ANNALES
POLONICI MATHEMATICI
XX (1968)

On the mean values of an integral function of two complex variables *

by SATYA NARAIN SRIVASTAVA (Lucknow)

1. Let

$$f(x, y) = \sum_{k,l}^{\infty} a_{k,l} x^k y^l$$

be an integral function of two complex variables x, y. Also, let

$$(1.1) \qquad \mu_p(r_1, r_2) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^p d\theta_1 d\theta_2 ,$$

$$(1.2) \qquad m_{p,q}(r_1, r_2) = \frac{1}{\pi^2 (r_1 r_2)^{q+1}} \int_0^{r_1} \int_0^{r_2} \int_0^{2\pi} \int_0^{2\pi} |f(\xi_1 e^{i\theta_1}, \xi_2 e^{i\theta_2})|^p \xi_1^q \xi_2^q d\xi_1 d\xi_2 d\theta_1 d\theta_2 ,$$

where p and q are any positive numbers.

We shall obtain some of the properties of the mean values $\mu_p(r_1, r_2)$ and $m_{p,q}(r_1, r_2)$. For simplicity, we have considered an integral function of two instead of several complex variables.

2. THEOREM 1. *If $f(x, y)$ is an integral function of two complex variables, differing from a constant, and a_1, a_2 $(0 < a_1, a_2 < 1)$ are constants, then*

$$(2.1) \qquad \lim_{r_1, r_2 \to \infty} \left\{ \frac{1}{m_{p,q}(r_1, r_2) - (a_1 a_2)^{q+1} m_{p,q}(a_1 r_1, a_2 r_2)} \right\} = 0 .$$

We first prove the following

LEMMA 1. *Let $f(x, y)$ be an integral function. Then, for*

$$0 < r_1' < R_1' < R_1 \quad \textit{and} \quad 0 < r_2' < R_2' < R_2 ,$$

* This work has been supported by a Senior Research Fellowship award of U.G.C., New Delhi (INDIA).

$$[\mu_p(R_1', r_2')\{(R_2')^{q+1}-(r_2')^{q+1}\}\{(R_1)^{q+1}-(R_1')^{q+1}\}+$$
$$+\mu_p(r_1', R_2')\{(R_2)^{q+1}-(R_2')^{q+1}\}\{(R_1')^{q+1}-(r_1')^{q+1}\}+$$
$$+\mu_p(R_1', R_2')\{(R_2)^{q+1}-(R_2')^{q+1}\}\{(R_1)^{q+1}-(R_1')^{q+1}\}]$$
$$\leqslant \left[\left(\frac{q+1}{2}\right)^2\{(R_1R_2)^{q+1}m_{p,q}(R_1, R_2)-(R_1'R_2')^{q+1}m_{p,q}(R_1', R_2')\}\right]$$
$$\leqslant [\mu_p(R_1, R_2')(R_2')^{q+1}\{(R_1)^{q+1}-(R_1')^{q+1}\}+$$
$$+\mu_p(R_1', R_2)(R_1')^{q+1}\{(R_2)^{q+1}-(R_2')^{q+1}\}+$$
$$+\mu_p(R_1, R_2)\{(R_2)^{q+1}-(R_2')^{q+1}\}\{(R_1)^{q+1}-(R_1')^{q+1}\}],$$

where p and q are any positive numbers.

Proof. From (1.1) and (1.2), we have

$$\frac{(r_1r_2)^{q+1}}{4}m_{p,q}(r_1, r_2) = \int_0^{r_1}\int_0^{r_2}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2, \tag{2.2}$$

where $\mu_p(\xi_1, \xi_2)$ is an increasing function of ξ_2 for a fixed value of ξ_1 and it is an increasing function of ξ_1 for a fixed value of ξ_2, whence it is an increasing function of both ξ_1 and ξ_2.

From (2.2) follows

$$(R_1R_2)^{q+1}m_{p,q}(R_1, R_2)-(R_1'R_2')^{q+1}m_{p,q}(R_1', R_2')$$
$$= 4\int_{R_1'}^{R_1}\int_0^{R_2'}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2+$$
$$+4\int_0^{R_1'}\int_{R_2'}^{R_2}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2+4\int_{R_1'}^{R_1}\int_{R_2'}^{R_2}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2.$$

Hence,

$$(R_1R_2)^{q+1}m_{p,q}(R_1, R_2)-(R_1'R_2')^{q+1}m_{p,q}(R_1', R_2')$$
$$\leqslant 4\mu_p(R_1, R_2')\frac{(R_2')^{q+1}}{(q+1)^2}\{(R_1)^{q+1}-(R_1')^{q+1}\}+$$
$$+4\mu_p(R_1', R_2)\frac{(R_1')^{q+1}}{(q+1)^2}\{(R_2)^{q+1}-(R_2')^{q+1}\}+$$
$$+4\mu_p(R_1, R_2)\frac{1}{(q+1)^2}\{(R_2)^{q+1}-(R_2')^{q+1}\}\{(R_1)^{q+1}-(R_1')^{q+1}\}.$$

