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Effective reduction of linear ordinary difference equations to equations with constant coefficients

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Abstract. Necessary and sufficient conditions on the coefficients of the general linear, homogeneous, nth order, ordinary difference equation are obtained so that it can be associated with a certain nth degree algebraic equation. The given equation is thus effectively reduced to a difference equation with constant coefficients, and is therefore solvable. The solutions are displayed explicitly in terms of the coefficients. The second order equation is treated in greater detail.

1. Introduction. Linear ordinary difference equations arise quite naturally in many branches of applied mathematics and engineering. There exist no general methods of solving these equations when their coefficients are variable, even for low orders $n > 1$.

The situation is much the same in the case of linear differential equations, but recently Breuer and Gottlieb [1] have characterized those differential equations which can be transformed into equations with constant coefficients by a transformation of the independent variable, and hence solved explicitly. In this paper we shall introduce a method, effectively equivalent to a change of the independent variable, to characterize a class of solvable difference equations of nth order.

Section 2 motivates the procedure to be used. In Section 3 we deduce necessary and sufficient conditions on the coefficients of the equations that are solvable by this method, and display their solutions explicitly. Section 4 treats the case $n = 2$ in greater detail, on account of its special importance.

2. Preliminary considerations. In the following we shall be interested in the most general nth order, linear, homogeneous difference equation, which has the form

$$(1) \qquad \sum_{k=0}^{n} P_k(j) y(j+k) = 0, \quad j = 0, 1, 2, \ldots$$

The coefficients $P_k(j)$ need not be real, but we shall assume $P_n(j)$ as well as $\sum_{k=0}^{n} P_k(j)$ to be different from zero for all j under consideration. By a solution of (1) we shall mean a function $y(j)$, defined for all relevant j and satisfying (1).

If the coefficients P_k in (1) are independent of j, the solution of (1) reduces to the solution of an nth degree algebraic equation, as does the solution of the associated differential equation with constant coefficients. However, no general methods are available to solve (1) when the P_k are functions of j. There are of course special cases in which (1) is equivalent to a difference equation with constant coefficients, under a transformation of the dependent variable $y(j)$ ([2], p. 584 or [4], p. 408). There are also various methods treated in standard texts such as [2], [3], and [4], which can be applied in special cases. On the whole, the situation is rather similar to the case of differential equations.

In this paper, we shall characterize a new class of difference equations of the form (1), which can be associated with an equation with constant coefficients via an nth degree algebraic equation, and hence solved explicitly in terms of its variable coefficients. To this end, we consider the recent work by Breuer and Gottlieb [1] on the most general nth order, linear, homogeneous differential equation of the form

$$\sum_{k=0}^{n} P_k(x)y^{(k)}(x) = 0, \tag{2}$$

which is the counterpart of (1). They have found necessary and sufficient conditions on the coefficients of (2) such that a transformation $\xi = \xi(x)$ of the independent variable x will carry (2) into a differential equation with constant coefficients for the function $y(x(\xi)) \equiv Y(\xi)$. The transformation is essentially unique, and $Y(\xi) = \exp(m\xi)$ is a typical solution, where m is a simple root of the associated algebraic equation of nth degree. The case of multiple roots is also treated in detail. It is clear that $y(x) = \exp(m\xi(x))$ is a typical solution of (2), where $\xi(x)$ is an explicit function of the coefficients $P_k(x)$.

The procedure outlined above is not directly applicable to the difference equation (1). If we try to change the variable j to, say, a new independent variable s, then when j changes by unity, s does not do so in general, and the transformed equation is not of the form (1), with or without constant coefficients. Evidently, then, a new line of attack has to be devised.

