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A difference scheme for an elliptic system of non-linear differential-functional equations with Dirichlet type boundary conditions. The existence and uniqueness of solution

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Abstract. We consider a system of second order differential-functional equations of elliptic type with boundary conditions of Dirichlet type.

We propose an implicit difference scheme for this problem and under certain assumptions we show the existence and uniqueness of solution of this scheme.

1. Introduction. Let $D = [0, X]^n \subset R^n$, $X < +\infty$.

We consider the following system of second order differential-functional equations of elliptic type

$$(1.1)\quad f_l(x, u(x), (u_l)_x(x), (u_l)_{xx}(x), u) = 0 \quad \text{for } x \in \operatorname{int} D \ (l = 1, \ldots, p)$$

with boundary conditions of Dirichlet type

$$(1.2)\quad u_l(x) = \varphi_l(x) \quad \text{for } x \in \partial D \ (l = 1, \ldots, p),$$

where

$$x = (x_i)_{i=1,\ldots,n}, \quad u = (u_\mu)_{\mu=1,\ldots,p}, \quad (u_l)_x = (\partial u_l/\partial x_i)_{i=1,\ldots,n},$$
$$(u_l)_{xx} = (\partial^2 u_l/\partial x_i \partial x_j)_{i,j=1,\ldots,n} \quad (l = 1, \ldots, p).$$

We will define an implicit difference scheme for problem (1.1), (1.2).

Under certain assumptions concerning the functions f_l $(l = 1, \ldots, p)$ and the net step h we show the existence of solution of this scheme ([1]).

The proof of the main result of this paper is based on the Banach iteration theorem – see [2]. The idea of construction of a suitable operator arises from [5].

The difference approximation of the mixed derivatives of the solution and the approximation of the functional argument goes by the method adopted from [3].

([1]) The convergence and stability of this scheme is considered in [1].

We use the notation applied by many authors dealing with difference schemes (A. Fitzke, Z. Kowalski, M. Malec, W. Niedoba, etc.).

2. ASSUMPTIONS H. We assume that

(1) the scalar functions $f_l\colon E \ni (x, y, t, w, z) \to f_l(x, y, t, w, z) \in \boldsymbol{R}$ $(l = 1, \ldots, p)$, where $E := D \times \boldsymbol{R}^p \times \boldsymbol{R}^n \times \boldsymbol{R}^{n^2} \times B(D)$, $x = (x_i)_{i=1,\ldots,n}$, $y = (y_\mu)_{\mu=1,\ldots,p}$, $t = (t_i)_{i=1,\ldots,n}$, $w = (w_{ij})_{i,j=1,\ldots,n}$, $B(D) := \{o = (o_1, o_2, \ldots, o_p) \mid o_l\colon D \to \boldsymbol{R}$ is a bounded function for $l = 1, \ldots, p\}$ are such that

$$(2.1)\quad f_l(x, y, q, w, z) - f_l(x, \bar{y}, \bar{q}, \bar{w}, z) = \sum_{\mu=1}^{p} \alpha_{l\mu}(y_\mu - \bar{y}_\mu) + \sum_{i=1}^{n} \beta_{li}(q_i - \bar{q}_i) + \sum_{i,j=1}^{n} \gamma_{lij}(w_{ij} - \bar{w}_{ij}) \quad (l = 1, \ldots, p)$$

for any two points $(x, y, q, w, z), (x, \bar{y}, \bar{q}, \bar{w}, z) \in E$, where $\alpha_{l\mu}$, β_{li}, γ_{lij} $(l, \mu = 1, \ldots, p;\ i, j = 1, \ldots, n)$ are functions defined for all points $(x, y, \bar{y}, q, \bar{q}, w, \bar{w}, z) \in D \times \boldsymbol{R}^{2p} \times \boldsymbol{R}^{2n} \times \boldsymbol{R}^{2n^2} \times B(D)$ and bounded in this set; For any two points (x, y, q, w, z), $(x, y, q, w, \bar{z}) \in E$ we have

$$(2.2)\quad f_l(x, y, q, w, z) - f_l(x, y, q, w, \bar{z}) = \varkappa_l \|z - \bar{z}\| \quad (l = 1, \ldots, p),$$

where $\varkappa_l$ $(l = 1, \ldots, p)$ are functions defined for $(x, y, q, w, z, \bar{z}) \in D \times \boldsymbol{R}^p \times \boldsymbol{R}^n \times \boldsymbol{R}^{n^2} \times [B(D)]^2$, bounded in this set and the norm $\|\ \|$ in the space $B(D)$ is defined by $\|z\| := \max_{1 \le l \le p} \{\sup_{x \in D} |z_l(x)|\}$;

