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# On a characterization of $L^p$-norm

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**Abstract.** Let $\varphi$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a bijective function. We prove that if $\varphi$ is continuous at an interior point of $\boldsymbol{R}_+$ and the functional $p_\varphi$: $\boldsymbol{R}^2 \to \boldsymbol{R}$ defined by the formula $p_\varphi(x) = \varphi^{-1}(\varphi(|x_1|) + \varphi(|x_2|))$, $x = (x_1, x_2)$, is positively homogeneous, then $\varphi(t) = ct^p$, $t \in \boldsymbol{R}_+$, for some $c > 0$ and $p \neq 0$. If, moreover, $p_\varphi$ is a norm on $\boldsymbol{R}^2$, then the same is true without any regularity conditions on $\varphi$.

Applying these results, we obtain a characterization of $L^p$-norm which extends some earlier results of A. C. Zaanen, W. Wnuk and the present author.

**Introduction.** Let $\varphi$ and $\gamma$ be the so-called *Orlicz functions*, i.e., $\varphi, \gamma$: $[0, \infty) \to [0, \infty)$ are continuous, strictly increasing, unbounded and $\varphi(0) = \gamma(0) = 0$. Zaanen [8] proved that the functional $p_\varphi$ defined on the Orlicz space $L^\varphi$ over the Lebesgue measure on real line by the formula

$$p_\varphi(x) = \varphi^{-1}\big(\int_R \varphi(|x(s)|)\,ds\big)$$

is homogeneous iff $\varphi$ is a power function (cf. also Zygmund [9], where no proof is given). This result has been generalized by Wnuk [6] who, using the multipliers of Orlicz functions, proved that if $L^\varphi(X, \Sigma, \mu)$ is an Orlicz space over the measure space $(X, \Sigma, \mu)$ with at least two disjoint sets of finite and positive measure and the functional $p_{\gamma,\varphi}$ defined by the formula

$$p_{\gamma,\varphi}(x) = \gamma\big(\int_X \varphi \circ |x|\,d\mu\big)$$

is homogeneous, then $\varphi(t) = \varphi(1)t^p$ and $\gamma(t) = \gamma(1)t^{1/p}$ for a $p \neq 0$. To ensure the correctness of the definition of the functional $p_{\gamma,\varphi}$ it is, in general, necessary to impose some additional conditions on the function $\varphi$ (e.g. the so-called $\Delta_2$ condition).

In the first part of this paper, we show that the homogeneity condition of the functional $p_{\gamma,\varphi}$ on $L^\varphi(X, \Sigma, \mu)$ can be reduced to the following one on the plane $\boldsymbol{R}^2$:

(1) $$\gamma\big(\varphi(|tx_1|) + \varphi(|tx_2|)\big) = t\gamma\big(\varphi(|x_1|) + \varphi(|x_2|)\big), \qquad x_1, x_2, t \in \boldsymbol{R}.$$

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Now the monotonicity and continuity of functions $\gamma$ and $\varphi$ as well as the $\Delta_2$ condition are unsuitable and ad hoc. In the detailed disscusion it is shown that of $\varphi$ is continuous at least at one point of the interval $(0, \infty)$ and (1) holds true for positive $x_1$, $x_2$, $t$, then $\varphi(t) = \varphi(1)t^p$, $\gamma(t) = \gamma(1)t^{1/p}$, where $p \neq 0$ and $t > 0$. The proof is quite elementary and is based on the Cauchy functional equation.

The main result of the second part says that if $p_{\gamma,\varphi}$ is a norm on $\boldsymbol{R}^2$ then, without any regularity assumptions, $\gamma$ and $\varphi$ have to be „conjugated" power functions.

