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Non-linear stationary parabolic boundary value problems in an infinite cylinder

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Abstract. This paper deals with the system of p quasi-linear differential equations

(1) $$D_t u - \sum_{|k|=2b} A_k(x) D_x^k u = F(x, t, D_x^\gamma u)$$

in the cylinder $Q_\infty = \Omega \times \langle 0, \infty)$ $(0 \leqslant |\gamma| \leqslant 2b-1)$ with data

(2) $$u|_{t=0} = 0, \quad x \in \Omega,$$

and

(3) $$\mathscr{B}_q(x, D_x) u|_{\Gamma_\infty} = 0$$

($\Gamma_\infty = \partial\Omega \times \langle 0, \infty)$), where $\mathscr{B}_q$ is a linear differential operator of order less or equal than $2b-1$ for $q = 1, \ldots, bp$. Using a priori estimations of the Green function, the existence theorem for problem (1), (2), (3) is established. The result is obtained in a locally convex topological space of sufficiently smooth Hölder functions.

1. Introduction. In the widely worked classic and non-classical theory of parabolic differential equations (see for instance [3]–[5]), various mixed problems have been studied on the finite time-cylinder $Q = \Omega \times \langle 0, T\rangle$, where Ω is a domain of the Euclidean space $\boldsymbol{R}_m$ and T is a fixed positive number. In paper [1] an initial-boundary value problem for the system

(i) $$D_t u - \sum_{|k|=2b} A_k(x, t) D_x^k u = F(x, t) u$$

with a non-linear differential operator $F(x, t)$ of order $\leqslant 2b-1$ was considered on Q. In this case, owing to the boundedness of the cylinder Q, the growth of the operator $F(x, t)$ may be very strong.

The topic of the present paper is the existence of solution of a stationary initial-boundary value problem for an equation of type (i) on the infinite cylinder $\Omega \times \langle 0, \infty)$. This question is studied in the complete topological space of locally bounded and locally Hölder continuous functions on $\Omega \times \langle 0, \infty)$. The conditions for solvability of the problem in question are limiting the growth of $F(x, t)$ much more than the conditions in [1]. However, they show the dependence of the growth of F by the selection of the time cylinder.

By the method described below we may solve the non-linear Cauchy problem for equation (i) on the unbounded sets $\boldsymbol{R}_m \times \langle 0, T\rangle$ or $\boldsymbol{R}_m \times \langle 0, \infty)$ in the class of locally Hölder continuous functions.

2. The formulation of problem. First of all we introduce notions and notations to be used throughout this paper.

a. The symbol Ω denotes a bounded domain of the real m-dimensional Euclidean space $\boldsymbol{R}_m$, $m \geqslant 2$, with the boundary $\partial\Omega$ and the diameter $\operatorname{diam}\Omega$. For $0 < T < \infty$ we define the bounded cylindrical domain $Q_T = \Omega \times \langle 0, T\rangle$ with the lateral surface $\Gamma_T = \partial\Omega \times \langle 0, T\rangle$, and for $T = \infty$ we put $Q_\infty = \Omega \times \langle 0, \infty)$ with $\Gamma_\infty = \partial\Omega \times \langle 0, \infty)$.

For any $x = (x_1, \ldots, x_m) \in \boldsymbol{R}_m$ and any multiindex or multiexponent $k = (k_1, \ldots, k_m)$, where k_i are non-negative integer, we write $|x| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$, $|k| = \sum_{i=1}^{m} k_i$, and $x^k = x_1^{k_1} \ldots x_m^{k_m}$ and $D_x^k = D_{x_1}^{k_1} \ldots D_{x_m}^{k_m} = \partial^{|k|}/\partial x_1^{k_1} \ldots x_m^{k_m}$. If k_0 is a non-negative integer, then $D_t^{k_0}$ denotes the differential operator $\partial^{k_0}/\partial t^{k_0}$.

Further, for any integer r, $t(r)$ means $\operatorname{Card}\{k = (k_1, \ldots, k_m)\colon |k| = r\}$ and $s = \sum_{r=0}^{2b-1} t(r)$ for $b \geqslant 1$. The Cartesian product $Q_\infty \times \prod_{i=1}^{s} \prod_{j=1}^{p} \{-\infty < u_j^i < \infty\}$ for $p \geqslant 1$ will be denoted by H_∞.

b. Let $(\mathscr{U}_m, \leqslant)$ and $(\mathscr{M}_m, \leqslant)$ denote the partially ordered set of all real $(p \times 1)$-vector functions $u(x) = \big(u_1(x), \ldots, u_p(x)\big)$ and the set of all real $(p \times p)$-matrix functions $A(x) = \big(a_{ij}(x)\big)_{i,j=1}^{p}$, respectively, with the range of definition in $\boldsymbol{R}_m$ and with the natural ordering.

By J and O we denote the $(p \times 1)$-unit vector and the zero vector. E_1 means the $(p \times p)$-matrix whose all elements are equal to 1 and E is the unit matrix. (u, v) is the scalar product of the vector functions u and v of $\mathscr{U}_m$. Finally, we write $|u(x)|^a = \big(|u_1(x)|^a, \ldots, |u_m(x)|^a\big)$ and $|A(x)|^a = \big(|a_{ij}(x)|^a\big)_{i,j=1}^{p}$ for any $a \in \boldsymbol{R}_1$.

c. Now we shall define some classes of vector and matrix functions.

A vector function $u \in (\mathscr{U}_{m_1+m_2}, \leqslant)$ mapping $\Omega_1 \times \Omega_2 \subset \boldsymbol{R}_{m_1} \times \boldsymbol{R}_{m_2}$ into $\boldsymbol{R}_p$ $(m_1, m_2, p \geqslant 1)$ is said to be *Hölder continuous* with respect to x on Ω_1 with the exponent ϱ $(0 < \varrho \leqslant 1)$ and uniformly with respect to λ on Ω_2 if and only if there is a constant $L = L(u) > 0$ such that for any pair of points $x, y \in \Omega_1$ and every $\lambda \in \Omega_2$ the relation

$$|u(x, \lambda) - u(y, \lambda)| \leqslant L\,|x - y|^\varrho$$

holds. The set of all such vector functions will be denoted by $H_\varrho(x, \Omega_1; \lambda, \Omega_2)$. From the above definition it is obvious that if Ω_1 is a bounded set and $0 < \varrho \leqslant \beta \leqslant 1$, then $H_\beta(x, \Omega_1; \lambda, \Omega_2) \subset H_\varrho(x, \Omega_1; \lambda, \Omega_2)$.

