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On the Dirichlet problem for linear elliptic equations in plane domains with corners*

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Abstract. In this paper we consider the first boundary value problem for linear second order elliptic equations in a plane domain Ω with corners. Conditions sufficient for the solutions to be of class $C^{m+2+\alpha}(\bar{\Omega})$ are given; $m \geq 0$ an integer and $0 < \alpha < 1$.

1. Introduction. In this paper we shall consider the Dirichlet problem for linear second order elliptic equations in domains with sectionally smooth boundaries. In Section 2 we introduce the problem and state some known results. Section 3 contains Theorem 2, which is the main result of this work. In Section 4 we study the problem in a circular sector. The proof of Theorem 2 follows from the sector case and will be given in Section 5.

2. The problem. In a bounded domain $\Omega \subset R^2$ with boundary Γ we consider the Dirichlet problem

$$a_{ij}(x)u_{ij} + a_i(x)u_i + a(x)u = f(x), \tag{2.1}$$

$$u = \varphi \quad \text{on } \Gamma. \tag{2.2}$$

Here $x = (x_1, x_2)$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ and we use the summation convention. We make the following assumption.

(A1) The coefficients of (2.1) belong to $C_{m+\alpha}(\bar{\Omega})$, where $m \geq 0$ is an integer, $0 < \alpha < 1$ and $\bar{\Omega}$ is the closure of Ω; cf. [1] for the definition of $C_{m+\alpha}$.

A known result is as follows (cf. [1]). Let u be a solution of (2.1), (2.2) and let (A1) be satisfied. Furthermore, assume that Γ can be represented parametrically by $C_{m+2+\alpha}$-functions and $\varphi \in C_{m+2+\alpha}(\Gamma)$. Then $u \in C_{m+2+\alpha}(\bar{\Omega})$.

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If Γ has a corner, this result may not hold. We now state another assumption.

(A2) Γ has a corner at 0 and elsewhere can be represented parametrically by $C_{m+2+\alpha}$-functions of the arc length s (measured from the corner point), and $\varphi \in C_{m+2+\alpha}(\Gamma \setminus \{0\}) \cap C_0(\Gamma)$.

Under assumptions (A1), (A2), for any solution of (2.1), (2.2) we have $u \in C_{m+2+\alpha}(\Omega_1) \cap C_0(\bar{\Omega})$ (cf. [1]), where Ω_1 is any compact subdomain of $\bar{\Omega}$ with positive distance from the corner. It was proved in [4] that in the neighbourhood of the corner, $u \in C_v$; here $v = \min(m+2+\alpha, \pi/\omega - \varepsilon)$ with arbitrarily small $\varepsilon > 0$,

$$\omega = \arctan\{[a_{11}(0)a_{22}(0) - a_{12}^2(0)]^{1/2}/[a_{22}(0)\cot\gamma - a_{12}(0)]\} \tag{2.3}$$

and γ is the angle at the corner. Note that $\omega = \gamma$ if the leading part of (2.1) is the Laplacian. For the Poisson equation analogous results in Sobolev spaces related to $L^p(\Omega)$ for every p were proved in [7]. For more references in the case of boundaries with corners see [7] and the list given at the end of this paper.

3. The main result of the paper. The Dirichlet problem for the Poisson equation has been extensively studied; cf. [5]–[9], [13] and [14]. The following result is known.

THEOREM 1. *Let Ω_{2r_0} be a sector with center at the origin and radius $2r_0$. Assume that this sector has an angle $\omega = \pi/q$, $q \geq 2$ an integer. In Ω_{2r_0} consider the Dirichlet problem*

$$\Delta w = F, \tag{3.1}$$

$$w = \varphi \quad \textit{on the boundary of } \Omega_{2r_0}. \tag{3.2}$$

If $F \in C_{m+\alpha}(\bar{\Omega}_{2r_0})$ and on each of the radii bounding the sector Ω_{2r_0} we have $\varphi \in C_{m+2+\alpha}$, then $w \in C_{m+2+\alpha}(\bar{\Omega}_{r_0})$ provided that at the corner the compatibility conditions imposed by (3.1) and (3.2) are satisfied.

We extend this result to the case of (2.1), (2.2), as follows.