(2) the functions $\alpha_{l\mu}$, β_{li}, γ_{lij}, $\varkappa_l$ $(l = 1, \ldots, p;\ \mu = 1, \ldots, p;\ i, j = 1, \ldots, n)$ satisfy the following conditions (for all admissible arguments):

$$(2.3)\quad L_1 \le \alpha_{ll} \le L < 0, \quad 0 \le \alpha_{l\mu} \le J \quad (l \ne \mu),$$

$$(2.4)\quad |\beta_{li}| \le \Gamma,$$

$$(2.5)\quad 0 < g \le \gamma_{lii} - \sum_{\substack{j=1 \\ j \ne i}}^{n} |\gamma_{lij}|,$$

$$(2.6)\quad \gamma_{lij} = \gamma_{lji},$$

$$(2.7)\quad |\gamma_{lii}| \le G,$$

$$(2.8)\quad L + J(p-1) + K < 0,$$

(2.9) for each pair of indices i, j $(i \ne j)$ the function γ_{lij} is either always non-positive, or always non-negative,

$$(2.10)\quad |\varkappa_l| \le K;$$

(3) there exists a solution $u(x)$ of problem (1.1), (1.2) such that $u \in C^2(D)$.

3. We introduce the uniform net in the cube D. Given a sequence of indices $M = (m_1, \ldots, m_n)$, $m_i = 0, 1, \ldots, N$ $(i = 1, \ldots, n)$, we denote by x^M the nodal point with the coordinates $x^M = (x_1^{m_1}, \ldots, x_n^{m_n})$, where $x_i^{m_i} = m_i h$ $(i = 1, \ldots, n)$, $0 < h = X/N$ and $N \ge 2$.

We introduce the following denotations:

$$Z := \{M\colon\ 0 \leqslant m_i \leqslant N, \quad i = 1, \ldots, n\},$$
$$Z_1 := \{M\colon\ 1 \leqslant m_i \leqslant N, \quad i = 1, \ldots, n\}, \tag{3.1}$$
$$Z_2 := \{M\colon\ 0 \leqslant m_i \leqslant N-1,\ i = 1, \ldots, n\}$$

and we introduce special symbols for the nodal points

$$-i(M) := (m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n) \quad (M \in Z_1),$$
$$i(M) := (m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n) \quad (M \in Z_2) \tag{3.2}$$
$$(i = 1, \ldots, n).$$

For any discrete function $a\colon Z \ni M \to a^M \in R$ the following operators are defined (see [4]):

(3.3)

$$a^{Mi} = 0.5\, h^{-1} (a^{i(M)} - a^{-i(M)}),$$
$$a^{-Mij} = 0.5\, h^{-2} (a^{i(M)} + a^{j(M)} + a^{-i(M)} + a^{-j(M)} - 2a^M - a^{i(-j(M))} - a^{-i(j(M))}),$$
$$a^{+Mij} = 0.5\, h^{-2} (-a^{i(M)} - a^{j(M)} - a^{-i(M)} - a^{-j(M)} + 2a^M + a^{i(j(M))} + a^{-i(-j(M))})$$
$$(M \in Z_1 \cap Z_2;\ i = 1, \ldots, n;\ j = 1, \ldots, n).$$

Every function $b = (b_1, \ldots, b_p) \in B(D)$ is approximated by $b^* = (b_1^*, \ldots, b_p^*) \in B(D)$, where

$$b_\mu^*(x) := \sum_{M \in Z} \chi_M(x)\, b_\mu^M,$$
$$\chi_M(x) := \begin{cases} 1 & \text{for } x \in I_M, \\ 0 & \text{for } x \notin I_M, \end{cases} \tag{3.3'}$$
$$I_M := \{x \in D\colon\ m_i h \leqslant x_i < (m_i + 1) h,\ i = 1, \ldots, n\},$$
$$b_\mu^M = b_\mu(x^M) \quad (\mu = 1, \ldots, p;\ M \in Z).$$

