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A generalization of Morera's Theorem

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The purpose of the present note is to obtain a generalization of Morera's Theorem by using the Weyl lemma. Let D be a region in the plane. A complex valued function φ is said to belong to the class C_D^∞ if φ possesses partial derivatives of all orders and the closure of the set of points where $\varphi \neq 0$ is contained in D. We use $z = x+iy$ to denote a point in the plane. We begin with the following well-known lemma of Weyl [1]:

LEMMA 1. *Let f be a function of class L^1 in D such that for every $\varphi \in C_D^\infty$*

$$\iint_D f \Delta\varphi \, dx dy = 0 . \tag{1}$$

Then f is equal almost everywhere to a harmonic function.

LEMMA 2. *Let f be a function of class L^1 in D such that for every $\varphi \in C_D^\infty$ we have*

$$\iint_D f \frac{\partial\varphi}{\partial\bar{z}} \, dx dy = 0 . \tag{2}$$

Then f is equal almost everywhere to an analytic function.

Proof. For every $\psi \in C_D^\infty$ we have $\partial\psi/\partial z \in C_D^\infty$, and so

$$\iint f \Delta\psi \, dx dy = 4 \iint f \frac{\partial}{\partial\bar{z}}\left[\frac{\partial\psi}{\partial z}\right] dx dy = 0 .$$

Thus by Lemma 1 we have $f(z) = g(z)$ almost everywhere with g a harmonic function. Hence for every $\varphi \in C_D^\infty$ we have

$$0 = \iint g \frac{\partial\varphi}{\partial\bar{z}} \, dx dy = -\iint \frac{\partial g}{\partial\bar{z}} \varphi \, dx dy .$$

Since $\partial g/\partial\bar{z}$ is continuous, this implies $\partial g/\partial\bar{z} = 0$, consequently g is analytic.

THEOREM 1. *Let $f \in L^1$ in D, and let*

$$\int_R f(z) dz = 0 \tag{3}$$

hold over almost every rectangle R. Then f is equal almost everywhere to an analytic function.

Proof. By almost every rectangle we mean, of course, all rectangles except for those of a set of measure zero in the 4-dimensional Euclidean space described by the coordinates of a pair of opposite corners of the rectangles. Without loss of generality, we may assume that D is itself a rectangle, say, $-a \leqslant x \leqslant a$, $-a \leqslant y \leqslant a$. Now for almost all points z_0 in D

$$\int_R f(z)dz = 0$$

for almost all rectangles having z_0 for a corner. Let z_0 be a point with this property, and suppose for convenience that z_0 is the origin. Define a function F on the y-axis be setting

$$F(y) = i\int_0^y f(i\eta)d\eta\,, \tag{4}$$

the function f being integrable on the y-axis by the choice of z_0. Then F is continuous, and the function F defined by

$$F(z) = F(y) + \int_0^x f(\xi + iy)\,d\xi \tag{5}$$

is measurable. Moreover,

$$|F(z)| \leqslant |F(y)| + \int_0^x |f(\xi+iy)|d\xi \leqslant M + \int_{-a}^{a} |f(z)|dx$$

where M is the maximum of $|F(y)|$ on $[-a, a]$. Hence

$$\iint_D |F(z)|dxdy \leqslant 4Ma^2 + 2a\iint |f(z)|dxdy\,,$$

and so $F \in L^1$ in D. Since the integral of f over almost any rectangle with a corner at the origin vanishes, we have

$$F(z) = F(x) + i\int_0^y f(x+i\eta)d\eta\,. \tag{6}$$

Let φ be an arbitrary function in C_D^∞. Then

$$\iint_D F\frac{\partial\varphi}{\partial\bar z}dxdy = \frac{1}{2}\iint_D F\frac{\partial\varphi}{\partial x}dxdy + \frac{i}{2}\iint F\frac{\partial\varphi}{\partial y}dxdy\,.$$

By (5) we have F absolutely continuous as a function of x for almost every y. Hence we may integrate by parts to obtain

$$\int_{-a}^{a} F(z)\frac{\partial\varphi}{\partial x}dx = -\int_{-a}^{a} f\varphi dx\,,$$

since φ vanishes near $x = \pm a$. A similar use of (6) gives

$$\int_{-a}^{a} F\frac{\partial\varphi}{\partial y}dy = -i\int_{-a}^{a} f\varphi dy .$$

Hence

$$\iint_D F\frac{\partial\varphi}{\partial \bar{z}}dxdy = -\iint f\varphi dxdy + \iint f\varphi dxdy = 0 .$$

Consequently, Lemma 2 applies and we have F equal to an analytic function g almost everywhere. Since F is continuous on almost every horizontal line, we have $F(z) = g(z)$ on almost every horizontal line. On such a line $f(z) = g'(z)$. Hence $f(z) = g'(z)$ almost everywhere in D.

THEOREM 2. *Let D be a region whose boundary contains an open arc C of class C^3. Let u be a harmonic function in D such that*

$$\iint_D \left\{\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right|\right\}dxdy < \infty . \tag{7}$$

Suppose that at almost every interior point p of C the limit of u as we approach p normally is zero. Then we obtain a continuous extension of u to $D\cup C$ by setting u identically zero on C.

Proof. We map D conformally on the upper half of the unit circle so that C goes into the real axis. Since the mapping function has a continuous derivative on C, it follows from condition (7) that in the upper half D^* of any circle $|z| < -\varepsilon$. We have

$$\iint_{D^*} \left\{\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right|\right\}dxdy < \infty \tag{8}$$

where we again write x and y for the variables in D^*.

Let

$$f(z) = \frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$

for $z \in D^*$, and

$$f(\bar{z}) = -\overline{f(z)}$$

for z in the reflection of D^* in the real axis. Thus setting D_1 for the circle $|z| < 1-\varepsilon$, we have $f \in L^1$ in D_1 by (8). Now the integral of f over every rectangle in D_1 lying in either the upper or lower half plane is zero since f is analytic there. Hence in order to apply Theorem 1 to f we need only show that

$$\int_R f(z)dz = 0$$

for almost every rectangle which is symmetric in the x-axis. For almost all such rectangles we have $\lim u = 0$ as we approach the real axis along the rectangle. Let R be a rectangle with this property and let R^+ denote the upper half and R^- the lower half of R. Then since

$$u(a) - u(b) = \operatorname{Re} \int_b^a f(z)\,dz \tag{9}$$

over any path in the upper half of D_1, we have

$$\operatorname{Re} \int_{R^+} f(z)\,dz = 0 \quad \text{and} \quad \operatorname{Re} \int_{R^-} f(z)\,dz = 0\,.$$

By reflection we have

$$\operatorname{Im} \int_{R^+} f\,dz = -\operatorname{Im} \int_{R^-} f\,dz\,,$$

and so

$$\int_R f\,dz = 0\,.$$

Thus Theorem 1 applies. Hence by choosing the values of f on the real axis suitably f becomes analytic. From this and (9) we see that u can be extended by reflection to be a harmonic function in D^1. Consequently, u is uniformly continuous in a neighbourhood of the real axis, and the eɹoəɥʇm follows.

Reference

[1] H. Weyl, *The method of orthogonal projection in potential theory*, Duke Math. Journal 7 (1940), p. 411-444.

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