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A convergence proof of a difference scheme for a parabolic system

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1. Consider the system of differential equations of second order

$$\frac{\partial y_i}{\partial x_0} = f_i\left(x, y, \frac{\partial y_i}{\partial x_1}, \ldots, \frac{\partial y_i}{\partial x_n}, \frac{\partial^2 y_i}{\partial x_1^2}, \ldots, \frac{\partial^2 y_i}{\partial x_n^2}\right), \quad i = 1, \ldots, m, \tag{1}$$

where $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$, $y = (y_1, \ldots, y_m) \in R^m$, with boundary conditions:

$$\begin{aligned} y_i(x) &= \varphi_{ij}(x), \quad x_j = 0, \ i = 1, \ldots, m, \ j = 0, 1, \ldots, n, \\ y_i(x) &= \psi_{ij}(x), \quad x_j = \tau, \ i = 1, \ldots, m, \ j = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

We shall replace the partial derivatives by suitable difference quotients, and thus get the system of difference equations (cf. (3)).

Z. Kowalski proved in [2], that, under suitable assumptions, the solutions of the difference system (3) tend to the solutions of (1) with b.c. (2). If we make use of a certain difference inequality (cf. [1]), similar to the differential one, a convergence proof becomes shorter than that of [2].

2. Notation. We shall consider the nodal points x^M of R^{n+1}, $0 \leqslant M \leqslant P$, where $M = (m_0, m_1, \ldots, m_n)$, $P = (p_0, p_1, \ldots, p_n)$ are given systems of integers m_i, p_i $(i = 0, 1, \ldots, n)$ and $0 \leqslant M \leqslant P$ denotes $0 \leqslant m_i \leqslant p_i$ $(i = 0, 1, \ldots, n)$; $x^M = (x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n})$, $x_0^{m_0} = m_0 k$, $x_i^{m_i} = m_i h$, $i = 1, 2, \ldots, n$, where $k = \tau/N_1$, $h = \tau/N$, are positive numbers. We shall also use the following multi-indices:

$$+M = (m_0+1, m_1, \ldots, m_n),$$

$$+jM = (m_0, m_1, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_n),$$

$$-jM = (m_0, m_1, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_n), \quad j = 1, 2, \ldots, n.$$

To each nodal point x^M there corresponds a system of m real numbers z_i^M.

Now we can introduce the difference quotients

$$z_i^{M\sim} = \frac{1}{k}(z_i^{+M} - z_i^M), \qquad z_i^{Mj} = \frac{1}{2\cdot h}(z_i^{+jM} - z_i^{-jM}),$$

$$z_i^{Mjj} = \frac{1}{h^2}(z_i^{+jM} - 2z_i^M + z_i^{-jM})$$

and the n-dimensional vectors

$$z_i^{MI} = (z_i^{M1}, \ldots, z_i^{Mn}), \qquad z_i^{MII} = (z_i^{M11}, z_i^{M22}, \ldots, z_i^{Mnn}).$$

We shall consider the difference scheme

$$z_i^{M\sim} = f_i(x^M, z^M, z_i^{MI}, z_i^{MII}), \tag{3}$$

where $z^M = (z_1^M, z_2^M, \ldots, z_m^M)$, with the boundary conditions

$$\begin{aligned} z_i^M &= \varphi_{ij}(x^M) \quad \text{for } m_j = 0 \ (j = 0, 1, \ldots, n,\ i = 1, \ldots, m), \\ z_i^M &= \psi_{ij}(x^M) \quad \text{for } m_j = N \ (j = 1, 2, \ldots, n,\ i = 1, \ldots, m). \end{aligned} \tag{4}$$

3. THEOREM. *Suppose that*

1° *functions* $f_i(x, s, q, w)$, $i = 1, \ldots, m$, $x \in R^{n+1}$, $s = (s_1, \ldots, s_m) \in R^m$, $q = (q_1, \ldots, q_n) \in R^n$, $w = (w_1, \ldots, w_n) \in R^n$, *are of the class* C^1 *in the domain* $D = [0, \tau]^{n+1} \times R^{m+2n}$;

2° *the derivatives of the functions* f_i *satisfy the relations*:

$$0 \leqslant \frac{\partial f_i}{\partial s_j} \leqslant L \quad (i = 1, \ldots, m,\ j = 1, \ldots, m;\ i \neq j),$$

$$\left|\frac{\partial f_i}{\partial q_j}\right| \leqslant \Gamma,$$

$$0 < g \leqslant \frac{\partial f_i}{\partial w_j} \leqslant G \quad (i = 1, \ldots, m,\ j = 1, \ldots, n),$$

the numbers h *and* k *being chosen so as to obtain*

$$\frac{g}{h} - \frac{\Gamma}{2} \geqslant 0,$$

$$1 + k\frac{\partial f_i}{\partial s_i} - 2kh^{-2}\sum_{j=1}^{n}\frac{\partial f_i}{\partial w_j} \geqslant 0, \quad i = 1, \ldots, m;$$

3° *the functions* $y_i(x)$ *are of class* C^2 *in the set* $E = [0, \tau]^{n+1}$, *satisfy system* (1) *with boundary conditions* (2), *and* z_i^M *satisfy* (3) *with conditions* (4). *Write* $u_i^M = y_i^M - z_i^M$.