Let Ω_1 be a domain in $\boldsymbol{R}_{m_1}$. If $u \in H_\varrho(x, \Delta_1; \lambda, \Omega_2)$ for any subset Δ_1 of Ω_1 such that $\bar{\Delta}_1 \subset \Omega_1$, then the vector function u is called *locally Hölder continuous* with respect to x on Ω_1 with the exponent ϱ and uniformly with respect to $\lambda \in \Omega_2$. The constant L depends of Δ_1. The set of all such vector functions will be denoted by $H_{\mathrm{loc},\varrho}(x, \Omega_1; \lambda, \Omega_2)$. If Ω_2 is a domain of $\boldsymbol{R}_{m_2}$ and $u \in H_\varrho(x, \Omega_1; \lambda, \Delta_2)$ for any set Δ_2 such that $\bar{\Delta}_2 \subset \Omega_2$, then u is called *Hölder continuous* with respect to x on Ω_1 with the exponent ϱ and locally uniformly with respect to λ on Ω_2. The constant L depends on Δ_2. The set of all such vector functions is denoted by $H_\varrho^{\mathrm{loc}}(x, \Omega_1; \lambda, \Omega_2)$. If Ω_1 and Ω_2 are domains and $u \in H_\varrho(x, \Delta_1; \lambda, \Delta_2)$ for any Δ_1 and Δ_2 such as above, then u is said to be *locally Hölder continuous* with respect to x on Ω_1 with the exponent ϱ and locally uniformly with respect to λ on Ω_2. The constant L depends on Δ_1 and Δ_2.

For an arbitrary integer $l \geqslant 0$ and a domain $\Omega \subset \boldsymbol{R}_m$ the Banach space of all vector functions $u \in (\mathscr{U}_m, \leqslant)$ defined on Ω into $\boldsymbol{R}_p$ such that the derivatives $D_x^k u$ up to the order l (including l) are continuous and bounded on Ω will be denoted by $C^l(\Omega)$. The norm in this space is defined by

$$\|u\|_{l,\Omega} = \max_{j=1,\ldots,p} \Big\{\sum_{i=0}^{l} \sum_{|k|=i} \sup_{\Omega} |D_x^k u_j(x)|\Big\}.$$

If $0 < a < 1$, then $C^{l+a}(\Omega)$ denotes the subspace of all vector functions $u \in C^l(\Omega)$ whose the derivatives of the l-th order satisfy the Hölder condition on Ω with the exponent a. In this case the norm is defined as the sum

$$\|u\|_{l,\Omega} + \max_{j=1,\ldots,p} \Big\{\sum_{|k|=l} \sup_{x,y \in \Omega} |D_x^k u_j(x) - D_x^k u_j(y)|/|x-y|^a\Big\}.$$

We say that the boundary $\partial\Omega$ belongs to the class C^{l+a} iff to each $\xi \in \partial\Omega$ there is $d > 0$ such that the part of the boundary $\partial\Omega$ contained in the ball $G = \{x \in \boldsymbol{R}_m: |x-\xi| \leqslant d\}$ may be explicitly expressed by an equation $x_m = g(x_1, \ldots, x_{m-1})$ in an orthonormal coordinate system $(0x_1 \ldots x_{m-1}x_m)$ in such a way that direction of the axis $0x_m$ coincides with that of the inner normal to $\partial\Omega$ at the point ξ and the scalar function g belongs to $C^{l+a}(G_0)$, where $G_0 = \{x' = (x_1, \ldots, x_{m-1}) \in \boldsymbol{R}_{m-1}: |x'| \leqslant d\}$.

Now we can formulate the stationary initial-boundary value problem in Q_∞.

Consider the system of $p \geqslant 1$ differential equations of $2b$-th order $(b \geqslant 1)$ with p unknown functions

$$(1) \qquad \mathscr{L}(x, D_x, D_t)u \overset{\mathrm{df}}{=} D_t u - \sum_{|k|=2b} A_k(x) D_x^k u = F(x, t, \ldots, D_x^\gamma u, \ldots), \quad (x, t) \in Q_\infty,$$

where $\gamma = (\gamma_1, \ldots, \gamma_m)$ is a multiindex such that $0 \leqslant |\gamma| \leqslant 2b-1$. The solution of (1) is required to fulfil the initial condition

$$u|_{t=0} = O, \quad x \in \Omega, \tag{2}$$

and the boundary conditions

$$\mathscr{B}_q(x, D_x)u\Big|_{\Gamma_\infty} \stackrel{\mathrm{df}}{=} \sum_{|k|\leqslant r_q} (B_k^{(q)}, D_x^k u)\Big|_{\Gamma_\infty} = 0 \tag{3}$$

for $r_q \leqslant 2b-1$ and $q = 1, \ldots, bp \cdot A_k(\cdot) = \big(a_k^{hj}(\cdot)\big)_{h,j=1}^r$ is a matrix function and $B_k^{(q)}(\cdot) = \big(b_k^{q1}(\cdot), \ldots, b_k^{qp}(\cdot)\big)$ is a vector function on Ω and $F = (f_1, \ldots, f_p)$ on H_∞.

Problem (1), (2), (3) just formulated will be solved in the following class of Hölder continuous functions:

Consider the non-decreasing parametric function $f_{B,\sigma}$ mapping the interval $\langle 0, \infty)$ into $\langle 0, \infty)$ with the properties

$$f_{B,\sigma}(t) \geqslant \int_0^t z^{-\sigma} e^{Bz} dz \tag{I}$$

for $t \in \langle 0, \infty)$ and $B > 0$ and $0 < \sigma < 1$ and

$$f_{B,\sigma}(0) > 0. \tag{II}$$

Further, for a real function v on $\boldsymbol{R}_{m+1}$ and for $0 < a < 1$ we put

$$\langle v(x, t)\rangle_{a,x} = |v(x, t) - v(y, t)| \cdot |x-y|^{-a}$$

and

$$\langle v(x, t)\rangle_{a,t} = |v(x, t) - v(x, t')|\, |t-t'|^{-a}.$$

Then for $b \geqslant 1$ and $0 < a < 1$ we define the linear space $C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$ of the vector functions $u(x, t) = \big(u_1(x, t), \ldots, u_p(x, t)\big)$ by the inequality

$$\begin{aligned} \|u\|^{f(B,\varkappa,\mu,\nu)}_{2b-1+a,Q_\infty} \stackrel{\mathrm{df}}{=} \max_{j=1,\ldots,p} \Big\{ \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{Q_\infty} [|D_x^k u_j(x, t)| f_{B,\varkappa}^{-1}(t)] + \\ + \sum_{|k|=2b-1} \sup_{\substack{(x,t),(y,t)\in Q_\infty \\ x\neq y}} [\langle D_x^k u_j(x, t)\rangle_{a,x} f_{B,\mu}^{-1}(t)] + \\ + \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{\substack{(x,t)(x,t')\in Q_\infty \\ t\neq t'}} [\langle D_x^k u_j(x, t)\rangle_{(2b-1+a-|k|)/2b,t} f_{B,\nu}^{-1}(|t-t'|) f_{B,\nu}^{-1}(t^*)]\Big\} < \infty, \end{aligned} \tag{4}$$

where the parameters $\varkappa$, μ and ν belong to the interval $(0, 1)$ and $t^* = \max(t, t')$.

