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Estimates of solutions of hyperbolic systems of differential equations in two independent variables

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This paper concerns the limitations of solutions of the system of differential equations

$$(*)\qquad \frac{\partial u_i}{\partial t}+\frac{\partial u_i}{\partial x}\lambda_i(x,t)=f_i(x,t,u_1,\dots,u_n),\qquad i=1,\dots,n.$$

We use the method of differential and integral inequalities. Similar investigations have been conducted in [3], [6].

In [1] the system $(*)$ was discussed by using the qualitative version of the method of integral inequalities.

1. To begin with we introduce the following condition:

(W_Ω) The functions $\sigma_i(t, y_1, \dots, y_n)$ $(i=1,\dots,n)$ are continuous on $\langle 0, a)\times\Omega$ and the inequalities $\bar{y}_k\leqslant\bar{\bar{y}}_k$ $(k\neq i)$ imply $\sigma_i(t,\bar{y}_1,\dots,y_i,\dots,\bar{y}_n)\leqslant\sigma_i(t,\bar{\bar{y}}_1,\dots,y_i,\dots,\bar{\bar{y}}_n)$.

T. Ważewski proved in [7] that if (W_Ω) is satisfied, then through every point $(t^0, y_1^0, \dots, y_n^0)$ there passes a unique right-hand maximum solution of the system

$$(1)\qquad y_i'=\sigma_i(t,y_1,\dots,y_n),\qquad i=1,\dots,n.$$

This maximum solution may be continued to the boundary. In what follows the components of the right-hand maximum solution of (1) which passes through $(0, y_1^0, \dots, y_n^0)$ are denoted by $\omega_i(t\colon y_1^0,\dots,y_n^0)$. Throughout the present paper we assume that these maximum solutions exist over the whole interval $\langle 0, a)$.

LEMMA 1 ([7]). *Let $\sigma_i(t,y_1,\dots,y_n)$ satisfy (W_Ω) and let the functions $\varphi_i(t)$ $(i=1,\dots,n)$ be continuous on $\langle 0,a)$. Suppose that $(\varphi_1(t),\dots,\varphi_n(t))\in\Omega$ for $t\in\langle 0,a)$ and $\varphi_i(0)\leqslant\eta_i$ $(i=1,\dots,n)$. If $\bar{D}_-\varphi_i(t)\leqslant\sigma_i(t,\varphi_1(t),\dots,\varphi_n(t))$ for $i=1,\dots,n$, $t\in(0,a)$ then $\varphi_i(t)\leqslant\omega_i(t\colon\eta_1,\dots,\eta_n)$ for $t\in\langle 0,a)$ and $i=1,\dots,n$.*

Let the functions $\bar{\varphi}(t)$, $\bar{\bar{\varphi}}(t)$ be continuous on $\langle 0, a)$ and $\bar{\varphi}(t) < \bar{\bar{\varphi}}(t)$ for $t \in \langle 0, a)$. Now define

$$R(\bar{\varphi}, \bar{\bar{\varphi}}) = \{(x, t)\colon\ 0 < t < a,\ \bar{\varphi}(t) < x < \bar{\bar{\varphi}}(t)\},$$
$$\bar{R}(\bar{\varphi}, \bar{\bar{\varphi}}) = \{(x, t)\colon\ 0 \leqslant t < a,\ \bar{\varphi}(t) \leqslant x \leqslant \bar{\bar{\varphi}}(t)\}.$$

The sequence of functions $u_1(x, t), \dots, u_n(x, t)$ which are continuous in $\bar{R}(\bar{\varphi}, \bar{\bar{\varphi}})$ is called a *regular solutions* of the system

$$\frac{\partial z_i}{\partial t} + \lambda_i(x, t)\frac{\partial z_i}{\partial x} = f_i(x, t, z_1, \dots, z_n), \qquad i = 1, \dots, n \tag{2}$$

if $u_i(x, t)$ $(i = 1, \dots, n)$ are continuously differentiable in $R(\bar{\varphi}, \bar{\bar{\varphi}})$ with regard to x and t and satisfy system (2) in $R(\bar{\varphi}, \bar{\bar{\varphi}})$. The set $R(\bar{\varphi}, \bar{\bar{\varphi}})$ is a proper one if the following conditions hold:

1° There exist two sequences $\{\bar{\varphi}_\nu(t)\}$, $\{\bar{\bar{\varphi}}_\nu(t)\}$ of functions which are continuously differentiable in $\langle 0, a)$ and

$$\bar{\varphi}(t) < \bar{\varphi}_\nu(t) < \bar{\bar{\varphi}}_\nu(t) < \bar{\bar{\varphi}}(t)$$

for $\nu = 1, 2, \dots$ and $t \in \langle 0, a)$.

