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Asymptotically expansible solutions of the Helmholtz equation *

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We consider in this paper complex functions of real variables $u(\boldsymbol{x}, k)$ defined for a k greater than a certain k_0 and an $\boldsymbol{x}$ belonging to a certain two- or three-dimensional domain D satisfying the equation

$$\nabla^2 u + k^2 u = 0 \tag{1}$$

with an asymptotic expansion in the form

$$u \sim e^{ikL(\boldsymbol{x})} \sum_{\nu=0}^{\infty} \frac{A_\nu(\boldsymbol{x})}{(ik)^\nu}, \tag{2}$$

where $L(\boldsymbol{x})$ and $A_\nu(\boldsymbol{x})$, $\nu = 0, 1, \ldots$ are complex functions of real variables defined for $\boldsymbol{x} \in D$. For real $L(\boldsymbol{x})$, expansions of this kind have been applied to a great number of problems [1].

Our present knowledge concerning the family of solutions admitting asymptotic expansions (2) is still unsatisfactory. From among recent works on this subject I will mention a very interesting one, [2], concerning expansions of a similar type.

The main result of the present paper consist in constructing a class of solutions admitting asymptotic expansions in question.

If we substitute in a formal manner expansion (2) in Eq. (1), then, by comparing the coefficients of the successive powers of ik, we obtain the following sequence of equations:

$$(\nabla L)^2 = 1, \tag{3}$$

$$2\nabla L \cdot \nabla A_\nu + \nabla^2 L A_\nu = -\nabla^2 A_{\nu-1}, \qquad \nu = 0, 1, \ldots \tag{4}$$

where A_{-1} should be assumed to be zero. For real L, Eqs (4) reduce along the line of the field ∇L to ordinary differential equations

$$2\dot{A}_\nu + \nabla^2 L A_\nu = -\nabla^2 A_{\nu-1}, \qquad \nu = 0, 1, \ldots \tag{5}$$

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This is one of the fundamental advantages of these and other similar asymptotic expansions [1]-[5].

A formal proceding, as mentioned above, will be justified if we assume that expression (2) is an asymptotic expansion of the function u, and, moreover, that an expression obtained by computing term by term the Laplacean of expression (2) and ordering the formal sum thus obtained according to the powers of ik is an asymptotic expansion of $\nabla^2 u$. Indeed, the function appearing on the left side of Eq. (1) vanishes identically, and thus its coefficients of asymptotic expansion also vanish.

We will now formulate conditions sufficient for the differentiability term by term of asymptotic expansion of type (2). We say that the function $u(\boldsymbol{x}, k)$ has asymptotic expansion in the domain D if for every $\boldsymbol{x} \in D$ and for every integer $N \geqslant 0$ we have:

$$\lim_{k\to\infty}(ik)^N\left(e^{-ikL}u-\sum_{\nu=0}^{N-1}\frac{A_\nu}{(ik)^\nu}\right)=A_N\,. \tag{6}$$

As to the domain D, we assume that in the case of two dimensions—it does not contain points of a certain circle with a radius greater than zero, and that in the case of three dimensions it does not contain points of a sphere with a radius also greater than zero. In the centre of such a circle or sphere we shall choose the origin of the system of coordinates. The positional vector $\boldsymbol{x}$ will be determined by means of its polar or spherical coordinates.

The following theorem holds:

If:

1. *for each k greater than a certain k_0 the function $u(\boldsymbol{x}, k)$ is an analytic function of coordinates in the domain D (i.e. it can be represented in the neighbourhood of every point $\boldsymbol{x} \in D$ as a sum of Taylor series in the coordinates),*

2. *for each $\boldsymbol{x} \in D$ and for each integer $N \geqslant 0$ transition to the limit* (6) *is uniform in a certain complex neighbourhood of the point $\boldsymbol{x}$ — that is in a neighbourhood formed by admitting complex values of coordinates (the function u is extended to cover such a neighbourhood by means of Taylor series),*

3. *the function $L(\boldsymbol{x})$ is an analytic function of coordinates in D, then:*

1. *the functions $A_\nu(\boldsymbol{x})$, $\nu = 0, 1, \ldots$ are analytic functions of coordinates in D,*

2. *by differentiating term by term the sum* (2) *with respect to the coordinates an arbitrary number of times, and by ordering the expressions obtained with respect to the powers of ik, we obtain asymptotic expansion of a corresponding derivative of the function $u(\boldsymbol{x}, k)$.*

