 we have an approximation for $f(z)$ for the square. The right-hand side of (1.8) has the value .3000054679.

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