In the same way, for every net function $c\colon Z \ni M \to c^M = (c_1^M, \ldots, c_p^M) \in \boldsymbol{R}^p$ we define c^*:

$$c_\mu^*(x) := \sum_{M \in Z} \chi_M(x)\, c_\mu^M \quad (\mu = 1, \ldots, p),$$

where $\chi_M(x)$ and I_M are defined by (3.3′).

Along with the system of differential-functional equations (1.1), (1.2) we consider the system of difference equations

$$f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) = 0 \quad (M \in Z_1 \cap Z_2;\ l = 1, \ldots, p), \tag{3.4}$$

with the boundary conditions

$$v_l^M = \varphi_l^M := \varphi_l(x^M) \quad \text{for } M \in Z \setminus (Z_1 \cap Z_2) \ (l = 1, \ldots, p), \tag{3.5}$$

where

$$v^M = (v_\mu^M)_{\mu=1,\ldots,p}, \quad v_l^{MI} = (v_l^{Mi})_{i=1,\ldots,n}, \quad v_l^{MIJ} = (v_l^{Mij})_{i,j=1,\ldots,n},$$

$$v_l^{Mij} = \begin{cases} v_l^{-Mij} & \text{for } i = j \text{ or } \gamma_{lij} \leqslant 0, \\ v_l^{+Mij} & \text{for } i \neq j \text{ and } \gamma_{lij} \geqslant 0. \end{cases}$$

System (3.4) and the boundary conditions (3.5) are generated by system (1.1) with boundary conditions (1.2). Let us denote by v and u the solutions of these problems, respectively. In general problem (3.4), (3.5) can have no solution because of non-linearity of the algebraic equations. The solution exists under certain additional assumptions, as we show below.

4. THEOREM 1. *If assumptions* H *and*

$$g/h - \Gamma/2 \geqslant 0 \tag{4.1}$$

are satisfied, then there exists exactly one solution of problem (3.4), (3.5).

Proof. Let q be the cardinality of Z and let $R^{q\times p} := \{V\colon V = (v_l^M)_{M\in Z, \, l=1,\ldots,p}\}$. Define

$$F\colon R^{q\times p} \ni V \to F(V) := C \in R^{q\times p}, \quad C = (c_l^M)_{M\in Z, \, l=1,\ldots,p}, \tag{4.2}$$

where

$$c_l^M := \begin{cases} f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) & \text{for } M \in Z_1 \cap Z_2, \\ (L + J(p-1) + K)(v_l^M - \varphi_l^M) & \text{for } M \in Z \setminus (Z_1 \cap Z_2) \end{cases} \quad (l = 1, \ldots, p) \tag{4.3}$$

and let Φ be a function such that

$$\Phi\colon R^{q\times p} \ni V \to \Phi(V) := h^s F(V) + V \in R^{q\times p}, \tag{4.4}$$

where

$$0 < 1 + h^s(L + J(p-1) + K), \tag{4.5}$$

$$h^s(h^{-2} 4nG - L - L_1) \leqslant 2. \tag{4.6}$$

Write

$$H := 1 + h^s(L + J(p-1) + K). \tag{4.7}$$

Then from (2.8) and (4.5) we have $H \in (0, 1)$. We shall show that

(4.8) $$\|\Phi(V)-\Phi(W)\|_1 \leqslant H\|V-W\|_1 \quad \text{for all } V, W \in R^{q\times p},$$

where $\|V\|_1 := \max\limits_{\substack{M\in Z\\ l=1,\ldots,p}} |v_l^M|$ $(V \in R^{q\times p})$.