In order to prove our theorem, let us observe that the function

$$v(\boldsymbol{x}, k) = e^{-ikL}u \tag{7}$$

is for $k > k_0$ an analytic function of coordinates in D. Furthermore, owing to the uniformity of the limit (6), all functions $A_\nu(\boldsymbol{x})$, $\nu = 0, 1, \ldots$, are analytic and

$$\lim_{k\to\infty}(ik)^N\left(d^n v - \sum_{\nu=0}^{N-1}\frac{d^n A_\nu}{(ik)^\nu}\right) = d^n A_N\,, \qquad N = 0, 1, \ldots, \tag{8}$$

where d^n denotes an arbitrary derivative of the order n with respect to the coordinates of $\boldsymbol{x}$. Thus not only the function v has an asymptotic expansion

$$v \sim \sum_{\nu=0}^{\infty}\frac{A_\nu}{(ik)^\nu} \tag{9}$$

but the derivative $d^n v$ has an expansion

$$d^n v \sim \sum_{\nu=0}^{\infty}\frac{d^n A_\nu}{(ik)^\nu} \tag{10}$$

as well, obtained from (9) by means of term by term differentiation. If we take into account (7), we find that derivatives of the function u have asymptotic expansions which can be computed in a manner indicated in the theorem.

We will now define the function

$$\begin{aligned} u(\boldsymbol{x}, k) &= k^{1/2}\int_{\mathcal{L}} a(\alpha)[B(\alpha)]^k \exp\big(ik\boldsymbol{v}(\alpha)\cdot\boldsymbol{x}\big)\,d\alpha \\ &= k^{1/2}\int_{\mathcal{L}} a(\alpha)[B(\alpha)]^k \exp\big(ikr\cos(\vartheta-\alpha)\big)\,d\alpha\,, \end{aligned} \tag{11}$$

where r and ϑ are polar coordinates of $\boldsymbol{x}$, the vector $\boldsymbol{v}(\alpha)$ has the polar co-ordinates: 1, α, the integration contour, depending on point $\boldsymbol{x}$, is presented in Fig. 1 (see p. 57) and the functions $a(\alpha)$ and $B(\alpha)$ satisfy the following conditions:

1. they are entire functions,
2. they are periodic with the period 2π,
3. there exist such numbers A and $\eta < \frac{1}{2}$ that, given sufficiently great $|\alpha|$, the following inequalities hold:

$$|a(\alpha)| < \exp(Ae^{|\alpha|})\,, \qquad |B(\alpha)| < \exp(\eta d e^{|\alpha|})\,, \tag{12}$$

where d denotes the distance of the domain D from the origin of the coordinates (the domain D satisfying identical condition as in the preceding theorem).

Given these assumptions, and k being an integer greater than a certain k_0, u is an analytic function of x defined in a unique manner in D and satisfying the Eq. (1) (1).

In order to prove the above theorem let us consider the function u_1 of two complex variables, r and ϑ, defined for r and ϑ in a complex neighbourhood respectively of r_0, ϑ_0, where r_0 and ϑ_0 denote the coordinates of a certain point $x_0 \in D$:

$$u_1(r, \vartheta, k) = k^{1/2} \int_{\mathcal{L}_0} a(a) B^k(a) \exp\big(ikr\cos(\vartheta - a)\big)\, da \tag{13}$$

the integration contour $\mathcal{L}_0$ in the above formula is fixed and the same as the contour $\mathcal{L}$ for the point x_0 in formula (11).

In virtue of inequalities (12) integral (13) is uniformly convergent in a certain (complex) neighbourhood of r_0, ϑ_0 for k sufficiently great. Thus, it is an analytic function. The function u_1 satisfies Eq. (1), as may be proved by differentiation under the integral sign. In a certain real neighbourhood of the point $x_0 \in D$ the function u_1 defined by formula (13) is identical with the function u defined by formula (11) since the integrals along the contours $\mathcal{L}_0$ and $\mathcal{L}$ are equal in virtue of inequalities (12) and the well-known Cauchy Theorem. It is evident that the function u for the integral k is single-valued.

Fairly large classes of functions $a(a)$ and $B(a)$ satisfy conditions (12). These conditions are satisfied in particular by functions with a finite order of increment.