In order to motivate the procedure we shall use here, let us consider the first order linear differential equation

$$P_1(x)y'(x) + P_0(x)y(x) = 0, \tag{3}$$

whose solution, for $P_1(x) \neq 0$, is given by

$$y(x) = \exp\left[-\int \frac{P_0(x)}{P_1(x)}\,dx\right]. \tag{4}$$

The, fact, emphasized in [1], that the conditions on $P_k(x)$ in (2) always leave $P_0(x)$ and $P_n(x)$ arbitrary (except for sign requirements), is reflected in the fact that (3) is always solvable. It is instructive to apply the transformation deduced in [1] (equation (21)), namely

$$\xi = \int \left[\frac{P_0(x)}{P_n(x)}\right]^{1/n} dx, \tag{5}$$

to (3). Thus, let $\xi = \int [P_0(x)/P_1(x)]dx$, and let $y(x) = Y(\xi)$. The chain rule gives

$$P_0(x)[Y'(\xi)+Y(\xi)] = 0. \tag{6}$$

The equation for $Y(\xi)$ has constant coefficients, and we find that $Y(\xi) = \exp m\xi$ with

$$m+1 = 0, \tag{7}$$

i.e., $m = -1$. Substitution of ξ from (5) yields the solution (4). In the nth order case, the transformation (5) leads to an algebraic equation of nth degree, with corresponding exponential solutions.

We consider next the first order linear difference equation

$$P_1(j)y(j+1)+P_0(j)y(j) = 0, \tag{8}$$

whose solution is given by

$$y(j) = \prod^{j-1} \left(\frac{-P_0(k)}{P_1(k)}\right). \tag{9}$$

The indicated product is on the index k, and since the lower limit is irrelevant, we have suppressed it. Suppose next that we look for a solution of (8) of the form

$$y(j) = \prod^{j-1} \big(1+mQ(k)\big), \qquad m \text{ constant}. \tag{10}$$

Substitution of (10) into (8) yields

$$y(j)\{P_1(j)[1+mQ(j)]+P_0(j)\} = 0. \tag{11}$$

In order that (11) define a constant m, independent of j, it is necessary and sufficient that the equation

$$P_1(j)+P_0(j)+mP_1(j)Q(j) = 0 \tag{12}$$

be proportional to (7). (Actually it should be proportional to $m+c = 0$, c constant, but we may then redefine m to get back to (7).) This yields

$$P_1(j)+P_0(j) = P_1(j)Q(j), \tag{13}$$

as well as

$$m = -1. \tag{14}$$

Substituting (13) and (14) into (10), we find that (10) is identical with (9). We may therefore associate (8) with the equation $y(j+1)+y(j) = 0$, via (7).

It becomes clear now that we may follow the same procedure for the general equation (1). We shall require solutions of the form (10), and this demand in turn will supply us with a set of necessary and sufficient conditions on the $P_k(j)$ such that m will be a solution of an nth degree algebraic equation, via which (1) may be associated with a corresponding equation with constant coefficients.

3. The equation of order n. In this section we shall state and prove the main theorem of this paper.

THEOREM 1. *Necessary and sufficient conditions that* (1) *possess a solution of the form* (10) *are that the coefficients* $P_k(j)$ *satisfy the following* n *equations*:

$$\sum_{k=s}^{n} \psi_{k,s}(j) P_k(j) = \beta_s \sum_{r=0}^{n} P_r(j), \qquad s = 1, 2, \ldots, n, \tag{15}$$

where β_s *are constants,* $\beta_n = 1$, *and* $\psi_{k,s}(j)$ *are given by*

$$\psi_{k,s}(j) = \sum_{0 \leqslant i_1 < i_2 < \ldots < i_s \leqslant k-1} Q(j+i_1) Q(j+i_2) \ldots Q(j+i_s), \tag{16}$$

in which

$$Q(j) = \frac{F(j-n+1)F(j-2n+1) \ldots F(j-kn+1)}{F(j-n)F(j-2n) \ldots F(j-kn)}, \qquad k \leqslant \frac{j}{n}, \tag{17}$$

where, in turn,

$$F(j) = \frac{\sum_{r=0}^{n} P_r(j)}{P_n(j)}, \; j \geqslant 0; \qquad F(j) = 1, \; j < 0. \tag{18}$$

Before we prove Theorem 1, we note that (15), (16), and (17), (18) are the counterparts of (17), (10)–(12), and (18), respectively, in [1].