Under these assumptions

$$|u_i^M| \leqslant \frac{\varepsilon(h)}{mL}(e^{mLkm_0}-1), \quad 0 \leqslant M \leqslant P \tag{5}$$

($\varepsilon(h)$ *being defined below* (cf. (7))

$$u_i^M \to 0 \quad \textit{as } h \to 0. \tag{6}$$

4. Proof. It can be verified that y_i satisfy at the nodal points x^M the following difference equations

$$y_i^{M\sim} = f_i(x^M, y^M, y_i^{MI}, y_i^{MII}) + \eta_i^M$$

with the boundary conditions

$$y_i^M = \varphi_{ij}(x^M) \quad \text{for } m_j = 0,\ j = 0, 1, \ldots, n,\ i = 1, 2, \ldots, m,$$

$$y_i^M = \psi_{ij}(x^M) \quad \text{for } m_j = N,\ j = 1, 2, \ldots, n,\ i = 1, \ldots, m,$$

where

$$\varepsilon(h) = \max_i |\eta_i^M| \to 0 \quad \text{as } h \to 0. \tag{7}$$

Hence by the mean value theorem, u_i^M satisfy the relations:

$$u_i^{M\sim} = y_i^{M\sim} - z_i^{M\sim} = f_i(x^M, y^M, y_i^{MI}, y_i^{MII}) - f_i(x^M, z^M, z_i^{MI}, z_i^{MII}) + \eta_i^M$$

$$= \sum_{j=1}^{m} \frac{\partial f_i}{\partial s_j}(-)u_j^M + \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}(-)u_i^{Mj} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial w_j}(-)u_i^{Mjj} + \eta_i^M.$$

Write

$$\frac{\partial f_i}{\partial s_j}(-) = a_{ij}^M, \quad \frac{\partial f_i}{\partial q_j}(-) = b_{ij}^M, \quad \frac{\partial f_i}{\partial w_j}(-) = c_{ij}^M.$$

Thus we can write

$$u_i^{M\sim} = \sum_{j=1}^{m} a_{ij}^M u_j^M + \sum_{j=1}^{n} b_{ij}^M u_i^{Mj} + \sum_{j=1}^{n} c_{ij}^M u_i^{Mjj} + \eta_i^M. \tag{8}$$

In virtue of assumptions 2° and 3° we have

$$0 \leqslant a_{ij}^M \leqslant L \quad (i \neq j), \quad |b_{ij}^M| \leqslant \Gamma, \quad 0 < g \leqslant c_{ij}^M \leqslant G,$$

$$1 + ka_{ii}^M - 2kh^{-2}\sum_{j=1}^{n} c_{ij}^M \geqslant 0.$$

Let us introduce values v^M

(9) $$v^M = \frac{\varepsilon(h)}{Lm}(e^{mLkm_0}-1)\,, \qquad 0 \leqslant M \leqslant P\,.$$

It is easy to verify that

(10) $$v^{M\sim} \geqslant Lmv^M + \varepsilon(h)\,.$$

It can be seen also that

(11) $$\sum_{j=1}^{m} a_{ij}^M v^M + \sum_{j=1}^{n} b_{ij}^M v^{Mj} + \sum_{j=1}^{n} c_{ij}^M v^{Mjj} + \eta_i^M \leqslant Lmv^M + \varepsilon(h) \leqslant v^{M\sim}\,,$$

because of $v^M \geqslant 0$ and $v^{MI} = 0$ and $v^{MII} = 0$.
We thus obtain

(12) $$v^{M\sim} \geqslant \sum_{j=1}^{m} a_{ij}^M v^M + \sum_{j=1}^{n} b_{ij}^M v^{Mj} + \sum_{j=1}^{n} c_{ij}^M v^{Mjj} + \eta_i^M\,,$$

(13) $$u_i^{M\sim} \leqslant \sum_{j=1}^{m} a_{ij}^M u_j^M + \sum_{j=1}^{n} b_{ij}^M u_i^{Mj} + \sum_{j=1}^{n} c_{ij}^M u_i^{Mjj} + \eta_i^M\,,$$

because of equality (8), and

(14) $$u_i^M = 0\,, \quad u_i^M \leqslant v^M \quad \text{for } m_j = 0 \ (j = 0, 1, \ldots, n)$$
$$\text{or } \ m_j = N \ (j = 1, \ldots, n)\,,$$

since the boundary values for y_i and z_i are equal, and $v^M \geqslant 0$. From (12), (13), (14) and Theorem 1 in paper [1] follows

(15) $$u_i^M \leqslant v^M \quad \text{for } 0 \leqslant M \leqslant P\,.$$

In a similar way we obtain

(16) $$(-v^M)^\sim \leqslant \sum_{j=1}^{m} a_{ij}^M(-v^M) + \sum_{j=1}^{n} b_{ij}^M(-v^M)^j + \sum_{j=1}^{n} c_{ij}^M(-v^M)^{jj} + \eta_i^M\,,$$

(17) $$u_i^{M\sim} \geqslant \sum_{j=1}^{m} a_{ij}^M u_j^M + \sum_{j=1}^{n} b_{ij}^M u^{Mj} + \sum_{j=1}^{n} c_{ij}^M u^{Mjj} + \eta_i^M\,,$$

because of (8), and

(18) $$u_i^M = 0\,, \quad u_i^M \geqslant -v^M \quad \text{for } m_j = 0 \ (j = 0, 1, \ldots, n)$$
$$\text{or } \ m_j = N \ (j = 1, \ldots, n)\,.$$

From (16), (17), (18) and Theorem 1 in [1] follows

(19) $$u_i^M \geqslant -v^M \quad \text{for } 0 \leqslant M \leqslant P .$$

(15) and (19) imply

$$|u_i^M| \leqslant v^M \quad (i = 1, \ldots, m) .$$

This ends the proof.

References

[1] A. Fitzke, *On a system of difference inequalities of parabolic type*, this fasc., pp. 299-302.

[2] Z. Kowalski, *A difference method for a non-linear system of parabolic differential equations without mixed derivatives*, Bull. Acad. Polon. Sci., ser, sci., math., astr. et phys., 15 (1967), pp. 683-689.

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