2° $\lim\limits_{\nu\to\infty}\bar{\varphi}_\nu(t) = \bar{\varphi}(t)$, $\lim\limits_{\nu\to\infty}\bar{\bar{\varphi}}_\nu(t) = \bar{\bar{\varphi}}(t)$ almost uniformly on $\langle 0, a)$.

3° Let $\bar{\lambda}(x, t) = \max\limits_i \lambda_i(x, t)$, $\underline{\lambda}(x, t) = \min\limits_i \lambda_i(x, t)$.

Then $\bar{\varphi}_\nu'(t) \geqslant \bar{\lambda}\big(\bar{\varphi}_\nu(t), t\big)$, $\bar{\bar{\varphi}}_\nu'(t) \leqslant \underline{\lambda}\big(\bar{\bar{\varphi}}_\nu(t), t\big)$ for $\nu = 1, 2, \dots$ and $t \in \langle 0, a)$.

Differential inequalities similar to those appearing in 3° have been considered in [2] in connection with a certain uniqueness problem.

Example 1. Suppose that $\lambda_i(x, t)$ are uniformly bounded in a set D which contains the segment $\langle a, b\rangle \times 0$. Define

$$A = \sup_{\substack{i=1,\dots,n\\(x,t)\in D}} \lambda_i(t, x), \qquad B = \inf_{\substack{i=1,\dots,n\\(x,t)\in D}} \lambda_i(x, t),$$
$$T = \{(x, t)\colon\ a + At < x < b + Bt,\ 0 < t < a\}.$$

Assume that $T \subset D$. The functions $\bar{\varphi}_\nu = (a + 1/\nu) + At$, $\bar{\bar{\varphi}}_\nu = (b - 1/\nu) + Bt$ satisfy 1°, 2° and 3° if $\bar{\varphi} = a + At$, $\bar{\bar{\varphi}} = b + Bt$.

Example 2. Let the functions $\lambda_i(x, t)$ $(i = 1, \dots, n)$ be continuous and define

$$\bar{\lambda}(x, t) = \max_i \lambda_i(x, t), \qquad \underline{\lambda}(x, t) = \min_i \lambda_i(x, t).$$

Denote by $\bar{\varphi}(t)$ $(\bar{\bar{\varphi}}(t))$ the right-hand maximum (minimum) solution of the equation

$$x' = \bar{\lambda}(x, t) \qquad (x' = \underline{\lambda}(x, t)) \tag{3}$$

passing through the point $(0, a)$ ($(0, b)$). Let the equation (3) have the left-hand uniqueness property and denote by $\bar{\varphi}_\nu(t)$ ($\bar{\bar{\varphi}}_\nu(t)$) the right-hand maximum (minimum) solution of (3) passing through the point $(0, a+1/\nu)$ ($(0, b-1/\nu)$). The functions $\bar{\varphi}_\nu, \bar{\bar{\varphi}}_\nu$ satisfy 1°, 2° and 3° with suitable a.

We now introduce the following notation: E_n denotes the n-dimensional space of points $(y_1, \ldots, y_n)$; E_n^+ denotes the set of all points $(y_1, \ldots, y_n)$ with non-negative coordinates $y_i \geqslant 0$.