THEOREM 2. *Let u be a solution of (2.1), (2.2), and assume that (A1) and (A2) hold. If ω in (2.3) is such that $\pi/\omega = q$, and integer ≥ 2, and at the corner the functions in (2.1) and (2.2) satisfy the compatibility conditions imposed by (2.1) and (2.2), then $u \in C_{m+2+\alpha}(\bar{\Omega})$.*

From [4] it follows that it is sufficient to consider the case $q \leq m+2$, and the case of a circular sector.

4. The case of a circular sector. We first consider the problem in a circular sector. In Section 5 we shall prove that the general case can be reduced to the present one by a suitable transformation. We make the following assumptions.

(A3) Let the sector be

$$\Omega_\sigma = \{(r, \theta) |\ r \leqslant \sigma < 1,\ \beta \leqslant \theta \leqslant \beta + \omega\},$$

where (r, θ) are the polar coordinates of $x = (x_1, x_2)$ and $\beta > 0$ is arbitrarily small, and $\omega = \pi/q$, $2 \leqslant q \leqslant m+2$. Let $w(x)$ be a bounded solution of (2.1) in Ω_σ, where the coefficients of (2.1) satisfy (A1) in Ω_σ and $a_{ij}(0) = \delta_{ij}$, the Kronecker delta. The boundary value ψ of w is continuous at the corner, and $\psi \in C_{m+2+\alpha}$ on the lines $\theta = \beta$ and $\theta = \beta + \omega$ $(0 < r \leqslant \sigma)$.

(A4) At the corner, $\psi(r, \theta)$ and $f(x)$ satisfy

$$\psi^{(k)}(0, \theta) \equiv d^k\psi/dr^k|_{r=0} = 0, \quad k = 0, 1, \ldots, p+2, \tag{4.1}$$

$$D^k f(0) = 0, \quad k = 0, 1, \ldots, p, \tag{4.2}$$

where $D^k f$ is any partial derivative of f of order k, and $0 \leqslant p \leqslant m$.

We now state a result from [4] which we are going to use.

THEOREM 3. *Let w be a bounded solution of* (2.1) *in Ω_σ and let assumption* (A3) *be satisfied. Then*

(a) *$w \in C_\nu(\Omega_{r_0})$, where $\nu = \min(m+2+\alpha, \pi/\omega - \varepsilon)$ and $\varepsilon > 0$ is arbitrarily small.*

(b) *If for some $\lambda \leqslant p+2+\alpha$ we have $|w(x)| \leqslant M_1 r^\lambda$ in Ω_{2r_0} and if* (A4) *is satisfied, then in Ω_{r_0} we have*

$$|D^k w| \leqslant M_2 r^{\lambda-k}, \quad k = 0, 1, \ldots, p+2.$$

We shall now see that the regularity of $D^q w$ $(q \leqslant m+2)$ can be improved by multiplying $D^q w$ by a suitable function. The details are as follows.

THEOREM 4. *Let w be a bounded solution of* (2.1) *in Ω_σ and let assumptions* (A3), (A4) *be satisfied. Suppose that for some integer p, $0 \leqslant p \leqslant m$ we have $w \in C_{p+2-\varepsilon}(\Omega_\sigma)$ and $|D^{p+2} w| \leqslant M_3 r^{-\varepsilon}$, $0 \leqslant \varepsilon < 1$. Then for any function $h \in C_\delta(\Omega_\sigma)$, $\varepsilon \leqslant \delta \leqslant 1$, vanishing at the corner point, we have $hD^{p+2} w \in C_\mu(\Omega_{r_0})$, where*

$$\mu = \begin{cases} \delta - \varepsilon & \text{if } p < m, \\ \min(\alpha, \delta - \varepsilon) & \text{if } p = m. \end{cases}$$

Proof. Consider any two points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ in Ω_{r_0} and suppose that $0 \leqslant r_2 \leqslant r_1 \leqslant r_0$. If $r_2 \leqslant r_1/2$, then $\overline{PQ} \geqslant r_1/2$ and

$$|h(P) D^{p+2} w(P) - h(Q) D^{p+2} w(Q)|/\overline{PQ}^\mu \leqslant 2M_3 M_4 r_1^{\delta-\varepsilon}/(r_1/2)^\mu \leqslant M_5, \tag{4.5}$$

since $|h(x)| \leqslant M_4 r^\delta$. We turn to the case $r_2 > r_1/2$. We first prove that $|D^{p+2}w(P) - D^{p+2}w(Q)|/\overline{PQ}^\mu \leqslant M_6 r_1{}^{-\mu-\varepsilon}$; from this we shall then readily derive an inequality of the form (4.5) [(4.12), below]. Consider the transformation