Also, we have

$$(R_1R_2)^{q+1}m_{p,q}(R_1, R_2)-(R_1'R_2')^{q+1}m_{p,q}(R_1', R_2')$$

$$\geqslant 4\int_{R_1'}^{R_1}\int_{r_2'}^{R_2'}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2+$$

$$+4\int_{r_1'}^{R_1'}\int_{R_2'}^{R_2}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2+4\int_{R_1'}^{R_1}\int_{R_2'}^{R_2}\mu_p(\xi_1, \xi_2)\xi_1^q\xi_2^q d\xi_1 d\xi_2$$

$$\geqslant \frac{4}{(q+1)^2}\mu_p(R_1', r_2')\{(R_2')^{q+1}-(r_2')^{q+1}\}\{(R_1)^{q+1}-(R_1')^{q+1}\}+$$

$$+\frac{4}{(q+1)^2}\mu_p(r_1', R_2')\{(R_2)^{q+1}-(R_2')^{q+1}\}\{(R_1')^{q+1}-(r_1')^{q+1}\}+$$

$$+\frac{4}{(q+1)^2}\mu_p(R_1', R_2')\{(R_2)^{q+1}-(R_2')^{q+1}\}\{(R_1)^{q+1}-(R_1')^{q+1}\}.$$

Combining the two inequalities we obtain the result.

Proof of Theorem 1. If we put $R_1 = r_1$, $R_1' = a_1 r_1$, $r_1' = \beta_1 r_1$, $R_2 = r_2$, $R_2' = a_2 r_2$ and $r_2' = \beta_2 r_2$ in Lemma 1, we obtain

$$[4\mu_p(a_1r_1, \beta_2r_2)(a_2^{q+1}-\beta_2^{q+1})(1-a_1^{q+1})+$$

$$+4\mu_p(\beta_1r_1, a_2r_2)(1-a_2^{q+1})(a_1^{q+1}-\beta_1^{q+1})+$$

$$+4\mu_p(a_1r_1, a_2r_2)(1-a_2^{q+1})(1-a_1^{q+1})]$$

$$\leqslant (q+1)^2\{m_{p,q}(r_1, r_2)-(a_1a_2)^{q+1}m_{p,q}(a_1r_1, a_2r_2)\}$$

$$\leqslant [4\mu_p(r_1, a_2r_2)a_2^{q+1}(1-a_1^{q+1})+4\mu_p(a_1r_1, r_2)a_1^{q+1}(1-a_2^{q+1})+$$

$$+4\mu_p(r_1, r_2)(1-a_2^{q+1})(1-a_1^{q+1})].$$

The result follows from the above inequalities.

3. THEOREM 2. *If $f(x, y)$ is an integral function, then*

$$\lim_{r_1,r_2\to\infty}\sup\frac{m_{p,q}(r_1, r_2)}{\{M(r_1, r_2)\}^p}\leqslant\lim_{r_1,r_2\to\infty}\sup\frac{m_{p,q}(r_1, r_2)}{\mu_p(r_1, r_2)}\leqslant\frac{4}{(q+1)^2}, \tag{3.1}$$

where

$$M(r_1, r_2) = \max_{|x|\leqslant r_1, |y|\leqslant r_2}|f(x, y)|$$

is the maximum modulus of $f(x, y)$ for $|x|\leqslant r_1$, $|y|\leqslant r_2$.

Proof. From (2.2) we have

$$m_{p,q}(r_1, r_2) = \frac{4}{(r_1 r_2)^{q+1}} \int_0^{r_1}\int_0^{r_2} \mu_p(\xi_1, \xi_2)\, \xi_1^q \xi_2^q\, d\xi_1\, d\xi_2$$

$$\leqslant \frac{4\mu_p(r_1, r_2)}{(r_1 r_2)^{q+1}} \int_0^{r_1}\int_0^{r_2} \xi_1^q \xi_2^q\, d\xi_1\, d\xi_2$$

$$= \frac{4}{(q+1)^2}\, \mu_p(r_1, r_2)\,,$$

since $\mu_p(\xi_1, \xi_2)$ is an increasing function of both ξ_1 and ξ_2. Taking limits, we get

$$\lim_{r_1, r_2 \to \infty} \sup \frac{m_{p,q}(r_1, r_2)}{\mu_p(r_1, r_2)} \leqslant \frac{4}{(q+1)^2}\,. \tag{3.2}$$

Also, from (1.1) we have

$$\mu_p(r_1, r_2) \leqslant \{M(r_1, r_2)\}^p\,. \tag{3.3}$$

Therefore, from (3.2) and (3.3) follows

$$\lim_{r_1, r_2 \to \infty} \sup \frac{m_{p,q}(r_1, r_2)}{\{M(r_1, r_2)\}^p} \leqslant \lim_{r_1, r_2 \to \infty} \sup \frac{m_{p,q}(r_1, r_2)}{\mu_p(r_1, r_2)} \leqslant \frac{4}{(q+1)^2}\,.$$

DEPARTMENT OF MATHEMATICS AND ASTRONOMY
LUCKNOW UNIVERSITY, LUCKNOW (INDIA)

Reçu par la Rédaction le 23. 2. 1966