THEOREM 1. *Let the functions $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ satisfy (W_{E_n}) and let $u_i(x, t)$ $(i = 1, \ldots, n)$ be a regular solution of (2). Suppose that $R(\bar{\varphi}, \bar{\bar{\varphi}})$ is a proper set. We assume that*

$$f_i(x, t, y_1, \ldots, y_n) \leqslant \sigma_i(t, y_1, \ldots, y_n), \quad i = 1, \ldots, n \tag{4}$$

for $(x, t) \in R(\bar{\varphi}, \bar{\bar{\varphi}})$ and arbitrary y_i. Then $u_i(x, t) \leqslant \omega_i(t\colon \eta_1, \ldots, \eta_n)$ for $i = 1, \ldots, n$, $(x, t) \in R(\bar{\varphi}, \bar{\bar{\varphi}})$ with $\eta_i = \max\limits_{\bar{\varphi}(0) \leqslant x \leqslant \bar{\bar{\varphi}}(0)} u_i(x, 0)$.

Proof. Define for a fixed ν

$$\varrho_i(t) = \max_{\bar{\varphi}_\nu(t) \leqslant x \leqslant \bar{\bar{\varphi}}_\nu(t)} u_i(x, t), \quad i = 1, \ldots, n.$$

There is an $\bar{x} \in \langle \bar{\varphi}_\nu(t), \bar{\bar{\varphi}}_\nu(t) \rangle$ such that $\varrho_i(t) = u_i(\bar{x}, t)$. Suppose that $\bar{\varphi}_\nu(t) < \bar{x} < \bar{\bar{\varphi}}_\nu(t)$. Then $(\partial u_i/\partial x)_{(\bar{x},t)} = 0$ and $(\partial u_i/\partial t)_{(\bar{x},t)} \geqslant \bar{D}_-\varrho_i(t)$. On the other hand $\varrho_i(t) = u_i(\bar{x}, t)$ and $u_k(\bar{x}, t) \leqslant \varrho_k(t)$ for $k \neq i$. Condition (W_{E_n}) and the above relations imply

$$\bar{D}_-\varrho_i(t) \leqslant \left(\frac{\partial u_i}{\partial t}\right)_{(\bar{x},t)} = f_i\big(\bar{x}, t, u_1(\bar{x}, t), \ldots, u_n(\bar{x}, t)\big) \leqslant \sigma_i\big(t, \varrho_1(t), \ldots, \varrho_n(t)\big). \tag{5}$$

Suppose now that $\bar{x} = \bar{\varphi}_\nu(t)$. Then $u_i\big(t, \bar{\varphi}_i(t)\big) = \varrho_i(t)$ and $u_i\big(t+h, \bar{\varphi}_\nu(t+h)\big) \leqslant \varrho_i(t+h)$ for $h < 0$ and h sufficiently small. Hence

$$\frac{d}{d\tau}\big[u_i\big(\tau, \bar{\varphi}_\nu(\tau)\big)\big]_{\tau=t} \geqslant \bar{D}_-\varrho_i(t).$$

But

$$\frac{d}{d\tau}\big[u_i\big(\tau, \bar{\varphi}_\nu(\tau)\big)\big]_{\tau=t} = \left(\frac{\partial u_i}{\partial t}\right)_{(\bar{x},t)} + \bar{\varphi}_\nu'(t)\left(\frac{\partial u_i}{\partial x}\right)_{(\bar{x},t)}$$

and $(\partial u_i/\partial x)_{(\bar{x},t)} \leqslant 0$, $\bar{\varphi}_\nu'(t) \geqslant \bar{\lambda}\big(\bar{\varphi}_\nu'(t), t\big) \geqslant \lambda_i\big(\bar{\varphi}_\nu(t), t\big)$. Making use of (W_{E_n}) we derive from the above relations the following inequality:

$$\begin{aligned} \bar{D}_-\varrho_i(t) &\leqslant \left(\frac{\partial u_i}{\partial t}\right)_{(\bar{x},t)} + \bar{\varphi}_\nu'(t)\left(\frac{\partial u_i}{\partial x}\right)_{(\bar{x},t)} \leqslant \left(\frac{\partial u_i}{\partial t}\right)_{(\bar{x},t)} + \lambda_i(\bar{x}, t)\left(\frac{\partial u_i}{\partial x}\right)_{(\bar{x},t)} \\ &= f_i\big(\bar{x}, t, u_1(\bar{x}, t), \ldots, u_n(\bar{x}, t)\big) \leqslant \sigma_i\big(t, \varrho_1(t), \ldots, \varrho_n(t)\big). \end{aligned} \tag{6}$$

In a similar way we can prove that $\bar{D}_-\varrho_i(t) \leqslant \sigma_i\big(t, \varrho_1(t), \ldots, \varrho_n(t)\big)$ in the case $\bar{x} = \bar{\bar{\varphi}}_\nu(t)$. By (5) and (6) we conclude that in every case

$\bar{D}_- \varrho_i(t) \leqslant \sigma_i(t, \varrho_1(t), \ldots, \varrho_n(t))$ for $i = 1, \ldots, n$ and $0 < t < a$. The assertion of our theorem follows from lemma 1.