(4.6) $$x_i = 2r_1 y_i/r_0, \quad i = 1, 2.$$

This transformation takes

$$\Omega_0 = \{(r, \theta) \mid r_1/2 \leqslant r \leqslant r_1, \ \beta \leqslant \theta \leqslant \beta+\omega\}$$

and

$$\Omega'_0 = \{(r, \theta) \mid r_1/4 \leqslant r \leqslant 2r_1, \ \beta \leqslant \theta \leqslant \beta+\omega\}$$

to

$$\Omega_1 = \{(\varrho, \theta) \mid r_0/4 \leqslant \varrho \leqslant r_0/2, \ \beta \leqslant \theta \leqslant \beta+\omega\}$$

and

$$\Omega'_1 = \{(\varrho, \theta) \mid r_0/8 \leqslant \varrho \leqslant r_0, \ \beta \leqslant \theta \leqslant \beta+\omega\},$$

respectively, where $\varrho = r_0 r/2r_1$. In Ω'_1 the function $v(y) = w(2r_1 y/r_0)$ satisfies the elliptic equation

(4.7) $$b_{ij}(y)\, v_{ij} + (2r_1/r_0)\, b_i(y)\, v_i + (2r_1/r_0)^2\, b(y)\, v = (2r_1/r_0)^2\, g(y),$$

where the coefficients of (4.7) are those of (2.1) after the transformation (4.6). On the two straight line segments Γ'_1 of the boundary of Ω'_1, the function $v(y)$ coincides with $\chi(\varrho, \theta) = \psi(2r_1\varrho/r_0, \theta)$. In Ω_1 and Ω'_1, Schauder's inequality yields

(4.8) $$\|v\|^{\Omega_1}_{p+2+\mu} \leqslant c\,[\|v\|^{\Omega'_1}_0 + (2r_1/r_0)^2 \|g\|^{\Omega'_1}_{p+\mu} + \|\chi\|^{\Gamma'_1}_{p+2+\alpha}].$$

Now,

(4.9)
$$\|v\|^{\Omega'_1}_0 = \|w\|^{\Omega'_0}_0 \leqslant M_7 r_1^{p+2-\varepsilon},$$
$$\|g\|^{\Omega'_1}_{p+\mu} = \sum_{0\leqslant k\leqslant p} \|D^k_1 g(y)\|^{\Omega'_1}_0 + H^{\Omega'_1}_\mu (D^p_1 g),$$

where D^k_1 is the derivative in the (y_1, y_2)-plane corresponding to D^k and $H^{\Omega'_1}_\mu(D^p_1 g)$ is the Hölder coefficient of $D^p_1 g$ with exponent μ in Ω'_1. From (4.6) and the definition $g(y) = f(2r_1 y/r_0)$ we have

$$D^k_1 g(y) = (2r_1/r_0)^k D^k f(x), \quad k = 0, 1, \ldots, p.$$

From (4.2) it also follows that in Ω'_0

$$|D^k f(x)| \leqslant M_8 r_1^{p-k+\mu}, \quad k = 0, 1, \ldots, p.$$

Thus in Ω'_1,

$$|D^k_1 g(y)| \leqslant M_9 r_1^{p+\mu}.$$

Using the definition of $H_\mu^{\Omega'_1}(D_1^p g)$ and a similar argument, we also obtain

(4.10) $$H_\mu^{\Omega'_1}(D_1^p g) \leqslant M_{10} r_1^{p+\alpha}.$$

Together,

(4.11) $$\|g\|_{p+\mu}^{\Omega'_1} \leqslant M_{11} r_1^{p+\alpha}.$$

In a similar way, by (4.3), we conclude that

$$\|\chi\|_{p+2+\mu}^{\Gamma'_1} \leqslant M_{12} r_1^{p+2+\alpha}.$$

From (4.8), (4.9) and (4.11) we obtain $\|v\|_{p+2+\mu}^{\Omega_1} \leqslant c_0 r_1^{p+2-\varepsilon}$. We now return to the x-plane. Noting that in Ω_0,

$$(2r_1/r_0)^{p+2}|D^{p+2}w| = |D_1^{p+2}v| \leqslant \|v\|_{p+2+\mu}^{\Omega_1}$$

and

$$(2r_1/r_0)^{p+2+\mu} H_\mu^{\Omega_0}(D^{p+2}w) = H_\mu^{\Omega_1}(D_1^{p+2}v) \leqslant \|v\|_{p+2+\mu}^{\Omega_1},$$

we see that in Ω_0

$$H_\mu^{\Omega_0}(D^{p+2}w) \leqslant M_6 r_1^{-\varepsilon-\mu}.$$

Using these estimates as well as the fact that, in the present case, $r_2 > r_1/2$, we finally arrive at the desired inequality