To this end take any two elements $V, W \in R^{q\times p}$ and write

$$D := \Phi(V)-\Phi(W) = (d_l^M)_{\substack{M\in Z\\ l=1,\ldots,p}}, \qquad Y := V-W = (y_l^M)_{\substack{M\in Z\\ l=1,\ldots,p}}.$$

By (2.1) and (2.2) we can write

(4.9) $$d_l^M = \begin{cases} h^{s-1}\sum\limits_{i=1}^{n}\Big[h^{-1}\Big(\gamma_{lii}-\sum\limits_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{lij}|\Big)+0.5\,\beta_{li}\Big]\,y_l^{i(M)}+ \\ +h^{s-1}\sum\limits_{i=1}^{n}\Big[h^{-1}\Big(\gamma_{lii}-\sum\limits_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{lij}|\Big)-0.5\,\beta_{li}\Big]\,y_l^{-i(M)}+ \\ +0.5\,h^{s-2}\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{lij}|\big(y_l^{i(s(l,i,j)j(M))}-y_l^{-i(-s(l,i,j)j(M))}\big)+ \\ +\Big[1+h^s\alpha_{ll}-h^{s-2}\sum\limits_{i,j=1}^{n}|\gamma_{lij}|-2h^{s-2}\sum\limits_{i=1}^{n}\Big(\gamma_{lii}-\sum\limits_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{lij}|\Big)\Big]\,y_l^M+ \\ +h^s\sum\limits_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}\,y_\mu^M+h^s\varkappa_l\|y^*\| \quad \text{for } M\in Z_1\cap Z_2, \\ H y_l^M \quad \text{for } M\in Z\setminus(Z_1\cap Z_2) \end{cases}$$

$(l = 1, \ldots, p)$, where

$$s(l,i,j) = \begin{cases} +1 & \text{for } \gamma_{lij}\geqslant 0, \\ -1 & \text{for } \gamma_{lij}\leqslant 0 \ (l=1,\ldots,p;\ i,j=1,\ldots,n), \end{cases}$$

and H is defined by (4.7). To simplify the notation we put

(4.10) $$\sigma_l := \sum_{i=1}^{n}\Big(\gamma_{lii}-\sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{lij}|\Big), \qquad \varrho_l := \sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{lij}|.$$

From the definition of the norm $\|\ \|_1$ it follows that there are $B\in Z$, $k\in\{1, 2, \ldots, p\}$ such that $|d_k^B| = \|D\|_1$.

If $B\in Z\setminus(Z_1\cap Z_2)$, then

$$|d_k^B| = H|y_k^B| \leqslant H\|Y\|_1.$$

If $B\in Z_1\cap Z_2$, then assumption (2) implies the inequalities

$$\sum_{i=1}^{n}\Big[h^{-1}\Big(\gamma_{kii}-\sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{kij}|\Big)+0.5\,\beta_{ki}\Big] \geqslant g/h-0.5\,\Gamma \geqslant 0,$$

(4.11)
$$\sum_{i=1}^{n}\left[h^{-1}\left(\gamma_{kii}-\sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{kij}|\right)-0.5\,\beta_{ki}\right]\geqslant g/h-0.5\,\Gamma\geqslant 0.$$

The definition of norms $\| \ \|$ and $\| \ \|_1$ gives

(4.12) $$\|y^*\| = \max_{1\leqslant l\leqslant p}\left\{\sup_{x\in D}\left|\sum_{M\in Z}\chi_M(x)\,y_l^M\right|\right\} = \max_{1\leqslant l\leqslant p}\left\{\max_{M\in Z}|y_l^M|\right\}\leqslant \|Y\|_1.$$

On account of (4.11), (4.12) and (2) we have the following estimation for d_k^B:

(4.13) $$\begin{aligned}|d_k^B| &\leqslant \Big[h^{s-2}\sigma_k+h^{s-1}\sum_{i=1}^{n}0.5\,\beta_{ki}+h^{s-2}\sigma_k-h^{s-1}\sum_{i=1}^{n}0.5\,\beta_{ki}+\\ &\quad +h^{s-2}\varrho_k+|1+h^s\alpha_{kk}-h^{s-2}\varrho_k-2h^{s-2}\sigma_k|\Big]\|Y\|_1+\\ &\quad +h^s\sum_{\substack{\mu=1\\ \mu\neq k}}^{p}\alpha_{k\mu}\,y_\mu^B+h^s|\varkappa_k|\,\|y^*\|\\ &\leqslant [2h^{s-2}\sigma_k+h^{s-2}\varrho_k+|1+h^s\alpha_{kk}-h^{s-2}\varrho_k-2h^{s-2}\sigma_k|]\,\|Y\|_1+\\ &\quad +h^sJ(p-1)\|Y\|_1+h^sK\,\|Y\|_1\\ &= [2h^{s-2}\sigma_k+h^{s-2}\varrho_k+h^sJ(p-1)+h^sK+|1+h^s\alpha_{kk}-\\ &\quad -h^{s-2}\varrho_k-2h^{s-2}\sigma_k|]\,\|Y\|_1.\end{aligned}$$

Now we shall examine the two cases:

(i) $1+h^s\alpha_{kk}-h^{s-2}\varrho_k-2h^{s-2}\sigma_k\geqslant 0$. Then

(4.14) $$\begin{aligned}|d_k^B| &\leqslant \big(1+h^s\alpha_{kk}+h^sJ(p-1)+h^sK\big)\|Y\|_1\\ &\leqslant \big(1+h^sL+h^sJ(p-1)+h^sK\big)\|Y\|_1 = H\,\|Y\|_1\end{aligned}$$

because of (2.3);

(ii) $1+h^s\alpha_{kk}-h^{s-2}\varrho_k-2h^{s-2}\sigma_k<0$. Then from (4.13) we obtain

(4.15) $$\begin{aligned}|d_k^B| &\leqslant \big(-1-h^s\alpha_{kk}+4h^{s-2}\sigma_k+2h^{s-2}\varrho_k+h^sJ(p-1)+h^sK\big)\|Y\|_1\\ &\leqslant \big(-1-h^s\alpha_{kk}+4h^{s-2}(\sigma_k+\varrho_k)+h^sJ(p-1)+h^sK\big)\|Y\|_1.\end{aligned}$$

From (2.7)

(4.16) $$\sigma_k+\varrho_k=\sum_{i=1}^{n}\left(\gamma_{kii}-\sum_{\substack{j=1\\ j\neq i}}^{n}|\gamma_{kij}|\right)+\sum_{\substack{i,j=1\\ i\neq j}}^{n}|\gamma_{kij}| = \sum_{i=1}^{n}\gamma_{kii}\leqslant nG,$$

so that

$$(4.17)\qquad |d_k^B| \leqslant \big(-1-h^s\alpha_{kk}+4h^{s-2}nG+h^sJ(p-1)+h^sK\big)\|Y\|_1$$
$$\leqslant \big(-1-h^sL_1+4h^{s-2}nG+h^sJ(p-1)+h^sK\big)\|Y\|_1$$
$$\leqslant \big(1+h^sL+h^sJ(p-1)+h^sK\big)\|Y\|_1 = H\|Y\|_1$$

because of (2), (4.6) and (4.16).

This proves (4.8). From the Banach theorem (see [2]) it follows that there exists exactly one vector $\bar{V} = (\bar{v}_l^M)_{\substack{M\in Z\\ l=1,\ldots,p}} \in R^{q\times p}$ such that $\Phi(\bar{V}) = \bar{V}$ and hence $F(\bar{V}) = 0$. The components $\bar{v}_l^M$ $(M\in Z;\ l = 1, \ldots, p)$ of the vector $\bar{V}$ are a solution of problem (3.4), (3.5) and this completes the proof of our theorem.

From the Banach theorem and Theorem 1 we also have the following

Remark 1. Under the assumptions of Theorem 1, for an arbitrary vector $V_0\in R^{q\times p}$ the sequence

$$(4.18)\qquad V_{m+1} := \Phi(V_m) \quad (m = 0, 1, 2, \ldots)$$

converges to the solution $\bar{V}$ of problem (3.4), (3.5). Moreover, the following estimation holds

$$(4.19)\qquad \|\bar{V}-V_m\|_1 \leqslant \frac{H^m}{1-H}\|V_1-V_0\|_1 = \frac{\|F(V_0)\|_1}{L+J(p-1)+K}H^m.$$

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