**1.** We are going to describe the set of all invertible functions $\varphi$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ ($\boldsymbol{R}_+ = [0, \infty)$). such that the functional $p_\varphi$: $\boldsymbol{R}^2 \to \boldsymbol{R}_+$ defined by the formula

$$p_\varphi(x) := \varphi^{-1}\big(\varphi(|x_1|)+\varphi(|x_2|)\big), \qquad x = (x_1, x_2)$$

satisfies all the conditions for the norm. In particular, we have

(i) $$\varphi^{-1}\big(\varphi(x_1)+\varphi(x_2)\big) = 0 \Leftrightarrow x_1 = x_2 = 0,$$

(ii) $$\varphi^{-1}\big(\varphi(tx_1)+\varphi(tx_2)\big) = t\varphi^{-1}\big(\varphi(x_1)+\varphi(x_2)\big), \qquad t > 0,$$

(iii) $$\varphi^{-1}\big(\varphi(x_1+y_1)+\varphi(x_2+y_2)\big) \leqslant \varphi^{-1}\big(\varphi(x_1)+\varphi(y_1)\big)+\varphi^{-1}\big(\varphi(x_2)+\varphi(y_2)\big)$$

for all $x_1, x_2, y_1, y_2 \in \boldsymbol{R}_+$.

Remark 1. From (i) it follows that $\varphi^{-1}\big(2\varphi(0)\big) = 0$ which gives $\varphi(0) = 0$. By (ii) the range of $\varphi$ has to be an unbounded semigroup in $\boldsymbol{R}_+$. In the sequel we assume that $\varphi(\boldsymbol{R}_+) = \boldsymbol{R}_+$.

Remark 2. If we replace $\varphi^{-1}$ by a function $\gamma$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ in (i)–(iii) then, maintaining all the properties of $\varphi$ established in Remark 1, from (i) we get $\gamma(0) = 0$ and, setting $x_1 = 1$, $x_2 = 0$ in (ii), we obtain $\gamma\big(\varphi(t)\big) = \gamma\big(\varphi(1)\big)t$. This implies that $\varphi$ has to be one-to-one. Since $\varphi(\boldsymbol{R}_+) = \boldsymbol{R}_+$, it follows that $\gamma(t) = c\varphi^{-1}(t)$, where $c := \gamma\big(\varphi(1)\big)$. Consequently, we have $p_{\gamma,\varphi} = cp_\varphi$ which shows that in this case $p_{\gamma,\varphi}$ is only an apparent generalization of the functional $p_\varphi$. One can consider even more general functionals $p_{\gamma,\varphi_1,\varphi_2}$: $\boldsymbol{R}^2 \to \boldsymbol{R}_+$ of the form $p_{\gamma,\varphi_1,\varphi_2}(x) = \gamma\big(\varphi_1(|x_1|)+\varphi_2(|x_2|)\big)$ for $x = (x_1, x_2)$, where $\gamma, \varphi_1, \varphi_2$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$. Note that if $\gamma$ is one-to-one, onto, $\gamma(0) = \varphi_1(0) = \varphi_2(0) = 0$ and $p_{\gamma,\varphi_1,\varphi_2}$ is positively homogeneous, then what can easily be verified

$$p_{\gamma,\varphi_1,\varphi_2}(x) = \gamma\big(\gamma^{-1}(c_1 x_1)+\gamma^{-1}(c_2 x_2)\big),$$

where $c_i = \gamma\big(\varphi_i(1)\big)$, $i = 1, 2$. Let $L$: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ be defined by $L(x_1, x_2) := (x_1/c_1, x_2/c_2)$. Hence, we have $p_{\gamma^{-1}} = p_{\gamma,\varphi_1,\varphi_2} \circ L$ and, since $L$ is linear, the functional $p_{\gamma^{-1}}$ is positively homogeneous.

In the sequel we assume:

(2) $$\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+ \text{ is one-to-one}, \qquad \varphi(\boldsymbol{R}_+) = \boldsymbol{R}_+ \quad \text{and} \quad \varphi(0) = 0.$$

DEFINITION 1. Let $\varphi$ satisfy (2). We define the functional $p_{\varphi,k}$: $\boldsymbol{R}_+^k \to \boldsymbol{R}_+$ by the formula

$$p_{\varphi,k}(x) := \varphi^{-1}(\varphi(x_1) + \ldots + \varphi(x_k)), \qquad x \in \boldsymbol{R}_+^k,\ k \in \boldsymbol{N}.$$

Remark 3. Note that for every invertible $\varphi$: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$; we have $p_{\varphi,1}(x) = x$, $x \in \boldsymbol{R}_+$.

Notation. Let $x = (x_1, \ldots, x_k) \in \boldsymbol{R}^k$. We write $x > 0$ iff $x_1 > 0, \ldots, x_k > 0$.