Applying a technique similar to that presented above one easily proves the following theorems:

THEOREM 2. *Let the functions* $\sigma_i(t, y_1, \ldots, y_n) \geqslant 0$ $(i = 1, \ldots, n)$ *satisfy* $(\mathrm{W}_{E_n^+})$ *and let* $u_i(x, t)$ $(i = 1, \ldots, n)$ *be a regular solution of* (2) *in a proper set* $R(\overline{\varphi}, \overline{\overline{\varphi}})$. *Suppose that* $|f_i(x, t, y_1, \ldots, y_n)| \leqslant \sigma_i(t, |y_1|, \ldots, |y_n|)$ $(i = 1, \ldots, n)$, $(x, t) \in R(\overline{\varphi}, \overline{\overline{\varphi}})$. *Then* $|u_i(x, t)| \leqslant \omega_i(t\colon \eta_1, \ldots, \eta_n)$ *for* $i = 1, \ldots, n$, $t \in \langle 0, a)$ *with* $\eta_i = \max_{\overline{\varphi}(0) \leqslant x \leqslant \overline{\overline{\varphi}}(0)} |u_i(x, 0)|$.

THEOREM 3. *Let the functions* $\sigma_i(t, y_1, \ldots, y_n) \geqslant 0$ $(i = 1, \ldots, n)$ *satisfy* $(\mathrm{W}_{E_n^+})$ *and let* $u_1^s(x, t), \ldots, u_n^s(x, t)$ $(s = 1, 2)$ *be a regular solution of the system*

$$\frac{\partial z_i}{\partial t} + \lambda_i(x, t) \frac{\partial z_i}{\partial x} = f_i^s(x, t, z_1, \ldots, z_n), \qquad i = 1, \ldots, n$$

in the proper set $R(\overline{\varphi}, \overline{\overline{\varphi}})$. *Suppose that*

$$|f_i^1(x, t, \bar{y}_1, \ldots, \bar{y}_n) - f_i^2(x, t, \bar{y}_1, \ldots, \bar{y}_n)| \leqslant \sigma_i(t, |\bar{y}_1 - \bar{\bar{y}}_1|, \ldots, |\bar{y}_n - \bar{\bar{y}}_n|)$$

for $i = 1, \ldots, n$ *and* $(x, t) \in R(\overline{\varphi}, \overline{\overline{\varphi}})$ *and arbitrary* $\bar{y}_i, \bar{\bar{y}}_i$. *Then* $|u_i^1(x, t) - u_i^2(x, t)| \leqslant \omega_i(t\colon \eta_1, \ldots, \eta_n)$ *for* $i = 1, \ldots, n$, $(x, t) \in R(\overline{\varphi}, \overline{\overline{\varphi}})$ *with* $\eta_i = \max_{\overline{\varphi}(0) \leqslant x \leqslant \overline{\overline{\varphi}}(0)} |u_i^1(x, 0) - u_i^2(x, 0)|$.

2. Theorem 3 shows that the following conclusion holds (1): if $f_i^1 = f_i^2 = f_i$ and $\omega_i(t\colon 0, \ldots, 0) \equiv 0$, then the Cauchy problem for system (2) has the uniqueness property. The uniqueness condition for the Cauchy problem for system (2) has been presented in [1]. The following assumptions have been introduced:

$$|f_i(x, t, \bar{y}_1, \ldots, \bar{y}_n) - f_i(x, t, \bar{\bar{y}}_1, \ldots, \bar{\bar{y}}_n)| \leqslant K(t) \sigma \Big(\sum_{j=1}^{n} |\bar{y}_j - \bar{\bar{y}}| \Big).$$

$K(t)$ is summable over every subinterval of $(0, a)$, the function $\sigma(z)$ is positive and increasing, and $\int_0^\eta dz/\sigma(z) = +\infty$.