(4.12) $$\begin{aligned} &|h(P)D^{p+2}w(P) - h(Q)D^{p+2}w(Q)|/\overline{PQ}^\mu \\ &\quad \leqslant |h(P)|\,|D^{p+2}w(P) - D^{p+2}w(Q)|/\overline{PQ}^\mu + \\ &\quad + |D^{p+2}w(Q)|\big(|h(P)-h(Q)|/\overline{PQ}^\delta\big)^{\mu/\delta}\big(|h(P)|+|h(Q)|\big)^{\varepsilon/\delta} \\ &\quad \leqslant M_4 r_1^\delta M_6 r_1^{-\varepsilon-\mu} + M_3 r_2^{-\varepsilon} M_4 (r_1^\delta + r_2^\delta)^{\varepsilon/\delta} \leqslant M_{15}. \end{aligned}$$

This completes the proof of Theorem 4.

We now prove Theorem 2 for a sector, in which case it takes the following form.

THEOREM 5. *Let w be a solution of* (2.1) *in Ω_σ and suppose that* (A3) *holds. Let $\pi/\omega = q$ be an integer and $2 \leqslant q \leqslant m+2$. Assume that the compatibility conditions at the corner are satisfied. Then $w \in C_{m+2+\alpha}$.*

Proof. Let $m \geqslant 0$. If $q = 2$, then from Theorem 3(a) it follows that $w \in C_{2-\varepsilon}(\Omega_{r_0})$. Consider the function $v = w - Y$, where

(4.13) $$\begin{aligned} Y(x) = \psi_\beta + \sum_{k=1}^{2} (x_1\cos\beta + x_2\sin\beta)^k \psi_\beta^k/k! + \\ + \sum_{k=1}^{2}\sum_{j=1}^{k} \big[\psi_{\beta+\omega}^k/j!\,(k-j)!\big](x_1\cos\beta + x_2\sin\beta)^{k-j}(-x_1\sin\beta + x_2\cos\beta)^j. \end{aligned}$$

Here $\psi_\beta = \psi(0, \beta) = \psi(0, \beta+\omega)$ and $\psi_\beta^k = d^k\psi(r, \beta)/dr^k|_{r=0}$ and $\psi_{\omega+\beta}^k$ is defined in a similar way. In Ω_σ the function v satisfies

$$(4.14) \qquad Lv = f_1 \equiv f - LY$$

and on the lines $\theta = \beta$ and $\theta = \omega+\beta$ it coincides with $\psi_1(r, \theta)$. The function ψ_1 vanishes at the corner point, together with its first and second derivatives in the directions of $\theta = \beta$ and $\theta = \beta+\omega$. It also follows from (4.14) that the compatibility conditions at the corner point give $f_1(0, 0) = 0$. Thus condition (A4) is satisfied with $p = 0$. For the simplicity of writing we shall still use the functions w, f and ψ, assuming (A4) with $p = 0$ to be satisfied. From Theorem 3(b) it then follows that in Ω_{r_0} we have $|D^2 w| \leqslant M_2 r^{-\varepsilon}$. We now write (2.1) in the form

$$(4.15) \qquad \Delta w = F(x) = f(x) - aw - a_i w_i - (a_{ij} - \delta_{ij}) w_{ij}.$$

To prove that $w \in C_{2+\varepsilon}(\Omega_{r_0})$, it is sufficient to show that $F \in C_\alpha$. Here C_α means $C_\alpha(\Omega_{r_0})$ and similarly we omit Ω_{r_0} until the end of this proof. The first three terms in F belong to C_α. Put $h(x) = a_{ij}(x) - \delta_{ij}$. Clearly $h \in C_\alpha$ and $h(0) = 0$. Using Theorem 4, we get $F \in C_{\alpha-\varepsilon}$. Thus $w \in C_{2+\alpha-\varepsilon}$, where $\varepsilon > 0$ is arbitrarily small. Before proceeding we note the following. If w has been shown to belong to $C_{p+2+\eta}$, $0 \leqslant p \leqslant m$, $0 \leqslant \eta < 1$, then the function $z = w - T_{p+2}$ satisfies in Ω_σ the equation