Remark 1. If we take Q_T, $0 < T < \infty$ instead of Q_∞ in (4), then the norms $\|u\|^{f(B,\varkappa,\mu,\nu)}_{2b-1+a,Q_T}$ and $\|U\|^{Q,p}_{2b-1+a}$ from [1] (p. 38) are mutually equivalent.

Remark 2. a. We immediately see that if $u \in C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$, then $D^k_x u \in H^{\mathrm{loc}}_\alpha(x,\ \Omega;\ t,\ \langle 0,\ \infty))$ for $|k| = 2b-1$ and $D^k_x u \in H_{\mathrm{loc},(2b-1+\alpha-|k|)/2b}(t,\ \langle 0,\ \infty);\ x,\ \Omega)$ for $|k| = 0, 1, \ldots, 2b-1$.

b. The derivatives $D^k_x u$ for $|k| = 2b-1$ can be continuously proceeded to $\boldsymbol{R}_m \times \langle 0, T\rangle$ for any $T \in (0, \infty)$ and then, using the mean value theorem and the relation

$$K_m \sum_{i=1}^{m} |x_i| \leqslant |x| \leqslant \sum_{i=1}^{m} |x_i| \tag{5}$$

for $x \in \boldsymbol{R}_m$ and $K_m \in (0, (1/\sqrt{2})^{m-1})$, one obtains: $D^k_x u \in H^{\mathrm{loc}}_1(x,\ \Omega;\ t, \langle 0,\ \infty))$ for $|k| = 0, 1, \ldots, 2b-2$.

c. By $\tilde{M}$ we shall denote the set of all matrix functions belonging to $(\mathscr{M}_m, \leqslant)$ whose rows and columns are vector functions of M.

3. Some assumptions and statements. The operators $\mathscr{L}$ and $\mathscr{B}_q$ and the boundary $\partial\Omega$ in problem (1), (2), (3) are required to satisfy the following assumptions:

ASSUMPTION (A). *System* (1) *is uniformly parabolic in the sense of I. G. Petrovskij, i.e., there is a constant $\delta > 0$ (independent of x) such that the roots $a_s(x, \xi)$ of the polynomial*

$$L(x; i\xi, a) = \det\mathscr{L}(x; i\xi, a) = \det\Big(\delta_{hj}a - i^{2b} \sum_{|k|=2b} a^{hj}_k(x)\,\xi^k\Big)^{p}_{h,j=1}$$

($i = \sqrt{-1}$ and $\delta_{h,j}$ is the Kronecker symbol) satisfy the inequality $\operatorname{Re} a_s(x, \xi) \leqslant -\delta|\xi|^{2b}$ *for $x \in \Omega$ and $\xi \in \boldsymbol{R}_m$.*

ASSUMPTION (B). *The operator $\mathscr{B}_q$ is connected with system* (1) *by the "uniform supplementary" condition: Let x be a point of $\partial\Omega$ and let $\nu(x) = (\nu_1(x), \ldots, \nu_m(x))$ be the unit vector of the inner normal to $\partial\Omega$ at the point x and let $\zeta(x)$ be a vector lying in the tangent plane to $\partial\Omega$ at x. Then for any $x \in \partial\Omega$ and for a complex number a with the properties* $\operatorname{Re} a \geqslant -\delta_1|\zeta|^{2b}$ *and* $|a|^2 + |\zeta|^{2b} > 0$ *($0 < \delta_1 < \delta$, δ_1 is an absolute constant) and for every vector $\zeta(x)$, the rows of the matrix*

$$\Big(i^{r_q} \sum_{|k|=r_q} b^{qj}_k(x)[\zeta(x) + \tau\nu(x)]^k\Big)^{bp,p}_{q=1,j=1} \hat{\mathscr{L}}(x,\ i(\zeta+\tau\nu),\ a),$$

where $\hat{\mathscr{L}} = L\mathscr{L}^{-1}$, are linearly independent vector functions of τ with respect to the module of the polynomial $M^+(x; \zeta, \tau, a) = \prod_{q=1}^{bp} (\tau - \tau^+_q(x; \zeta, a))$; *$\tau^+_q$ for $q = 1, \ldots, bp$ are the roots of the polynomial $L(x; i(\zeta+\tau\nu), a)$ (in τ) with positive imaginary parts.*

ASSUMPTION ($D_{l+\alpha}$). *Let $l \geqslant 0$ be an integer and let $0 < \alpha < 1$. The $(p \times p)$-matrix coefficients $A_k(x)$ of* (1) *belong to $\tilde{C}^{l+\alpha}(\Omega)$ for $|k| = 2b$ and $B^{(q)}_k(x)$*

$\in C^{l+2b-r_q+a}(\partial\Omega)$ *for* $|k| \leqslant r_q \leqslant 2b-1$ *and* $q = 1, \ldots, bp$. *The boundary* $\partial\Omega$ *belongs to the class* C^{l+2b+a}.

The estimations of the Green matrix of the operator $\mathscr{L}$ and its derivatives on the infinite cylinder Q_∞ are established by

THEOREM 1 (Eidelman and Ivasišen [2]). *Let assumptions* (A), (B) *and* (D_{l+a}) *be fulfilled. Then there exists the Green matrix function* $G(x, t; \xi, \tau) \in \mathscr{M}_{2(m+1)}$ *of problem* (1), (2), (3) (*with* $F = 0$). *For* $0 \leqslant \tau < t_0 < t < \infty$ *and* $x, y, \xi \in \boldsymbol{R}_m$ ($r = 1/(2b-1)$) *we have*

$$(6) \qquad |D_t^{k_0} D_x^k G(x, t; \xi, \tau)| \leqslant C(t-\tau)^{-(m+2bk_0+|k|)/2b} \exp\{A(t-\tau) - c|x-\xi|^{2br}/(t-\tau)^r\} E_1$$

if $2bk_0 + |k| \leqslant 2b + l$;

$$(7) \qquad |D_t^{k_0} D_x^k G(x, t; \xi, \tau) - D_t^{k_0} D_x^k G(y, t; \xi, \tau)| \leqslant C|x-y|^a (t-\tau)^{-(m+2bk_0+|k|+a)/2b} \exp\{A(t-\tau) - c|x^* - \xi|^{2br}/(t-\tau)^r\} E_1$$

if $2bk_0 + |k| = 2b + l$ *and* $|x^* - \xi| = \min(|x-\xi|, |y-\xi|)$;

$$(8) \qquad |D_t^{k_0} D_x^k G(x, t; \xi, \tau) - D_t^{k_0} D_x^k G(x, t_0; \xi, \tau)| \leqslant C(t-t_0)^{(2b(1-k_0)+l-|k|+a)/2b}(t_0-\tau)^{-(m+2b+l+a)/2b} \times \exp\{A(t-\tau) - c|x-\xi|^{2br}/(t-\tau)^r\} E_1$$

if $l < 2bk_0 + |k| \leqslant 2b + l$. A, C, c *are positive constants independent of* x, y, t, t_0 *and* ξ, τ.