Proof of Theorem 1. Let (1) possess a solution of the form (10). Substitution of (10) into (1) yields

$$P_0(j) + \sum_{k=1}^{n} P_k(j) \prod_{l=0}^{k-1} [1 + mQ(j+l)] = 0. \tag{19}$$

Expanding the product in (19), we find

$$\prod_{l=0}^{k-1} [1 + mQ(j+l)] = 1 + \sum_{s=1}^{k} \psi_{k,s}(j) m^s, \tag{20}$$

where $\psi_{k,s}(j)$ is given by (16). Putting (20) into (19), we obtain

$$\sum_{r=0}^{n} P_r(j) + \sum_{k=1}^{n} P_k(j) \sum_{s=1}^{k} \psi_{k,s}(j) m^s = 0. \tag{21}$$

Interchanging the order of summation in (21) leads to

$$\sum_{r=0}^{n} P_r(j) + \sum_{s=1}^{n} m^s \sum_{k=s}^{n} \psi_{k,s}(j) P_k(j) = 0. \tag{22}$$

Now if (22) is to define a constant m independent of j, it is necessary and sufficient that (22) be proportional to an algebraic equation of the nth degree, i.e. proportional to

$$m^n + \sum_{s=1}^{n-1} \beta_s m^s + 1 = 0, \tag{23}$$

where $\beta_n = 1$ without loss of generality. This proves the necessity of (15) and (16) in Theorem 1.

To prove (17), we put $s = n$ in (15) to find

$$\psi_{n,n}(j) = \frac{\sum_{r=0}^{n} P_r(j)}{P_n(j)}, \tag{24}$$

since $\beta_n = 1$. Using (16) and (18), we may write (24) in the form

$$\prod_{l=0}^{n-1} Q(j+l) = F(j), \quad j = 0, 1, 2, \ldots, \tag{25}$$

from which it must be shown that $Q(j)$ are given by (17). To this end, define $Z(j) = \log Q(j)$ and $c(j) = \log F(j)$. Then (25) becomes

$$\sum_{l=0}^{n-1} Z(j+l) = c(j). \tag{26}$$

Equation (26) has constant coefficients and is easily solved. Indeed, the general solution is given by

$$Z(j) = \sum_{k=1}^{n-1} a_k \omega_k^j + Z^*(j), \tag{27}$$

where a_k are arbitrary constants, ω_k are the nth roots of unity, other than unity itself, and $Z^*(j)$ is any particular solution of (26). It is easily seen, however, that the terms in ω_k contribute nothing to the solution of (25) except multiplying the left-hand side by unity. Consequently, we are only interested in any one particular solution of (26), and we might as well forget about $Z(j)$ and get a particular solution for $Q(j)$

directly from (25). For that purpose, it is entirely sufficient to define $Q(j)$ recursively as follows:

$$(28)\qquad \begin{aligned} &Q(0) = Q(1) = \ldots = Q(n-2) = 1, \\ &Q(j+n-1) = \frac{F(j)}{Q(j)Q(j+1)\ldots Q(j+n-2)}, \quad j = 0, 1, 2, \ldots \end{aligned}$$

A little reflection will show that (28) are identical with (17). This completes the proof of the necessity of the conditions in Theorem 1. The sufficiency is proved by straightforward substitution, completing the proof.

It is part of the import of Theorem 1 that demanding $y(j)$ to be of the form (10) effectively reduces (1), via (23), to the nth order difference equation

$$y(j+n)+\sum_{k=1}^{n-1} \beta_k y(j+k)+y(j) = 0.$$

An immediate consequence of Theorem 1 is the following:

COROLLARY 1. *Necessary and sufficient conditions that* (1) *possess n linearly independent solutions of the form* (10) *are*:

(i) *The conditions of Theorem* 1 *hold, and*

(ii) *The algebraic equation* (23) *has n distinct roots with β_s given by* (15).