Now we can prove the following

LEMMA 1. *If $p_{\varphi,2}$ is positively homogeneous, i.e.,*

$$p_{\varphi,2}(tx) = tp_{\varphi,2}(x), \qquad x > 0,\ t > 0, \tag{3}$$

*then for every $k \in \boldsymbol{N}$ the functional $p_{\varphi,k}$ is positively homogeneous, i.e.,*

$$p_{\varphi,k}(tx) = tp_{\varphi,k}(x), \qquad x > 0,\ t > 0. \tag{4}$$

Proof. For $k = 1$ the lemma is trivial. Suppose that (4) holds for some $k \geqslant 1$. Then for $x = (x_1, \ldots, x_{k+1}) \in \boldsymbol{R}_+^{k+1}$ we have in view of assumption (3)

$$\begin{aligned} p_{\varphi,k+1}(tx) &= \varphi^{-1}([\varphi(tx_1) + \ldots + \varphi(tx_k)] + \varphi(tx_{k+1})) \\ &= \varphi^{-1}\big(\varphi\big[\varphi^{-1}(\varphi(tx_1) + \ldots + \varphi(tx_k))\big] + \varphi(tx_{k+1})\big) \\ &= \varphi^{-1}\big(\varphi\big(p_{\varphi,k}(t(x_1, \ldots, x_k))\big) + \varphi(tx_{k+1})\big) \\ &= \varphi^{-1}\big(\varphi\big(tp_{\varphi,k}(x_1, \ldots, x_k)\big) + \varphi(tx_{k+1})\big) \\ &= t\varphi^{-1}\big(\varphi\big(p_{\varphi,k}(x_1, \ldots, x_k)\big) + \varphi(x_{k+1})\big) \\ &= tp_{\varphi,k+1}(x) \end{aligned}$$

and induction completes the proof.

In this paper the crucial part is played by the following result concerning some functional equations.

LEMMA 2 ([2]). *Suppose that $a, b, \alpha, \beta$ are positive real numbers and $a \neq 1$, $\alpha \neq 1$. If one of the numbers $\ln b/\ln a$, $\ln\beta/\ln\alpha$ is irrational, $\gamma$: $(0, \infty) \to (0, \infty)$ is continuous at least at one point and*

$$\gamma(at) = \alpha\gamma(t), \qquad \gamma(bt) = \beta\gamma(t), \qquad t > 0, \tag{5}$$

*then there exist $c > 0$ and $p \in \boldsymbol{R}$, $p \neq 0$, such that $\gamma(t) = ct^p$, $t > 0$.*

Remark 4. Note that if $\gamma$ satisfies equations (5), then the function $\psi := \gamma/\gamma(1)$ satisfies the following system of equations $\psi(at) = \psi(a)\psi(t)$, $\psi(bt) = \psi(b)\psi(t)$. Hence easily follows that $\psi(a^n b^m) = \psi(a^n)\psi(b^m)$ for all integers $n$ and $m$. This is a multiplicative Cauchy functional equation on a restricted domain. (For the simple proof of Lemma 2, see [2].)

The main result of this section reads as follows:

THEOREM 1. *If the functional* $p_{\varphi,2}\colon \boldsymbol{R}_+^2 \to \boldsymbol{R}_+$ *is positively homogeneous and* $\varphi$ *or* $\varphi^{-1}$ *is continuous in at least one point of the interval* $(0, \infty)$, *then there exist* $c > 0$ *and* $p \in \boldsymbol{R}$, $p \neq 0$, *such that* $\varphi(t) = ct^p$, $t > 0$.

Proof. Suppose first that $\varphi$ is continuous at least in one point of the interval $(0, \infty)$. From the assumption we have

$$\varphi^{-1}(\varphi(tx_1)+\varphi(tx_2)) = t\varphi^{-1}(\varphi(x_1)+\varphi(x_2)), \qquad x_1 > 0,\ x_2 > 0,\ t > 0.$$

Setting here $x_1 = x_2 = 1$ and $a := \varphi^{-1}(2\varphi(1))$, we obtain

$$\varphi(at) = 2\varphi(t), \qquad t > 0. \tag{6}$$

By Lemma 2 the functional $p_{\varphi,3}$ is positively homogeneous, i.e.,

$$\varphi^{-1}(\varphi(tx_1)+\varphi(tx_2)+\varphi(tx_3)) = t\varphi^{-1}(\varphi(x_1)+\varphi(x_2)+\varphi(x_3))$$

for positive $x_1, x_2, x_3, t$. Setting $x_1 = x_2 = x_3 = 1$ and $b := \varphi^{-1}(3\varphi(1))$, we get

$$\varphi(bt) = 3\varphi(t), \qquad t > 0. \tag{7}$$

Since $\ln 3/\ln 2$ is irrational, Lemma 2 completes the proof in this case.