Remark 3. In our considerations we shall often use instead of estimation (6) its modified form

$$(6') \qquad \begin{aligned} |D_t^{k_0} D_x^k G(x, t; \xi, \tau)| &\leqslant C(t-\tau)^{-\mu} |x-\xi|^{2b\mu-(m+2bk_0+|k|)} \times \\ &\quad \times [|x-\xi|^{2b}/(t-\tau)]^{(m+2bk_0+|k|-2b\mu)/2b} \times \\ &\quad \times \exp\{-c|x-\xi|^{2br}/(t-\tau)^r\} e^{A(t-\tau)} E_1 \\ &\leqslant K(t-\tau)^{-\mu} |x-\xi|^{2b\mu-(m+2bk_0+|k|)} e^{A(t-\tau)} E_1 \end{aligned}$$

for $0 \leqslant \tau < t < \infty$ and $x, \xi \in \boldsymbol{R}_m$, $\xi \neq x$ and $\mu \leqslant (m+2bk_0+|k|)/2b$, where K is a positive constant independent of x, t, ξ and τ. If we have $|x-\xi|^{2b}/(t-\tau) \geqslant \varepsilon > 0$, then this estimation holds for any $\mu \in (-\infty, \infty)$.

Remark 4. If we consider the finite interval $\langle 0, T\rangle$, $T > 0$, instead of $\langle 0, \infty)$, then estimations (6), (7) and (8) reduce to the estimations of Theorem 2 in [1].

An obvious consequence of Remark 3 and Theorem 3 of [1] is

THEOREM 2. *Let assumptions* (A), (B), (D_{l+a}) *be satisfied and let* $\Phi \in C^0(Q_T) \cap H_a^{\mathrm{loc}}(x, \Omega; t, \langle 0, \infty))$ *be a* $(p \times 1)$-*vector function bounded in the* *norm* $\|\cdot\|_{0,Q_T}$ *for any cylinder* Q_T, $T \in (0, \infty)$. *Then the vector function*

$$u(x, t) = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau)\Phi(\xi, \tau)d\xi$$

is a solution of the linear equation $\mathscr{L}(x; D_x, D_t)u = \Phi(x, t)$ *on* Q_∞ *satisfying conditions* (2) *and* (3).

This last theorem enables us to investigate the solution of the nonlinear problem (1), (2), (3) in the unbounded cylinder Q_∞ by the Tichonov fixed point

THEOREM 3 (Tichonov [7]). *Let* (P, τ) *be a complete locally convex linear topological space and let* S *be a bounded closed and convex subset of* P. *Let* $\mathfrak{A}$ *be a continuous mapping defined on* S *into itself such that the closure of* $\mathfrak{A}S$ *is compact in* (P, τ). *Then the equation* $\mathfrak{A}u = u$ *has at least one solution in* S.

In the following text the symbol L will always denote positive constants.

4. The existence of a solution. To derive the fundamental theorem we introduce some properties of the space $P(Q_\infty) = C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(B,\varkappa,\mu,\nu)}(Q_\infty)$ and the Green function G and the integro-differential operator

$$\mathfrak{A}(x, t)u = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau)F[\xi, \tau, \ldots, D_x^\nu u(\xi, \tau), \ldots]d\xi. \tag{9}$$

It is obvious that the space $P(Q_\infty)$ with the norm defined in (4) does not form a Banach space. The sequence of functionals $\{\sigma_n\}_{n=1}^\infty$ defined by

$$\sigma_n(u) = \|u\|^{f(B,\varkappa,\mu,\nu)}_{2b-1+a,Q_n}, \qquad n = 1, 2, \ldots,$$

defines a countable monotone family of seminorms on $P(Q_\infty)$ satisfying the axiom of separation, that is, for any $u_0 \in P(Q_\infty)$, $u_0 \neq O$ there is n_0 such that $\sigma_{n_0}(u_0) \neq 0$. The linear space $P(Q_\infty)$, topologized by the family of seminorms $\{\sigma_n\}_{n=1}^\infty$ in the usual way, the sets

$$N(O, n, \varepsilon) = \{u \in P(Q_\infty)\colon \sigma_n(u) < \varepsilon\}, \qquad n = 1, 2, \ldots, \ \varepsilon > 0,$$

forming a local neighbourhood baze at zero is a locally convex linear topological Hausdorff space. Denote it by $(P(Q_\infty), \tau)$, where τ is the topology just defined.

LEMMA 1. *The space* $(P(Q_\infty), \tau)$ *is complete.*

Proof. Since the topology τ is defined by the countable family of seminorm $\{\sigma_n\}_{n=1}^\infty$, it is sufficient to show that the space $(P(Q_\infty), \tau)$ is sequentially complete for $p = 1$. Let $\{u_s(x, t)\}_{s=1}^\infty$ be a fundamental sequence of real functions of $(P(Q_\infty), \tau)$, that is, $u_l - u_s \in N(O, n, \varepsilon)$ for any $l, s > s_0(n, \varepsilon)$ (s_0 is a fixed positive integer) and any neighbourhood

$N(O, n, \varepsilon)$. Hence

$$|D_x^k u_l(x, t) - D_x^k u_s(x, t)| \leqslant \varepsilon f_{B,\varkappa}(t) \leqslant \varepsilon f_{B,\varkappa}(n)$$

for $|k| = 0, 1, \ldots, 2b-1$ on any finite cylinder Q_n. Consequently there is a function $u \in C^{2b-1}(Q_\infty)$ such that $\lim_{s\to\infty} D_x^k u_s(x, t) = D_x^k u(x, t)$ at every point $(x, t) \in Q_\infty$ for $|k| = 0, 1, \ldots, 2b-1$. Letting $s \to \infty$ in the relations

$$|D_x^k u_s(x, t)| f_{B,\varkappa}^{-1}(t) \leqslant L \quad \text{for } k = 0, 1, \ldots, 2b-1,$$

$$|D_x^k u_s(x, t) - D_x^k u_s(y, t)| f_{B,\mu}^{-1}(t) \leqslant L|x-y|^a \quad \text{for } |k| = 2b-1$$

and

$$|D_x^k u_s(x, t) - D_x^k u_s(x, t')| f_{B,\nu}^{-1}(|t-t'|) f_{B,\nu}^{-1}(t^*) \leqslant L|t-t'|^{(2b-1+a-|k|)/2b} \quad \text{for } |k| = 0, 1, \ldots, 2b-1,$$

we get $u \in P(Q_\infty)$. From the inequality $\sigma_n(u_l - u_s) < \varepsilon$ we easily obtain $u_s - u \in N(O, n, L\varepsilon)$ for all $s > s_0(n, \varepsilon)$, and this guarantees the convergence of the sequence $\{u_s(x, t)\}_{s=1}^\infty$ to $u(x, t)$ in the topology τ.