Let us now consider a particular case when the function $B(a) \equiv 1$ and let us expand the function $a(a)$ into a series (uniformly convergent in every bounded domain):

$$a(a) = \sum_{n=-\infty}^{n=+\infty} a_n e^{ina}. \tag{14}$$

Substituting this series in formula (11) we arrive at:

$$u(x, k) = -\frac{k^{1/2}}{\pi} \sum_{-\infty}^{+\infty} a_n i^n H_n^{(1)}(kr) e^{in\vartheta}. \tag{15}$$

The above function is uniquely defined not only for integral values of k as in the general case, but also for every k greater than a certain k_0. The function u defined by formula (15) has an asymptotic expansion of type (2), whence the function $L(x) = r$ (this can be proved by means of the well-known expansions of Hankel functions). The expansion under

(1) The function u satisfies also the radiation condition, as may be proved by finding its asymptotic expression for $|x| \to \infty$.

discussion can be differentiated term by term, as follows from the general theorem given below. As a solution of the Helmholtz equation, expression (15) is a general one. That is every solution regular outside a sphere and satisfying the radiation condition can be represented by (15). This expression has been obtained as a particular case of formula (11) for $B(a) = 1$. The question arises whether formula (11) is not unnecessarily complicated. This, indeed, is the case when one definite value of the parameter k is considered. We will, however, avail ourselves of formula (11) in investigating asymptotic properties when k is variable or, strictly speaking, when $k \to \infty$.

With assumptions so far made the functions $u(\boldsymbol{x}, k)$ defined by formula (11) will generally have no asymptotic expansions of the type discussed, defined in D. Additional assumptions which we make in order to ensure the existence of an asymptotic expansion of the function u will apply principally to the function $B(a)$. It will be helpful to introduce a function $b(a)$ such that

$$B(a) = e^{ib(a)}. \tag{16}$$

Then we can rewrite formula (11) in the following form:

$$\begin{aligned} u(\boldsymbol{x}, k) &= k^{1/2}\int_{\mathcal{L}} a(a)\exp\left(ik[\boldsymbol{v}(a)\cdot\boldsymbol{x} + b(a)]\right) da \\ &= k^{1/2}\int_{\mathcal{L}} a(a)\exp\left(ik[r\cos(\vartheta - a) + b(a)]\right) da\,. \end{aligned} \tag{17}$$

The domain D will now be defined more precisely. Namely, let it consist of points lying outside the curve defined by the equation $r = f(\vartheta)$, where $f(\vartheta)$ is a positive continuous function with the period 2π. We also assume that the curve C is convex.

The assumptions made in defining the function u by means of formula (11) are still valid. They concern the function $b(a)$ in virtue of (16). As a conclusion of [5] we shall now obtain the following theorem.

If we assume additionally that:

1. *the function $b(a)$ for real values of the argument is also real,*
2. *its derivative for real a satisfies the inequality:*

$$-f(a+\pi/2) < b'(a) < f(a-\pi/2)\,, \tag{18}$$

3. *there exists a number λ_0 such that $b''(a)$ satisfies for real a the inequality:*

$$-\boldsymbol{v}''(a)\cdot\boldsymbol{x}_C - b''(a) \geqslant \lambda_0 > 0\,, \tag{19}$$

where $\boldsymbol{x}_C$ is the point (defined in a unique manner) of intersection of the curve C with the semi-axis defined by the conditions

$$\boldsymbol{v}'(a)\cdot\boldsymbol{x} + b'(a) = 0\,, \tag{20}$$

$$|\vartheta - a| < \pi/2\,, \tag{21}$$

then conditions (20) *and* (21) *define the implicit real function* $a = a_0(\boldsymbol{x})$, *which is single-valued and analytic in* D, *and the following inequality is satisfied in* D:

$$-\boldsymbol{v}''(a_0)\cdot\boldsymbol{x} - b''(a_0) \geqslant \lambda_0 > 0 . \tag{22}$$

It follows from the above theorem that for $\boldsymbol{x} \in D$ there exists, on a horizontal section of the path $\mathcal{L}$, precisely one zero point of the function $\boldsymbol{v}'(a)\cdot\boldsymbol{x} + b'(a)$ — the saddle point — and that it is a single zero.

We will now determine the function

$$L(\boldsymbol{x}) = \boldsymbol{v}(a_0)\cdot\boldsymbol{x} + b(a_0) . \tag{23}$$

It is a real function, analytic in D and satisfying Eq. (3) [2].