It is obvious that if the conditions of Corollary 1 hold, then the solutions are given by (10), where m is any one of the n distinct roots of (23).

The case where (23) has multiple roots offers no special difficulty and can be handled by the standard methods of reducing the order of the equation when one or more solutions are known. We shall treat this case in some detail for the difference equation of second order, to which we shall turn our attention in the next section.

4. The equation of order 2. The difference equation of order 2 is by far the most important one from the point of view of the applications, and it seems worthwhile to specialize the results of Section 3 to this case. This will yield the most general from of the second order equation whose solutions are of the form (10).

The equation under consideration is

$$(29)\qquad P_2(j)y(j+2)+P_1(j)y(j+1)+P_0(j)y(j) = 0, \quad j = 0, 1, 2, \ldots$$

For future reference we recall that (29) can always be written in the self-adjoint form

$$(30)\qquad \Delta[p(j)\Delta y(j)]+q(j)y(j) = 0,$$

where $\Delta y(j) = y(j+1) - y(j)$, and where

(31) $\quad P_2(j) = p(j+1), \quad P_1(j) = -[p(j)+p(j+1)], \quad P_0(j) = p(j)+q(j).$

It will be seen that the form (30) will help to simplify the calculations considerably.

Next we set $n = 2$ in (15) and (16) to obtain the following 2 equations:

$$Q(j)Q(j+1)P_2(j) = \sum_{r=0}^{2} P_r(j), \tag{32}$$

$$Q(j)P_1(j) + [Q(j)+Q(j+1)]P_2(j) = \beta_1 \sum_{r=0}^{2} P_r(j). \tag{33}$$

Making use of (31), the left-hand side of (33) becomes

$$\begin{aligned}[P_1(j)+P_2(j)]Q(j)+P_2(j)Q(j+1) &= Q(j+1)p(j+1) - Q(j)p(j) \\ &= \Delta[Q(j)p(j)].\end{aligned} \tag{34}$$

Similarly, using (31) for the right-hand side of (33) as well as for (32), we find that we must have

$$Q(j)Q(j+1) = \frac{q(j)}{p(j+1)}, \tag{35}$$

and

$$\Delta[Q(j)p(j)] = \beta_1 q(j). \tag{36}$$

The right-hand side of (35) is $F(j)$, defined in (18), as stated in (25). We could use (17) to find $Q(j)$, and use this in (10) to determine $y(j)$. However, since $p(j)$ and $q(j)$ are not independent, it is more profitable to work with the system (35), (36) directly.

There are two cases to consider, according as $\beta_1 = 0$ or $\beta_1 \neq 0$ in (36), where β_1 of course enters the algebraic equation for m, which for $n = 2$ is obtained from (23) as

$$m^2 + \beta_1 m + 1 = 0. \tag{37}$$

Suppose first that $\beta_1 = 0$. Then (36), (35) and (37) yield

$$Q(j) = \frac{a}{p(j)}, \quad q(j) = \frac{a^2}{p(j)}, \quad m = \pm i, \tag{38}$$

where a is a constant, not necessarily real. Replacing a by ia, we obtain the following theorem.

THEOREM 2. *Two linearly independent solutions of the equation*

$$\Delta[p(j)\Delta y(j)] - \frac{a^2}{p(j)} y(j) = 0, \tag{39}$$

are given by

$$(40)\qquad y(j) = \prod^{j-1}\left[1 \pm \frac{a}{p(l)}\right].$$

If a is complex, the solution (40) *can be expressed in terms of the appropriate sines and cosines in the usual way.*

Suppose next that $\beta_1 \neq 0$. Then (36) yields

$$(41)\qquad Q(j) = \frac{\beta_1}{p(j)} \sum^{j-1} q(k),$$

and (35) in turn gives

$$(42)\qquad p(j) = \beta_1^2 \frac{\sum^{j} q(k) \sum^{j-1} q(l)}{q(j)}.$$

Moreover, m is given by (37) as

$$(43)\qquad m = \frac{-\beta_1 \pm \sqrt{\beta_1^2 - 4}}{2},$$

where we assume for the time being that $\beta_1^2 \neq 4$.