LEMMA 2. *Let* $(x, t), (y, t), (x, t'), t < t'$ *be points of* Q_∞ *and let* $|k| = 0, 1, \ldots, 2b-1$ *and* $\beta \in (0, 1)$. *If hypotheses* (A), (B) *and* (D_a) *hold, then*

$$(10) \qquad I_{1,k}(x, t) = \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau)| d\xi \leqslant L f_{A,\varkappa}(t) E_1$$

for $0 < \varkappa < (m+|k|)/2b$ $(\varkappa < 1)$ *and*

$$(11) \qquad I_{2,k}(x, y, t) = \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau) - D_x^k G(y, t; \xi, \tau)| d\xi \leqslant L|x-y|^\beta f_{A,\mu}(t) g(|x-y|) E_1 \leqslant L|x-y|^\beta f_{A,\mu}(t) E_1$$

for $\mu \in (0, 1)$ *such that* $(|k|+1)/2b \leqslant \mu \leqslant (m+|k|)/2b$ *if* $0 \leqslant |k| < 2b-2$ *and* $(2b-1+\beta)/2b \leqslant \mu$ *if* $|k| = 2b-1$, *where* $g(z) = z^{1-\beta+2b(\mu-1)[|k|/(2b-1)]}$. (The expression $[x]$ in the exponent denotes the integer for which $[x] \leqslant x < [x]+1$.)

If conditions (A), (B) *and* (D_{2b-1+a}) *are satisfied, then*

$$(12) \qquad I_{3,k}(x, t, t') = \int_0^t d\tau \int_\Omega |D_x^k G(x, t; \xi, \tau) - D_x^k G(x, t'; \xi, \tau)| d\xi + \int_t^{t'} d\tau \int_\Omega |D_x^k G(x, t'; \xi, \tau)| d\xi$$
$$\leqslant L(t'-t)^{(2b-1+\beta-|k|)/2b} f_{A,\nu}(t'-t) f_{A,\nu}(t') h(t'-t) E_1 \leqslant L(t'-t)^{(2b-1+\beta-|k|)/2b} f_{A,\nu}(t'-t) f_{A,\nu}(t') E_1$$

for $(4b-1+\beta)/4b \leqslant \nu < 1$, *where* $h(z) = z^{(2b\nu-2b+1-\beta)/2b} f_{B,\nu}^{-1}(z)$.

Proof. Estimation (10) follows directly by (6'). For $0 \leqslant |k| \leqslant 2b-2$ inequality (11) can be determined from (6') with use of the mean value theorem. Indeed, there is $x_i^* = (y_1, \ldots, y_{i-1}, \zeta_i, x_{i+1}, \ldots, x_m) \in \boldsymbol{R}_m$ $(y = (y_1, \ldots, y_m))$ such that

$$|D_x^k G(x, t; \xi, \tau) - D_x^k G(y, t; \xi, \tau)| \leqslant \sum_{i=1}^{m} |x_i - y_i| |D_x^{k(i)} G(x_i^*, t; \xi, \tau)|,$$

where $k(i) = (k_1, \ldots, k_{i-1}, k_i+1, k_{i+1}, \ldots, k_m)$ and ζ_i lies between x_i and y_i and $|x-y| > |x_i^* - x|$. Hence, in view of (5), we obtain for $(|k|+1)/2b < \mu \leqslant (m+|k|)/2b$ $(\mu < 1)$

$$I_{2,k}(x, y, t) \leqslant L|x-y| f_{\Delta,\mu}(t) \max_{i=1,\ldots,m} \int_{\Omega} |x_i^* - \xi|^{2b\mu-(m+|k(i)|)} d\xi E_1,$$

which proves (11).

Let now $|k| = 2b-1$. Divide the domain Ω into two subset $S_1 = \{\xi \in \Omega\colon |x-\xi| > 2|x-y|\}$ and $S_2 = \Omega - S_1$. From the inequality $|\xi - x| > 2|x-y|$ we get $|x-\xi| < 2|x_i^* - \xi|$, whence

$$I_{2,k}(x, y, t) \leqslant L f_{\Delta,\mu}(t) \Big\{ 2^{m+2b-2b\mu} \int_{S_1} |x-y| |x-\xi|^{2b\mu-(m+2b)} d\xi +$$
$$+ \int_{S_2} [|x-\xi|^{2b\mu-(m+2b-1)} + |y-\xi|^{2b\mu-(m+2b-1)}] d\xi \Big\} E_1.$$

Both integrals converge for $(2b-1+\beta)/2b \leqslant \mu < 1$ and so (11) is true for $|k| = 2b-1$, too.

To prove the third estimation, we put $S_3 = \{\xi \in \Omega\colon |\xi - x| > |t'-t|^{1/2b}\}$ and $S_4 = \Omega - S_3$. By the mean value theorem we find $\tilde{t} \in (t, t')$ such that for $|k| = 0, 1, \ldots, 2b-1$ and $(2b-1+\beta)/2b < (4b-1+\beta)/4b \leqslant \nu < 1$ and $0 < \tau < |k|/2b$

$$\begin{aligned} I_{3,k}(x, t, t') \leqslant & \int_0^t d\tau \int_{S_3} |D_x^k G(x, t; \xi, \tau) - D_x^k G(x, t'; \xi, \tau)| d\xi + \\ & + \int_t^{t'} d\tau \int_{S_3} |D_x^k G(x, t'; \xi, \tau)| d\xi + \int_0^t d\tau \int_{S_4} |D_x^k G(x, t; \xi, \tau)| d\xi + \\ & + \int_0^{t'} d\tau \int_{S_4} |D_x^k G(x, t'; \xi, \tau)| d\xi \\ \leqslant & L \Big\{ (t'-t)(t'-t)^{(2b\nu-2b-|k|)/2b} \int_0^t (\tilde{t}-\tau)^{-\nu} e^{\Delta(\tilde{t}-\tau)} d\tau + \\ & + (t'-t)^{(2b\lambda-|k|)/2b} f_{\Delta,\lambda}(t'-t) + (t'-t)^{(2b\nu-|k|)/2b} f_{\Delta,\nu}(t) + \\ & + (t'-t)^{(2b\nu-|k|)/2b} f_{\Delta,\nu}(t') \Big\} E_1. \end{aligned}$$

Since $\tau < \nu < 1$, we easily see that $(t'-t)^{\lambda-\nu} f_{A,\lambda}(t'-t) < f_{A,\nu}(t'-t)$. Hence and by the monotonicity of $f_{A,\nu}$ one obtains

$$I_{3,k}(x, t, t') \leqslant L(t'-t)^{(2b\nu-|k|)/2b} f_{A,\nu}(t') E_1.$$

Finally, relation (12) follows from the boundedness of the function

$$(t'-t)^{(2b\nu-2b+1-\beta)/2b} \Big(\int\limits_0^{t'-t} z^{-\nu}\, e^{Az}\, dz \Big)^{-1}$$

for $(t, t') \in \langle 0, \infty) \times \langle 0, \infty)$.