If $\varphi^{-1}$ is continuous at a point of the interval $(0, \infty)$, then from (6) and (7) we obtain that the function $\gamma := \varphi^{-1}$ satisfies equations (5) with $a = 2$, $b = 3$, $\alpha = \varphi^{-1}(2\varphi(1))$ and $\beta = \varphi^{-1}(3\varphi(1))$. This concludes the proof of the theorem.

From the proof of Theorem 1 one can easily get the following

COROLLARY 1. *If* $p_{\varphi,3}$ *is positively homogeneous in two directions* $(1, 1, 0)$ *and* $(1, 1, 1)$ *and* $\varphi$ *or* $\varphi^{-1}$ *is continuous in an interior point of* $\boldsymbol{R}_+$, *then* $\varphi(t) = ct^p$, $c > 0$, $p \in \boldsymbol{R}$, $p \neq 0$.

Remark 5. Now we show how to reduce the homogeneity condition of the functional $p_{\gamma,\varphi}$ defined on $L^\varphi(X, \Sigma, \mu)$ to the one we have just considered.

Let $(X, \Sigma, \mu)$ be a measure space with two disjoint sets of positive finite measure. Denote these sets by $A$ and $B$ and their characteristic functions by $\chi_A$ and $\chi_B$, respectively. Let $\varphi$ and $\gamma$ map $\boldsymbol{R}_+$ into itself and suppose that $\varphi$ satisfies condition (2). Then the functional

$$p_{\gamma,\varphi}(x) := \gamma\Big(\int_X \varphi \circ |x|\, d\mu\Big)$$

is correctly defined for all simple functions $x = a\chi_A + b\chi_B$. Suppose now that $p_{\gamma,\varphi}(tx) = tp_{\gamma,\varphi}(x)$ for $t > 0$ and $x = a\chi_A + b\chi_B$ with $a > 0$, $b > 0$, i.e., that

$$\gamma\Big(\int_X \varphi(ta\chi_A + tb\chi_B)\, d\mu\Big) = t\gamma\Big(\int_X \varphi(a\chi_A + b\chi_B)\, d\mu\Big)$$

which can be written in the following form:

(8) $$\gamma(\varphi(ta)\mu(A)+\varphi(tb)\mu(B)) = t\gamma(\varphi(a)\mu(A)+\varphi(b)\mu(B)).$$

Taking $b = 0$, we get

(9) $$\gamma(\varphi(ta)\mu(A)) = t\gamma(\varphi(a)\mu(A)), \quad t > 0.$$

This implies that there are $c_0 > 0$, $c > 0$ such that

(10) $$\gamma(t) = c_0\varphi^{-1}(ct), \quad t > 0.$$

Setting this into (9), we obtain

$$\varphi^{-1}(c\varphi(ta)\mu(A)) = t\varphi^{-1}(c\varphi(a)\mu(A)).$$

Similarly, taking $a = 0$ in (8) and next applying (10), we obtain

$$\varphi^{-1}(c\varphi(tb)\mu(B)) = t\varphi^{-1}(c\varphi(b)\mu(B)).$$

Take now arbitrarily $x_1, x_2 \in \boldsymbol{R}_+$. It follows from (2) that there exist $a, b \in \boldsymbol{R}_+$ such that

$$x_1 = \varphi^{-1}(c\varphi(a)\mu(A)) \quad \text{and} \quad x_2 = \varphi^{-1}(c\varphi(b)\mu(B)).$$

Therefore, the last two relations can be written in the following form

$$c\varphi(ta)\mu(A) = \varphi(tx_1), \quad c\varphi(tb)\mu(B) = \varphi(tx_2), \quad t > 0.$$

Hence from (8) and (10) we get

$$\varphi^{-1}(\varphi(tx_1)+\varphi(tx_2)) = t\varphi^{-1}(\varphi(x_1)+\varphi(x_2))$$

for $x_1, x_2 \in \boldsymbol{R}_+$ and $t > 0$, which was to be shown.