$$(4.16) \qquad Lz = f_1 \equiv f - LT_{p+2}.$$

Here, T_{p+2} is the sum of the first terms of the Maclaurin expansion of w, up to and including terms of order $p+2$. The function z vanishes at the corner together with all its partial derivatives of order not exceeding $p+2$. Thus it follows that the boundary value ψ_1 of z on the lines $\theta = \beta$ and $\theta = \beta+\omega$ satisfies (4.1). It also follows from (4.1) that f_1 satisfies (4.2). Without loss of generality and for the sake of simplicity in writing, whenever w has been proved to belong to $C_{p+2+\eta}$ we shall assume that (A4) is satisfied and that $|D^k w| \leqslant M r^{p+2-k+\eta}$, $k = 0, \ldots, p+2$. We now show that from $w \in C_{2+\alpha-\varepsilon}$ it follows that $w \in C_{2+\alpha}$. Take $h = a_{ij} - \delta_{ij} \in C_\alpha$. Since (A4) with $p = 0$ is satisfied and $|D^2 w| \leqslant A_1 r^{\alpha-\varepsilon} \leqslant A_2$, Theorem 4 with $\delta = \alpha$ and $\varepsilon = 0$ gives $(a_{ij} - \delta_{ij}) w_{ij} \in C_\alpha$. Using Theorem 1 we obtain $w \in C_{2+\alpha}$. If $m = 0$; then $q = 2$. This case has been discussed. Let $m > 0$ and $q \geqslant 2$. Suppose that it has been shown that $w \in C_{p+2-\varepsilon}$, where $0 \leqslant p \leqslant m$ and $\varepsilon = 1-\alpha$ if $q < p+2$ while ε is arbitrarily small if $q = p+2$. Since the coefficients of (2.1) belong to $C_{m+\alpha}$, it follows that $f - aw - a_i w_i \in C_{p+\alpha}$. To show that $(a_{ij} - \delta_{ij}) w_{ij} \in C_{p+\mu}$ $(\mu \leqslant \alpha)$, it is sufficient to prove that $(a_{ij} - \delta_{ij}) D^{p+2} w \in C_\mu$. As it was mentioned above, we may assume that $D^k w(0) = 0$, $k = 0, 1, \ldots, p+1$, and $|D^k w(x)| \leqslant A_3 r^{p+2-k-\varepsilon}$, $k = 0, 1, \ldots, p+1$. We also assume that (A4) with p replaced by $p-1$ is satisfied. Thus Theorem 3(b) gives $|D^{p+2} w(x)| \leqslant A_4 r^{-\varepsilon}$. Using Theorem 4 with $h = a_{ij} - \delta_{ij} \in C_1$ we finally conclude that $(a_{ij} - \delta_{ij}) D^{p+2} w \in C_\alpha$. This completes the proof of Theorem 5.

We now prove Theorem 2 by showing that the general case may be reduced to the case just studied.

5. Proof of Theorem 2. To prove the theorem it is sufficient to show that $u \in C_{m+2+\alpha}(N)$, where

$$N = \{(x_1, x_2) \mid (x_1, x_2) \in \bar{\Omega},\ x_1^2 + x_2^2 \leqslant \sigma_0^2\}.$$

This follows from the known result that $u \in C_{m+2+\alpha}(\bar{\Omega} \backslash N)$. Without loss of generality, assume that the corner point is at the origin and the two curves bounding the corner are represented by $x_1 = g_2(x_2)$ and $x_2 = g_1(x_1)$, where $g_1(0) = g_2(0) = g_1'(0) = 0$ and $g_2'(0) = \cot \gamma$. We transform the equation

$$a_{ij}(0) u_{ij} = 0 \tag{5.1}$$

to canonical form. The new angle after the transformation is independent of the transformation used and is given by

$$\omega = \arctan \{[a_{11}(0) a_{22}(0) - a_{12}^2(0)]^{1/2} / [a_{22}(0) \cot \gamma - a_{12}(0)]\}. \tag{5.2}$$

This transformation is of class C_∞ (cf. (3.5)–(3.6) in [4]) and transforms the domain N to a domain N_0 bounded by two straight-line segments Γ_1 and Γ_2 making angles β and $\beta + \omega$ with the horizontal line, and by a curve joining the two non-coinciding end points of these segments. In this domain the transformed function w is a solution of an equation of the form (2.1) with all the conditions of Theorem 5 being satisfied. Thus in a subdomain Ω_{r_0} of N_0 we have $w \in C_{m+2+\alpha}$. Noting that the transformation used is of class C_∞ and its Jacobian at (0, 0) has the value $[a_{11}(0) a_{22}(0) - a_{12}^2(0)]^{-1}$ (cf. [4]), we conclude that in a subdomain $N \subset \bar{\Omega}$ we have $u \in C_{m+2+\alpha}$. This completes the proof of Theorem 2.