LEMMA 3. *Let conditions* (A), (B), (D_{2b-1+a}) *be fulfilled and let the vector function* F *be continuous and bounded in the norm* $\|\cdot\|_{0,H_\infty}$. *Then there is a real number* $R > 0$ *such that* $\mathfrak{A}(x, t)P(Q_\infty) \subset S_R = \{u \in P(Q_\infty)$: $\|u\|_{2b-1+a,Q_\infty}^{f(A,\varkappa,\mu,\nu)} \leqslant R\}$. (The parameters $\varkappa$, μ and ν are defined as in Lemma 2.)

Proof. Let $u \in P(Q_\infty)$ and $\|F\|_{0,H_\infty} \leqslant L$. Using (9) and Lemma 2 for $\beta = a$ we have for $(x, t), (y, t), (x, t') \in Q_\infty$ and $t < t'$

$$|D_x^k \mathfrak{A}(x, t)u| \leqslant L I_{1,k}\,(x, t) J \leqslant pL f_{A,\varkappa}(t) J$$

and

$$|D_x^k \mathfrak{A}(x, t)u - D_x^k \mathfrak{A}(x, t')u| \leqslant L I_{3,k}(x, t, t') J$$
$$\leqslant pL(t'-t)^{(2b-1+a-|k|)/2b} f_{A,\nu}(t'-t) f_{A,\nu}\,(t') J$$

for $|k| = 0, 1, \ldots, 2b-1$ and

$$|D_x^k \mathfrak{A}(x, t)u - D_x^k \mathfrak{A}(y, t)u| \leqslant L I_{2,k}(x, y, t) J \leqslant pL\,|x-y|^a f_{A,\mu}(t) J$$

for $|k| = 2b-1$. Thus it is sufficient to take $R \geqslant pL\{2s + t(2b-1)\}$.

Now we may formulate the existence theorem.

THEOREM 4. *Let conditions* (A), (B), (D_{2b-1+a}) *be satisfied and let the right-hand side* F *of* (1) *be a continuous vector function bounded in the norm* $\|\cdot\|_{0,\tilde H_\infty}$, *where* $\tilde H_\infty = Q_\infty \times \prod_{i=1}^{s} \prod_{j=1}^{p} \{-R f_{A,\varkappa}(t) \leqslant u_j^i \leqslant R f_{A,\varkappa}(t)\} \subset H_\infty$. ($R$ is the constant occurring in Lemma 3.) *Further, let the Hölder condition*

$$(13) \qquad |F(x, t, \ldots, u^\gamma, \ldots) - F(y, t, \ldots, v^\gamma, \ldots)|$$
$$\leqslant \Big\{ q(t)\,|x-y|^\beta + \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} q^\gamma,\ |u^\gamma - v^\gamma|^{\beta_\gamma} \Big\} J$$

be fulfilled for $\beta, \beta_\gamma \in (0, 1\rangle$ *and* $(x, t, \ldots, u^\gamma, \ldots), (y, t, \ldots, v^\gamma, \ldots) \in \tilde H_\infty$, *where* $q^\gamma(t) = \big(q_1^\gamma(t), \ldots, q_p^\gamma(t)\big)$ *and* $q_j^\gamma(t) \geqslant 0$, $q\,(t) > 0$ *for* $j = 1, \ldots, p$ *and* $|\gamma| = 0, 1, \ldots, 2b-1$ *are bounded and integrable real functions on* $\langle 0, T\rangle$ *for every* $T > 0$. *Then problem* (1), (2), (3) *has at least one solution* u

belonging to $C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(\Delta,\varkappa,\mu,\nu)}(Q_\infty)$ *for which* $\|u\|^{f(\Delta,\varkappa,\mu,\nu)}_{2b-1+\alpha,Q_\infty} \leqslant R_0$, *where* $R_0 \geqslant R$. *Here the parameter* $\varkappa < 1$ *is such that* $0 < \varkappa < (m+|k|)/2b$ *for* $|k| = 0, 1, \ldots, 2b-1$ *and* $\mu \in (0, 1)$ *is such that* $(|k|+1)/2b \leqslant \mu \leqslant (m+|k|)/2b$ *for* $|k| = 0, 1, \ldots, 2b-2$ *and* $\mu \geqslant (2b-1+\beta)/2b$ *for* $|k| = 2b-1$ *and* $(4b-1+\beta)/4b \leqslant \nu < 1$.

Proof. According to Lemma 1, $\left(C^{2b-1+\alpha,(2b-1+\alpha)/2b}_{x,t,f(\Delta,\varkappa,\mu,\nu)}(Q_\infty), \tau\right)$ is a complete locally convex linear topological Hausdorff space. The ball S_R is a bounded closed and convex set in the topology τ and the operator $\mathfrak{A}(x, t)$ given by (9) maps S_R into itself (see Lemma 3). In virtue of (13) and Remark 2b, for any $v \in S_R$ the vector function $F_v(x, t) = F[x, t, \ldots, D^\gamma_x v(x, t), \ldots]$ satisfies the inequality

$$|F_v(x, t) - F_v(y, t)|$$
$$\leqslant \Big\{q(t)|x-y|^\beta + \sum_{i=0}^{2b-2} \sum_{|\gamma|=i} \sum_{j=1}^{p} q^\gamma_j(t) L^{\gamma\beta}(t)\,|x-y|^{\beta_\gamma} +$$
$$+ \sum_{|\gamma|=2b-1} \sum_{j=1}^{p} q^\gamma_j(t)\,[Rf_{\Delta,\mu}(t)\,|x-y|^\alpha]^{\beta_\gamma}\Big\} J,$$

where $L(t) > 0$ is a bounded function on every interval $\langle 0, T\rangle$ for $T \in (0, \infty)$. Consequently $F_v \in H^{\mathrm{loc}}_\varrho\big(x, \Omega; t, \langle 0, \infty)\big)$, where $\varrho = \min(\beta, \beta_\gamma, \alpha\beta_\gamma) < 1$. Moreover, $F_v \in C^0(Q_\infty)$, and so problem (1), (2), (3) and the operator equation $\mathfrak{A}(x, t)u = u$ are mutually equivalent on S_R.

The existence of a solution of $\mathfrak{A}(x, t)u = u$ will be proved by the Tichonov fixed point theorem.

First of all we establish the continuity of the operator $\mathfrak{A}(x, t)$.