Remark 6. Under the assumption that $\varphi\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is strictly increasing Theorem 1 has been presented in [4] and in the present version in [3] without proofs.

In connection with Theorem 1 the following problem is open.

PROBLEM. Is the continuity of $\varphi$ or $\varphi^{-1}$ at one point essential?

**2.** In this section we consider the properties of functions $\varphi$ for which (iii) holds, i.e., the functional $p_\varphi\colon \boldsymbol{R}_+^2 \to \boldsymbol{R}_+$ is subadditive. In particular, we desire to prove that (iii) implies the continuity of $\varphi$ or $\varphi^{-1}$ at least at one point of the interval $(0, \infty)$. We will show that in fact the function $\varphi^{-1}$ has this property. As far as $\varphi$ is concerned, the situation is a little more complicated.
We start with the following

LEMMA 3. *Suppose that $\varphi$ satisfies conditions* (2). *If the functional $p_\varphi := p_{\varphi,2}$ is subadditive in $\boldsymbol{R}_+^2$, i.e.,* (iii) *holds, then $\gamma := \varphi^{-1}$ is subadditive in $\boldsymbol{R}_+$.*

Proof. Setting in (iii) $x_1 = t$, $x_2 = 0$, $y_1 = 0$, $y_2 = s$, $t \geq 0$, $s \geq 0$, we have $\varphi^{-1}(\varphi(t)+\varphi(s)) \leq t+s$, i.e., $\gamma(t+s) \leq \gamma(t)+\gamma(s)$ which completes the proof.

Remark 7. Note that there are subadditive functions $\gamma\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ which are one-to-one, onto and such that $\gamma(0) = 0$ but not continuous at any point of $\boldsymbol{R}_+$. To see this take a Hamel base $H$ of the linear space $\boldsymbol{R}$ over the field $Q$ of rational numbers and arbitrary function $\alpha\colon H \to H$ which is one-to-one, onto and not of the form $\alpha(h) = ch$ for $h \in H$. For each $t \in R$ let

$$t = \sum_{h \in H} g_h(t)h, \qquad g_h(t) \in Q,$$

be the representation of $t$ in the base $H$. (Evidently, this sum is finite, because $g_h(t) \neq 0$ only for a finite set of elements $h \in H$.) From the uniqueness of this representation it follows that

$$g_h(t+s) = g_h(t)+g_h(s), \qquad t, s \in \boldsymbol{R},\ h \in H.$$

The function $A\colon \boldsymbol{R} \to \boldsymbol{R}$ defined by the formula

$$A(t) := \sum_{h \in H} g_h(t)\alpha(h)$$

is additive one-to-one mapping of $\boldsymbol{R}$ onto $\boldsymbol{R}$. One can easily verify that $\gamma\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$, $\gamma(t) := |A(t)|$, $t > 0$, has the desired properties. Moreover, $\gamma$ is unbounded on every interval contained in $\boldsymbol{R}_+$ and it is non-measurable.

We shall need the following well-known properties of subadditive functions.

LEMMA 4. *Suppose that* $\gamma\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *is subadditive and bounded on every compact interval contained in* $\boldsymbol{R}_+$. *Then*

(i) *there exists* $\beta := \lim_{t \to \infty} \gamma(t)/t$ *and* $\beta = \inf_{t > 0} \gamma(t)/t$ *and*

(ii) *if* $\lim_{t \to 0+} \gamma(t) = 0$ *then for every* $t > 0$ *there exist the one-sided limits* $\gamma(t-)$, $\gamma(t+)$ *and*

$$\gamma(t+) \leq \gamma(t) \leq \gamma(t-).$$

*Moreover, the set* $D := \{t > 0\colon \gamma(t-) \neq \gamma(t+)\}$ *is at most countable.*

The proof of (i) and the first part of (ii) one can find in [1], pp. 244–248. The countability of the set $D$ follows from Young's theorem [7] (cf. also [5], p. 134).