Let us now consider a certain point $\boldsymbol{x}_0 \in D$ with coordinates r_0 and ϑ_0. In a sufficiently small neighbourhood of this point the function u can be expressed by means of formula (13). We shall now bring this formula to the form

$$e^{-ikL}u = k^{1/2}\int_{\mathcal{L}} a(a)e^{ik\psi(a,r,\vartheta)}da , \tag{24}$$

where the function ψ can be defined by means of two equivalent formulas,

$$\begin{aligned}\psi &= r\cos(\vartheta - a) + b(a) - L \\ &= b(a) - b(a_0) - b'(a_0)\sin(a - a_0) - [b''(a_0) + \lambda][1 - \cos(a - a_0)] ,\end{aligned} \tag{25}$$

where

$$\lambda = -\boldsymbol{v}''(a_0)\cdot\boldsymbol{x} - b''(a_0) .$$

For arbitrary complex, r, ϑ from a sufficiently small neighbourhood of r_0, ϑ_0, the expansion of the function ψ at the point a_0 into the Taylor series with respect to a begins with a square term.

We will now determine a certain path S leading through the point a_0 on the complex plane a satisfying the equation

$$\operatorname{Re}\psi = 0 \tag{26}$$

and the condition that from a_0 the $\operatorname{Im}\psi$ should increase. This is the path of the steepest ascent of $\operatorname{Im}\psi$ and of the steepest descent of the function $|e^{ik\psi}|$. We consider the function

$$v_1 = k^{1/2}\int_{S} a(a)e^{ik\psi}da . \tag{27}$$

As regards the path S there are two possibilities: either it ends in infinity or at a singular point of the function $b(a)$. Considered from the standpoint

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[2] This is the simplest and the most important case of radiated fields; the case of complex a_0 corresponds, when $\operatorname{Im}L > 0$ for sufficiently great $|x|$, to evanescent fields. The inequality $\operatorname{Im}L < 0$ is not interesting from the physical point of view.

of greatest importance to us, two particular cases deserve special mention: when the path S ends in infinity and approaches the path $\mathcal{L}_0$ in such a manner that expressions (24) and (27) are equal in a certain neighbourhood of r_0, ϑ_0 and — the other case — when the path S ends at a singular point of the function $b(a)$ lying in finity, but it can be supplemented by the path S_1 (or by several such paths) emerging from that singular point, such that Eq. (26) holds, $\mathrm{Im}\,\psi$ remains positive, and the path $\mathcal{L}_0$ can be replaced by $S+S_1$. In either case it can be expected that, instead of asymptotically expanding expression (24), we can expand expression (27), obtaining the same result.

By introducing a new variable of integration:

$$t = (a-a_0)\sqrt{\frac{\psi}{i(a-a_0)^2}} \tag{28}$$

we can transform the last expression to a form convenient for calculation:

$$v_1 = k^{1/2} \int_{-\infty}^{+\infty} a[a(t,r,\vartheta)]a'(t,r,\vartheta)e^{-kt^2}dt. \tag{29}$$

Eq. (28) defines an implicit function $a(t,r,\vartheta)$ which is analytic at the point $t=0$, $r=r_0$, $\vartheta=\vartheta_0$.

Bearing in mind the situation outlined above, we shall prove the following theorem.

If, in addition to the assumptions so far made concerning the functions $a(a)$ and $b(a)$, we assume also that for each pair of numbers r_0, ϑ_0 constituting the coordinates of the point $x_0 \in D$ there exist a complex neighbourhood E and a finite number k_0 such that:

1. *there exists a finite number*

$$M = \sup_{\substack{(r,\vartheta)\in E \\ x>0}} \left| \int_{-x}^{x} a(a)a'e^{-k_0t^2}dt \right|, \tag{30}$$

2. *for natural N and natural k*

$$e^{-ikL}u - v_1 = o(k^{-N}), \qquad k\to\infty \tag{31}$$

uniformly with respect to r and ϑ.

Then the function u defined by formula (17) *has an asymptotic expansion in the form of* (2) *in the entire domain D. The functions: $L(r,\vartheta)$ and $A_\nu(r,\vartheta)$, $\nu = 0, 1, \ldots$ are analytic functions of the coordinates; this expansion admits the term by term differentiation. The successive functions A_ν are respectively defined by the successive even coefficients of the Maclaurin expansion with respect to t of the function $a(a)a'$ appearing in formula* (29).