Let $\{u_s(x, t)\}_{s=1}^\infty$ be a sequence of elements $u_s(x, t) = \big(u^s_1(x, t), \ldots \ldots, u^s_p(x, t)\big)$ of S_R such that $u_s \to u_0$ in the topology τ; $u_0(x, t) = \big(u^0_1(x, t), \ldots \ldots, u^0_p(x, t)\big) \in S_R$. Then to any neighbourhood $N(O, n, \varepsilon) \in \tau$ there is a positive integer $s_0(n, \varepsilon)$ such that for all $s > s_0$ the relation $u_s - u_0 \in N(O, n, \varepsilon)$ holds. Hence

$$|D^k_x u_s(x, t) - D^k_x u_0(x, t)| \leqslant f_{\Delta,\varkappa}(n)\,\varepsilon J \tag{14}$$

for $(x, t) \in Q_n$ and $|k| = 0, 1, \ldots, 2b-1$. From hypothesis (13) we get for $(x, t), (y, t), (x, t') \in Q_n$ and $|k| = 2b-1$

$$|D^k_x\mathfrak{A}(x, t)u_s - D^k_x\mathfrak{A}(x, t)u_0 - D^k_x\mathfrak{A}(y, t)u_s + D^k_x\mathfrak{A}(y, t)u_0|$$
$$\leqslant I_{2,k}(x, y, t)\,C(n)\,J$$

and for $|k| = 0, 1, \ldots, 2b-1$

$$|D^k_x\mathfrak{A}(x, t)u_s - D^k_x\mathfrak{A}(x, t)u_0| \leqslant I_{1,k}(x, t)\,C(n)\,J$$

and

$$|D_x^k\mathfrak{A}(x,t)u_s - D_x^k\mathfrak{A}(x,t)u_0 - D_x^k\mathfrak{A}(x,t')u_s + D_x^k\mathfrak{A}(x,t')u_0| \leqslant I_{3,k}(x,t,t')C(n)J,$$

where $C(n) = \sup_{Q_n} \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} \big(q^\gamma(t),\, |D_x^\gamma u_s(\xi,\tau) - D_x^\gamma u_0(\xi,\tau)|^{\beta_\gamma}\big)$. Using Lemma 2 for $\beta = \alpha$ and estimation (14) we have

$$\|\mathfrak{A}(x,t)u_s - \mathfrak{A}(x,t)u_0\|_{2b-1+\alpha,Q_n}^{f(A,\varkappa,\mu,\nu)} < L(n)\varepsilon,$$

i.e., $\mathfrak{A}(x,t)u_s - \mathfrak{A}(x,t)u_0 \in N\big(O, n, L(n)\varepsilon\big)$ for $s > s_0(n,\varepsilon)$. The constant $L(n) > 0$ depends only of n and so the operator $\mathfrak{A}(x,t)$ is continuous in the topology τ.

To prove the compactness of $\overline{\mathfrak{A}(v,t)S_R}$ in τ we use the well-known lemma of Dunford (see [6]).

Put $v_s(x,t) = \big(v_1^s(x,t), \ldots, v_p^s(x,t)\big) \in \mathfrak{A}(x,t)S_R \subset S_R$ for $s = 1, 2, \ldots$ Then there exists a sequence of elements $u_s(x,t) = \big(u_1^s(x,t), \ldots, u_p^s(x,t)\big) \in S_R$ such that $v_s = \mathfrak{A}(x,t)u_s$ for $s = 1, 2, \ldots$ Since $\|v_s\|_{2b-1+\alpha,Q_\infty}^{f(A,\varkappa,\mu,\nu)} \leqslant R$, the sequence of derivatives $\{D_x^k v_j^s(x,t)\}_{s=1}^\infty$ is uniformly bounded on Q_n for any $n = 1, 2, \ldots$ and $j = 1, \ldots, p$ and $|k| = 0, 1, \ldots, 2b-1$. On account of the assumption $\|F\|_{0,\tilde{H}_\infty} < L$ for $|k| = 0, 1, \ldots, 2b-1$ we have

$$|D_x^k v_s(x,t) - D_x^k v_s(y,t')| \leqslant |D_x^k\mathfrak{A}(x,t)u_s - D_x^k\mathfrak{A}(y,t)u_s| + \\ + |D_x^k\mathfrak{A}(y,t)u_s - D_x^k\mathfrak{A}(y,t')u_s| \leqslant L[I_{2,k}(x,y,t) + I_{3,k}(y,t,t')]J.$$

Thus Lemma 2 guarantees the equicontinuity of the sequence $\{D_x^k v_j^s(x,t)\}_{s=1}^\infty$. Then there is a subsequence $\{v_{s_l}(x,t)\}_{l=1}^\infty = \{\mathfrak{A}(x,t)u_{s_l}\}_{l=1}^\infty$ of the sequence $\{v_s(x,t)\}_{s=1}^\infty$ and $v_0(x,t) = \big(v_1^0(x,t), \ldots, v_p^0(x,t)\big)$ such that $\|D_x^k v_{s_l} - D_x^k v_0\|_{0,Q_n} \to 0$ as $l \to \infty$ for $n = 1, 2, \ldots$ and $|k| = 0, 1, \ldots, 2b-1$. Consequently the sequence $\{D_x^k v_{s_l}(x,t)\}_{l=1}^\infty$ converges to $D_x^k v_0(x,t)$ at every point $(x,t) \in Q_\infty$.

Letting, for fixed $(x,t), (y,t), (x,t') \in Q_\infty$, $l \to \infty$ in the inequalities

$$|D_x^k v_0(x,t) - D_x^k v_0(y,t)| \\ \leqslant |D_x^k v_0(x,t) - D_x^k v_{s_l}(x,t)| + LI_{2,k}(x,y,t)J + |D_x^k v_{s_l}(y,t) - D_x^k v_0(y,t)|$$

for $|k| = 2b-1$ and

$$|D_x^k v_0(x,t)| \leqslant |D_x^k v_0(x,t) - D_x^k v_{s_l}(x,t)| + LI_{1,k}(x,t)J,$$

$$|D_x^k v_0(x,t) - D_x^k v_0(x,t')| \\ \leqslant |D_x^k v_0(x,t) - D_x^k v_{s_l}(x,t)| + LI_{3,k}(x,t,t')J + |D_x^k v_{s_l}(x,t') - D_x^k v_0(x,t')|$$

for $|k| = 0, 1, \ldots, 2b-1$, we immediately obtain that v_0 belongs to $C_{x,t,f(A,\varkappa,\mu,\nu)}^{2b-1+\alpha,(2b-1+\alpha)/2b}(Q_\infty)$. It remains to show the convergence of the sequence $\{v_s(x,t)\}_{s=1}^\infty$ to $v_0(x,t)$ in the topology τ.