The monotonicity of $\sigma(z)$ ensures the use of the method of integral inequalities developed in [1]. Moreover, in [1] appears the assumption that the ordinary differential equations $x' = \lambda_i(x, t)$ $(i = 1, \ldots, n)$ satisfy the uniqueness condition. This assumption has been introduced in order to replace (2) by a suitable system of integral equations. Applying the method of differential inequalities we can prove, just as in [4], a certain uniqueness condition without making use of any integral equations.

(1) For a similar uniqueness property see [7].

THEOREM 4. *Let $R(\overline{\varphi}, \overline{\overline{\varphi}})$ be a proper set and let the function $\sigma(t, u)$ satisfy the following condition:*

$\sigma(t, u) \geqslant 0$ is continuous for $t \in (0, a)$, $u \geqslant 0$ and for every $\varrho \in (0, a)$ the unique function $\omega(t)$ which satisfies the equation $u' = \sigma(t, u)$ in $(0, \varrho)$ and the condition $\lim_{t\to 0+} \omega(t) = 0$ is the function identically equal to zero $\omega(t) \equiv 0$.

Suppose that

$$|f_i(x, t, \overline{y}_1, \ldots, \overline{y}_n) - f_i(x, t, \overline{\overline{y}}_1, \ldots, \overline{\overline{y}}_n)| \leqslant \sigma(t, \max_j |\overline{y}_j - \overline{\overline{y}}_j|)$$

for $i = 1, \ldots, n$ and $(x, t) \in R(\overline{\varphi}, \overline{\overline{\varphi}})$. Then the Cauchy problem for system (2) satisfies the uniqueness property in $R(\overline{\varphi}, \overline{\overline{\varphi}})$.

3. We will discuss some integral equations connected with system (2). We begin with the following

LEMMA 2 ([5], th. 1). *Suppose that the functions $\sigma_i(t, y_1, \ldots, y_n) \geqslant 0$ $(i = 1, \ldots, n)$ are continuous on $\langle 0, a) \times E_n^+$. We assume that for every i the function $\sigma_i(t, y_1, \ldots, y_n)$ increases in $y_1, y_2, \ldots, y_n$. It is supposed that for every point $(\eta_1, \ldots, \eta_n)$, $\eta_i \geqslant 0$ the right-hand maximum solution $\omega_i(t\colon \eta_1, \ldots, \eta_n)$ of the system $y_i' = \sigma_i(t, y_1, \ldots, y_n)$ exists in the whole interval $\langle 0, a)$. Let the non-negative continuous functions $\varphi_1(t), \ldots, \varphi_n(t)$ satisfy in $\langle 0, a)$ the inequalities*

$$\varphi_i(t) \leqslant \eta_i + \int_0^t \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\, d\tau\,, \qquad i = 1, \ldots, n.$$

Then $\varphi_i(t) \leqslant \omega_i(t\colon \eta_1, \ldots, \eta_n)$ for $i = 1, \ldots, n$, $t \in \langle 0, a)$.

Let the functions $\lambda_i(x, t)$ $(i = 1, \ldots, n)$ be continuous and suppose that every differential equation $x' = \lambda_i(x, t)$ satisfies the uniqueness condition. The solution of $x' = \lambda_i(x, t)$ passing through (x_0, t_0) is denoted by $x_i(t\colon x_0, t_0)$. Suppose we are given two continuously differentiable functions $\overline{\varphi}(t), \overline{\overline{\varphi}}(t)$ such that $\overline{\varphi}'(t) > \lambda_i\big(\overline{\varphi}(t), t\big)$, $\overline{\overline{\varphi}}'(t) < \lambda_i\big(\overline{\overline{\varphi}}(t), t\big)$ for $i = 1, \ldots, n$ and $t \in \langle 0, a)$. Assume that $\overline{\varphi}(0) = a < b = \overline{\overline{\varphi}}(0)$. Suppose that $\overline{\varphi}(t) < \overline{\overline{\varphi}}(t)$ for $t \in \langle 0, a)$. We define

$$R(\overline{\varphi}, \overline{\overline{\varphi}}) = \{(x, t)\colon\ 0 < t < a,\ \overline{\varphi}(t) < x < \overline{\overline{\varphi}}(t)\}\,.$$