For the proof it will be more convenient to introduce an even function

$$2f(t) = a[\alpha(t)]\alpha'(t) + a[\alpha(-t)]\alpha'(-t) . \tag{32}$$

We will expand this function into the Maclaurin series with respect to t. It should be mentioned here that, by slightly modifying the Lagrange method of expansion, we can effectively obtain the coefficients of our series, namely

$$f_{2\nu} = \frac{1}{(2\nu)!}\frac{\partial^{2\nu}}{\partial\alpha^{2\nu}}\left\{\left[\frac{i(\alpha-\alpha_0)^2}{\psi}\right]^{(2\nu+1)/2} a(\alpha)\right\}_{\alpha=\alpha_0}, \qquad \nu = 0, 1, \ldots \tag{33}$$

By choosing sufficiently small δ we obtain

$$e^{-ikL}u - 2k^{1/2}\int_0^\infty e^{-kt^2}\sum_{\nu=0}^{N} f_{2\nu}t^{2\nu}\,dt \tag{34}$$

$$= o(k^{-N}) + 2k^{1/2}\int_0^\delta e^{-kt^2} f^{(2N+2)}(\eta t)\,t^{2N+2}dt + 2k^{1/2}\int_\delta^\infty e^{-kt^2}\Big[f(t) - \sum_{\nu=0}^{N} f_{2\nu}t^{2\nu}\Big]dt ,$$

where η is contained between zero and unity.

We shall define the functions $A_\nu(r,\vartheta)$, $\nu = 0, 1, \ldots$ as follows:

$$(ik)^{-\nu}A_\nu = 2f_{2\nu}k^{1/2}\int_0^\infty e^{-kt^2}t^{2\nu}dt = f_{2\nu}\pi^{1/2}\frac{(2\nu-1)(2\nu-3)\ldots 1}{2^\nu k^\nu} \tag{35}$$

and introduce the function

$$F_N(x, r, \vartheta) = 2\int_0^x e^{-k_0t^2}\Big[f(t) - \sum_{\nu=0}^{N} f_{2\nu}t^{2\nu}\Big]dt , \qquad N = 0, 1, \ldots \tag{36}$$

From (34) we obtain the following evaluation (to obtain the evaluation of the last term we use integration by parts):

$$\left|k^N\left(e^{-ikL}u - \sum_{\nu=0}^{N}\frac{A_\nu}{(ik)^\nu}\right)\right| \leqslant o(1) + \frac{M_N}{k} + M_N' k^{1/2+N}e^{-k\delta^2} \tag{37}$$

where M_N and M_N' are constants independent either of k or of r and ϑ, namely:

$$M_N = \pi^{1/2}\frac{(2N+1)(2N-1)\ldots 1}{2^{N+1}}\sup_{0\leqslant t\leqslant\delta}|f^{(2N+2)}(t)| ,$$
$$M_N' = e^{k_0\delta^2}\sup_{\substack{x>\delta\\(r,\vartheta)\in E_1}}|F_N| . \tag{38}$$

Such numbers exist if we choose a sufficiently small complex neighbourhood $E_1 \subset E$ of the point r_0, ϑ_0. The transition to the limit (6) is thus uniform and it follows from this fact that our theorem holds.

The assumptions made for this last theorem are not especially convenient. It can be expected that an investigation of the behaviour of the path S according to the choice of the function $b(\alpha)$ would enable us to simplify these assumptions. However, in the case discussed above, when $B(\alpha) \equiv 1$ and the corresponding expression is given by (15), the verification of the assumptions for the last theorem presents no difficulties. A drawing of the path S for this case is given in Fig. 1.

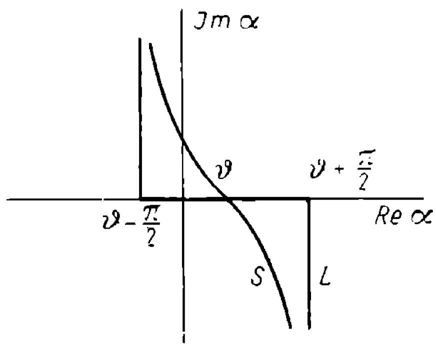

Fig. 1

The coefficients of the asymptotic expansion can be computed effectively by means of Eqs (25), (33), (35). We find for instance that $A_0 = \left(\frac{2\pi}{i\lambda}\right)^{1/2} a(\alpha_0)$. Since $\nabla^2 L = \lambda^{-1}$, we also find directly that A_0 satisfies Eq. (5).

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