Denote by $S^*_{R,n}$ the completion of the set $\mathfrak{A}(x,t)S_R$ in the norm

$$\|u\|^t_{2b-1,Q_n} = \max_{j=1,\dots,p}\Big\{\sum_{i=0}^{2b-1}\sum_{|k|=i}\sup_{Q_n}|D^k_x u_j(x,t)|\Big\}$$

for $n = 1,2,\dots$ The sequence $\{S^*_{R,n}\}_{n=1}^\infty$ possesses the following properties:

(a) $v_0 \in S^*_{R,n}$ for any $n = 1,2,\dots$

(b) The non-void intersection $S^*_R = \bigcap_{n=1} S^*_{R,n}$ is a subset of $C^{2b-1+a,(2b-1+a)/2b}_{x,t,f,(A,\varkappa,\mu,\nu)}(Q_\infty)$. Indeed, for $v(x,t) = \big(v_1(x,t),\dots,v_p(x,t)\big) \in S^*_R$ there is a sequence $\{w_s(x,t)\}_{s=1}^\infty \subset \mathfrak{A}(x,t)S_R$ such that $\|w_s - v\|^t_{2b-1,Q_n} \to 0$ as $s\to\infty$ for any $n = 1,2,\dots$ Thus $D^k_x w_s$ converges to $D^k_x v$ at every point $(x,t)\in Q_\infty$. Hence, using the same considerations as above for v_0, we obtain that $v \in C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$ $(\supset S^*_R)$.

(c) By Lemma 2 for $v\in S^*_R$

$$\lim_{x\to y}|D^k_x v_j(x,t) - D^k_x v_j(y,t)|f^{-1}_{A,\mu}(t)\,|x-y|^{-a} = 0$$

uniformly with respect to $t\in\langle 0,\infty)$ for $|k| = 2b-1$ and

$$\lim_{t\to t'-}|D^k_x v_j(x,t) - D^k_x v_j(x,t')|f^{-1}_{A,\nu}(t'-t)f^{-1}_{A,\nu}(t')\times(t'-t)^{-(2b-1+a-|k|)/2b} = 0\,.$$

uniformly with respect to $x\in\Omega$ for $|k| = 0,1,\dots,2b-1$ and $j = 1,\dots,p$.

In view of (c), to each $\varepsilon>0$ and $n = 1,2,\dots$ we find $\delta(\varepsilon,n)>0$ such that for every $j = 1,\dots,p$ and $l = 1,2,\dots$ and (x,t), $(x,t')\in Q_n$, $t<t'$,

$$\langle D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t)\rangle_{a,x} f^{-1}_{A,\mu}(t) < \varepsilon$$

if $|k| = 2b-1$ and $0<|x-y|<\delta$ and

$$\langle D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t)\rangle_{(2b-1+a-|k|)/2b,t}\, f^{-1}_{A,\nu}(t'-t)f^{-1}_{A,\nu}(t') < \varepsilon$$

if $|k| = 0,1,\dots,2b-1$ and $0<t'-t<\delta$.

From the relation: $\|v_{s_l} - v_0\|^t_{2b-1,Q_n}\to 0$ as $l\to\infty$, follows the existence of a positive integer $l_0(\varepsilon,n)$ such that for $l>l_0(n,\varepsilon)$ and $|x-y| \geqslant \delta$ we have on Q_n

$$\begin{aligned}\langle D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t)\rangle_{ax} f^{-1}_{A,\mu}(t)&\\ \leqslant f^{-1}_{A,\mu}(0)\,\delta^{-a}\max_{j=1,\dots,p}\Big\{&\sup_{(x,t)\in Q_n}|D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t)| +\\ &+\sup_{(y,t)\in Q_n}|D^k_x v^{s_l}_j(y,t) - D^k_x v^0_j(y,t)|\Big\} < \varepsilon\end{aligned}$$

if $|k| = 2b-1$. For $t'-t\geqslant\delta$

$$\begin{aligned}\langle D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t)\rangle_{(2b-1+a-|k|)/2b,t} f^{-1}_{A,\nu}(t'-t)f^{-1}_{A,\nu}(t')&\\ \leqslant f^{-2}_{A,\nu}(0)\,\delta^{-(2b-1+a-|k|)/2b}\max_{j=1,\dots,p}\Big\{&\sup_{(x,t)\in Q_n}|D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t)| +\\ &+\sup_{(x,t')\in Q_n}|D^k_x v^{s_l}_j(x,t) - D^k_x v^0_j(x,t')|\Big\} < \varepsilon\end{aligned}$$

if $|k| = 0, 1, \ldots, 2b-1$. Finally,

$$
\begin{aligned}
&\|v_{s_l} - v_0\|_{2b-1+a,Q_n}^{f(A,\varkappa,\mu,\nu)} \\
&\quad \leqslant \max_{j=1,\ldots,p} \Big\{ \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_{Q_n} |D_x^k v_j^{s_l}(x,t) - D_x^k v_j^0(x,t)| f_{A,\varkappa}^{-1}(t) + \\
&\qquad + \sum_{|k|=2b-1} \max\Big[\sup_{\substack{(x,t)(y,t)\in Q_n \\ 0<|x-y|<\delta}} \langle D_x^k v_j^{s_l}(x,t) - D_x^k v_j^0(x,t) \rangle_{a,x} f_{A,\mu}^{-1}(t); \\
&\qquad\qquad \sup_{\substack{(x,t)(y,t)\in Q_n \\ |x-y|\geqslant\delta}} \langle D_x^k v_j^{s_l}(x,t) - D_x^k v_j^0(x,t) \rangle_{a,x} f_{A,\mu}^{-1}(t) \Big] + \\
&\qquad + \sum_{i=0}^{2b-1} \sum_{|k|=i} \max\Big[\sup_{\substack{(x,t),(x,t')\in Q_n \\ 0<t'-t<\delta}} \langle D_x^k v_j^{s_l}(x,t) - D_x^k v_j^0(x,t) \rangle_{(2b-1+a-|k|)/2b,t} \times \\
&\qquad\qquad\qquad \times f_{A,\nu}^{-1}(t'-t) f_{A,\nu}^{-1}(t'); \\
&\qquad\qquad \sup_{\substack{(x,t),(x,t')\in Q_n \\ t'-t\geqslant\delta}} \langle D_x^k v_j^{s_l}(x,t) - D_x^k v_j^0(x,t) \rangle_{(2b-1+a-|k|)/2b,t} f_{A,\nu}^{-1}(t'-t) f_{A,\nu}^{-1}(t') \Big\} \\
&\quad < \varepsilon [f_{A,\varkappa}^{-1}(0)s + t(2b-1) + s]
\end{aligned}
$$

for $l > l_0(n, \varepsilon)$ and arbitrary $n = 1, 2, \ldots$ Hence

$$v_{s_l} - v_0 \in N(O, n, \varepsilon[f_{A,\varkappa}^{-1}(0)s + t(2b-1) + s]) \qquad \text{for } l > l_0(n, \varepsilon),$$

which concludes the proof of the theorem.

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