Collecting the results we have proved the following theorem:

THEOREM 3. *The most general form of* (30) *having two linearly independent solutions of the form* (10), *where* $p(j)q(j) \neq$ *constant, is*

$$(44)\qquad \Delta\left[\beta_1^2 \frac{\sum^{j} q(k) \sum^{j-1} q(l)}{q(j)} \Delta y(j)\right] + q(j) y(j) = 0.$$

The solutions of (44) *are given by*

$$(45)\qquad y(j) = \prod^{j-1}\left[1 + \frac{m}{\beta_1}\left(\frac{q(l)}{\sum^{l} q(k)}\right)\right],$$

where the two values of m are given by (43), *in which it is assumed that* $\beta_1^2 \neq 4$.

Theorem 3 may be given an alternative formulation by expressing $q(j)$ in terms of $p(j)$. Indeed, if we expand (36) and divide it by (35), we obtain

$$(46)\qquad \frac{1}{Q(j+1)p(j+1)} - \frac{1}{Q(j)p(j)} = -\frac{\beta_1}{p(j)},$$

which is first order linear in $1/Q(j)p(j)$. Solving (46), we reach

$$(47)\qquad Q(j) = \frac{-1/\beta_1}{p(j) \sum^{j-1} \frac{1}{p(k)}},$$

and (35) then yields

$$q(j) = \frac{(1/\beta_1)^2}{p(j)\sum^{j}\frac{1}{p(k)}\sum^{j-1}\frac{1}{p(l)}}. \tag{48}$$

It easily follows now that Theorem 3 has the following equivalent formulation.

THEOREM 4. *The most general form of* (30) *having two linearly independent solutions of the form* (10), *where* $p(j)q(j) \neq$ *constant, is*

$$\Delta[p(j)\Delta y(j)] + \frac{(1/\beta_1)^2}{p(j)\sum^{j}\frac{1}{p(k)}\sum^{j-1}\frac{1}{p(l)}}\,y(j) = 0. \tag{49}$$

The solutions of (49) *are given by*

$$y(j) = \prod^{j-1}\left[1 - \frac{m/\beta_1}{p(l)\sum^{l-1}\frac{1}{p(k)}}\right], \tag{50}$$

where the two values of m *are again given by* (43), *in which it is assumed that* $\beta_1^2 \neq 4$.

Equations (41) and (47) are of course consistent and can easily be shown to be consistent with (17). Comparison of (41) and (47) also yields

$$\sum^{j-1} q(k)\sum^{j-1}\frac{1}{p(l)} = -\frac{1}{\beta_1^2}, \tag{51}$$

which is another necessary and sufficient condition for (30) to have solutions of the form (45) or (50).

We note that Theorems 2, 3, and 4 are the counterparts of cases (iii) and (i) of Corollary 4.1 in [1].

There remains the case $\beta_1^2 = 4$ to be considered. In this case we find from (43) that $m/\beta_1 = -1/2$, and one solution is given from (45), say, by

$$y(j) = \prod^{j-1}\left[1 - \frac{q(l)}{2\sum^{l} q(k)}\right]. \tag{52}$$

Standard methods then show the other solution to be given by

$$Y(j) = y(j)\sum^{j-1}\frac{\prod^{l-1}[P_0(k)/P_2(k)]}{y(l)y(l+1)}, \tag{53}$$

where $P_0(j)$ and $P_2(j)$ are given in terms of $p(j)$ and $q(j)$, by (31). This case is the counterpart of case (ii) of Corollary 4.1 in [1].

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