Remark. In [1], it is generally assumed that all functions considered are measurable. But, in particular, in the proofs of (i) and (ii) the authors use the boundedness of the function on every compact interval in $\boldsymbol{R}_+$. This property is a consequence of measurability and subadditivity of $\gamma$ (cf. [1], p. 241, Theorem 7.4.1).

Now we prove the following

LEMMA 5. *Suppose that $\gamma$: $\mathbf{R}_+ \to \mathbf{R}_+$ is onto. If there exists a $c > 0$ such that*

$$(11) \qquad \gamma(t+s) \geq c\gamma(t), \quad t, s > 0,$$

*then $\lim_{t \to 0+} \gamma(t) = 0$. If, moreover, $\gamma$ is subadditive, then it is bounded on every compact interval in $\mathbf{R}_+$.*

Proof. Suppose that the first part of the lemma is false. Then there are $\varepsilon > 0$ and a sequence $t_n > 0$, $n = 1, 2, \ldots$, such that

$$\lim_{n \to \infty} t_n = 0 \quad \text{and} \quad \gamma(t_n) \geq \varepsilon/c, \quad n = 1, 2, \ldots$$

Now inequality (11) implies

$$\gamma(t_n + s) \geq \varepsilon, \quad s > 0, \ n = 1, 2, \ldots$$

Consequently, $\gamma(t) \geq \varepsilon$ for all $t > 0$. This is a contradiction as $\gamma$ is assumed to be onto. Thus $\lim_{t \to 0+} \gamma(t) = 0$.

From the just proved part of the lemma it follows that there is a $\delta > 0$ such that $\gamma(t) < 1$ for all $t \in [0, \delta)$. For an arbitrary $a > 0$ choose a positive integer $n$ such that $a < n\delta$. Now, for every $t \in [0, a]$ we have $0 \leq t/n < \delta$ and, consequently,

$$\gamma(t) = \gamma(nt/n) \leq n\gamma(t/n) \leq n,$$

which completes the proof.

Now we can prove the main result of the paper.

THEOREM 2. *Suppose that $\varphi$: $\mathbf{R}_+ \to \mathbf{R}_+$ is one-to-one, onto and $\varphi(0) = 0$. If the functional $p_\varphi$: $\mathbf{R}^2 \to \mathbf{R}_+$ defined by the formula*

$$p_\varphi(x) := \varphi^{-1}\big(\varphi(|x_1|) + \varphi(|x_2|)\big), \quad x = (x_1, x_2),$$

*is a norm on $\mathbf{R}^2$, then $\varphi(t) = ct^p$, $t \geq 0$, where $c > 0$, $p \geq 1$, and*

$$p_\varphi(x) = (|x_1|^p + |x_2|^p)^{1/p}.$$

Proof. Since all norms on $\mathbf{R}^2$ are equivalent, there is a $c > 0$ (in fact $c \leq 1$) such that

$$p_\varphi(t, s) = \varphi^{-1}\big(\varphi(t) + \varphi(s)\big) \geq c(t+s), \quad t, s > 0.$$

Hence, for $\gamma := \varphi^{-1}$, we get $\gamma(t+s) \geq c\big(\gamma(t) + \gamma(s)\big)$ for $t, s > 0$, and, in particular, inequality (11) holds. Now, by Lemmas 3, 5 and 4 it follows that $\gamma = \varphi^{-1}$ is continuous at least at one point. Therefore, Theorem 1 applies and this completes the proof.

From Theorem 2 and Remark 5 we obtain the following

COROLLARY 2. *Let* $(X, \Sigma, \mu)$ *be a measure space and suppose that there are* $A, B \in \Sigma$ *disjoint and of finite measure. If the functional*

$$p_\varphi(x) := \varphi^{-1}\Big(\int_X \varphi \circ |x|\, d\mu\Big)$$

*is a norm on the two-dimensional linear space of all simple functions of the form* $x = a\chi_A + b\chi_B$, $a, b \in \mathbf{R}$, *then* $\varphi(t) = ct^p$, $t > 0$ $(c > 0,\ p \geqslant 1)$, *and*

$$p_\varphi(x) = \Big(\int_X |x|^p\, d\mu\Big)^{1/p}.$$

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