We conclude this section by a theorem which follows from [1], [4] and Theorem 2 of this paper.

THEOREM 6. *Let $\Omega \subset R^2$ be a bounded domain whose boundary Γ consists of a finite number of curves $\Gamma_1, \ldots, \Gamma_k$, $k \geqslant 2$. Let $\Gamma_i \in C_{m+2+\alpha}$, $i = 1, 2, \ldots, k$, and suppose that Γ_i and Γ_{i+1} intersect at O_i making an angle γ_i, $0 < \gamma_i < 2\pi$. Assume that u satisfies* (2.1) *in Ω and on Γ coincides with $\varphi \in C_{m+2+\alpha}(\Gamma \backslash \bigcup O_i) \cap C_0(\Gamma)$. If* (A1) *is satisfied and the necessary compatibility conditions at the corners hold, then $u \in C_{m+2+\alpha}(\Omega_1)$, where Ω_1 is a compact subdomain of $\bar{\Omega}$ with positive distance from those corners satisfying neither* (i) *nor* (ii):

(i) $\pi/\omega_i > m+2+\alpha$.

(ii) π/ω_i *is an integer.*

In the neighbourhood of such an "excluded" corner we have $u \in C_{\pi/\omega_i - \varepsilon}$, where $\varepsilon > 0$ is arbitrarily small.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations, satisfying general boundary conditions*, Comm. Pure Appl. Math. 12 (1959), p. 623–727.

[2] A. Avantaggiati, M. Troisi, *Spazi di Sobolev con peso e problemi ellittici in un angolo I, II, III*, Ann. Mat. Pura Appl. 95 (1973), p. 361–408; 97 (1973), p. 207–252; 99 (1974), p. 1–64.

[3] A. Azzam, *Schauder estimations of the solution of Dirichlet's problem of second order elliptic equations in sectionally smooth domains*, Vestnik Moskov. Univ. ser. 1, Mat. 5 (1981), p. 29–33.

[4] —, *Smoothness properties of solutions of the Dirichlet problem for elliptic equations in regions with corners on the boundary*, Ann. Polon. Math. 40 (1980), p. 81–93.

[5] G. Dziuk, *Das Verhalten von Lösungen semilinearer elliptischer Systeme an Ecken eines Gebietes*, Math. Z. 159 (1978), p. 89–100.

[6] V. V. Fufaev, *On the Dirichlet problem for regions with corners*, Dokl. Akad. Nauk SSSR 131 (1960), p. 37–39.

[7] P. Grisvard, *Behavior of solutions of an elliptic boundary value problem in a polygonal or polyhedral domain*. In: B. Hubbard, *Numerical solutions of partial differential equations*, Vol. III, Academic Press, New York 1976, p. 207–274.

[8] M. Hanna, and K. Smith, *Some remarks on the Dirichlet problem in piecewise smooth domains*, Comm. Pure Appl. Math. 20 (1967), p. 577–593.

[9] K. Ibuki, *Dirichlet problem for elliptic equations of the second order in a singular domain of* R^2, J. Math. Kyoto Univ. 14 (1) (1974), p. 55–71.

[10] V. A. Kondratev, *Boundary value problems for elliptic equations in domains with conical or angular points*, Tran. Moscow Mat. Soc. 16 (1967), p. 227–313.

[11] —, *The smoothness of a solution of Dirichlet's problem for second order elliptic equations in a region with a piecewise smooth boundary*, Differ. Uravn. 10 (1970), p. 1831–1843.

[12] S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Providence, Amer. Math. Soc. (1963).

[13] E. A. Volkov, *On the differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle*, Trudy Mat. Inst. Steklov 77 (1965), p. 89–112.

[14] —, *Differential properties of solutions of boundary value problems for the Laplace equation on polygons*, ibidem 77 (1965), p. 113–142.

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