It is a simple matter to verify that for $(x_0, t_0) \in R(\overline{\varphi}, \overline{\overline{\varphi}})$ and $i = 1, \ldots, n$ the integral $x_i(t\colon x_0, t_0)$ meets the segment $\langle a, b \rangle$, i.e. $x_i(0\colon x_0, t_0) \in \langle a, b\rangle$. Assume now that the functions $f_i^s(x, t, z_1, \ldots, z_n)$ $(i = 1, \ldots, n,\ s = 1, 2)$ are continuous for $t \in \langle 0, a)$ and arbitrary $z_1, \ldots, z_n$. We can then write the system of integral equations

$$(7) \qquad z_i(x, t) = z_i\big(x_i(0\colon x, t), 0\big) + \\ + \int_0^t f_i^s\Big(x_i(\tau\colon x, t), \tau, z_1\big(x_i(\tau\colon x, t), \tau\big), \ldots, z_n\big(x_i(\tau\colon x, t), \tau\big)\Big)\, d\tau\,.$$

THEOREM 5. *Let the functions* $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, \ldots, n)$ *satisfy the assumptions of lemma* 2. *Suppose that*

$$(8)\quad |f_i^1(x, t, \bar{y}_1, \ldots, \bar{y}_n) - f_i^2(x, t, \bar{\bar{y}}_1, \ldots, \bar{\bar{y}}_n)| \leqslant \sigma_i(t, |\bar{y}_1 - \bar{\bar{y}}_1|, \ldots, |\bar{y}_n - \bar{\bar{y}}_n|)$$

for $i = 1, \ldots, n$ *and* $(x, t) \in R(\bar{\varphi}, \bar{\bar{\varphi}})$. *Let the continuous functions* $z_1^s(x, t), \ldots, z_n^s(x, t)$ $(s = 1, 2)$ *satisfy* (7). *Then*

$$|z_i^1(x, t) - z_i^2(x, t)| \leqslant \omega_i(t\colon \eta_1, \ldots, \eta_n)$$

for $i = 1, \ldots, n$ *and* $(x, t) \in R(\bar{\varphi}, \bar{\bar{\varphi}})$ *with* $\max\limits_{\langle a,b\rangle} |z_i^1(x, 0) - z_i^2(x, 0)| = \eta_i$.

Proof. We first define

$$\varrho_i(t) = \max_{\bar{\varphi}(t) \leqslant x \leqslant \bar{\bar{\varphi}}(t)} |z_i^1(x, t) - z_i^2(x, t)|\,.$$

By (7) and (8) we have

$$(9)\quad |z_i^1(x, t) - z_i^2(x, t)| \leqslant \eta_i + \int_0^t \sigma_i\big(\tau, \big|z_1^1\big(x_i(\tau\colon x, t), \tau\big) - z_1^2\big(x_i(\tau\colon x, t), \tau\big)\big|, \ldots, \big|z_n^1\big(x_i(\tau\colon x, t), \tau\big) - z_n^2\big(x_i(\tau\colon x, t), \tau\big)\big|\big)\,d\tau\,.$$

Observe now that

$$\bar{\varphi}(\tau) \leqslant x_i(\tau\colon x, t) \leqslant \bar{\bar{\varphi}}(\tau)$$

and consequently $\big|z_j^1\big(x_i(\tau\colon x, t), \tau\big) - z_j^2\big(x_i(\tau\colon x, t), \tau\big)\big| \leqslant \varrho_j(\tau)$. It follows from (9) and from the monotonicity of σ_i that

$$(10)\quad \varrho_i(t) \leqslant \eta_i + \int_0^t \sigma_i\big(\tau, \varrho_1(\tau), \ldots, \varrho_n(\tau)\big)\,d\tau\,.$$

The assertion of our theorem now follows from lemma 2.

It is easy to formulate analogous theorems concerning limitations of solutions or the absolute values of solutions of systems of the form (7).

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