ANNALES
POLONICI MATHEMATICI
XXIX (1975)

Investigation of a certain system of singular integral equations of arbitrary power

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Abstract. Let B be a non-empty set of real numbers of arbitrary power. If to each $\beta \in B$ there corresponds exactly one function $f_\beta(t)$ defined on the arc L_0 ($L_0 = L - \{a, b\}$, where $L = ab$ denotes a non-closed smooth arc with and-points a and b), then we say that on L_0 there is defined a system of functions $\{f_\beta(t)\}_{\beta \in B}$ of the power of the set B.

Assume that we are given system (6) of singular integral equations of arbitrary power, where $\{f_\beta(t)\}_{\beta \in B}$ is a system of unknown functions.

The authors prove the existence of solutions of the system (6), applying the Schauder–Tichonov theorem. It appears that if assumptions 1°–3° are fulfilled and the constants of the problem satisfy condition (21), then system (6) has at least one solution belonging to the class $\mathfrak{H}(M, \alpha, \varphi)$.

Introduction. Let $L = \breve{ab}$ denote a non-closed smooth arc with end-points a and b lying in the complex plane, and let $L_0 = L - \{a, b\}$. For each $t \in L_0$, let t^* denote that of the end-points a and b for which le. $\breve{tt^*}$ = min (le. $\breve{at}$, le. $\breve{tb}$) (le. denotes length). Analogously, for any $t_1 \in L_0$ let t_1^* denote that element of the set $\{a, b\}$ for which le. $\breve{t_1 t_1^*}$ = min(le. $\breve{at_1}$, le. $\breve{t_1 b}$). If t and t_1 are two points of the set L_0, then we assume that le. $\breve{tt^*} \leqslant$ le. $\breve{t_1 t^*}$.

Let $\varphi(\sigma)$ be any increasing and continuous function defined in the interval $\langle 0, +\infty)$, satisfying the condition $\varphi(0) = 0$ and the condition

$$\forall_{\nu \geqslant 1} \forall_{\sigma \geqslant 0} \varphi(\nu\sigma) \leqslant \nu\varphi(\sigma). \tag{1}$$

Let Φ denote the set of all functions $\varphi(\sigma)$ satisfying the above conditions.

We consider a subset Φ^* of the set Φ. Namely, Φ^* contains all functions $\varphi(\sigma)$ satisfying the conditions

$$\exists_{A_1 > 0} \forall_{\alpha \in \langle 0,1)} \int_0^{+\infty} \frac{\varphi(\sigma)\, d\sigma}{\sigma(1+\sigma)^{1-\alpha}} \leqslant A_1, \tag{2}$$

$$\exists_{A_2 > 0} \forall_{s \geqslant 0} \int_0^{s} \frac{\varphi(\sigma)}{\sigma} d\sigma \leqslant A_2 \varrho(s), \tag{3}$$

$$\exists_{A_3>0}\ \forall_{0<a\leqslant b} \int_a^b \frac{\varphi(\sigma)}{\sigma^2}\,d\sigma \leqslant A_3\frac{\varphi(a)}{a}. \tag{4}$$

Definition ([4], [5]). By the class $\mathfrak{H}(M, \alpha, \varphi)$ we understand the set of all complex functions $h(t)$ of a real variable t defined and continuous in the set L_0 which satisfy the inequalities

$$\begin{aligned} |t-t^*|^\alpha |h(t)| &\leqslant M, \\ |t-t^*|^\alpha |h(t)-h(t_1)| &\leqslant M\varphi(|t-t_1|/|t-t^*|), \end{aligned} \tag{5}$$

where $\varphi\in\Phi$, and $M>0$ and $\alpha\in\langle 0, 1)$ denote given numbers.

Problem. Let B be a non-empty set of real numbers of arbitrary power. If to each $\beta\in B$ there corresponds exactly one function $f_\beta(t)$ defined on the arc L_0, then we say that on L_0 there is defined a system of functions $\{f_\beta(t)\}_{\beta\in B}$ of the power of the set B.

Assume that we are given a system of singular integral equations of arbitrary power

$$f_\beta(t) = h_\beta(t) + \int_L \frac{K_\beta[\tau, \{f_\gamma(\tau)\}_{\gamma\in B}]}{\tau - t}\,d\tau, \qquad \beta\in B, \tag{6}$$

where $\{f_\beta(t)\}_{\beta\in B}$ is a system of unknown functions, and $\{h_\beta(t)\}_{\beta\in B}$ with $\{K_\beta\}_{\beta\in B}$ are systems of given functions.

We make the following assumptions:

1° The real function $\varphi(\sigma)$ belongs to the set Φ^*.

2° The functions $h_\beta(t)$, $\beta\in B$, are of class $\mathfrak{H}(M_h, \alpha, \varphi)$, so that they satisfy the inequalities

$$\begin{aligned} |t-t^*|^\alpha |h_\beta(t)| &\leqslant M_h, \\ |t-t^*|^\alpha |h_\beta(t)-h_\beta(t_1)| &\leqslant M_h\varphi(|t-t_1|/|t-t^*|), \end{aligned} \tag{7}$$

where $M_h>0$ and $\alpha\in\langle 0, 1)$ denote given numbers.

3° The functions $K_\beta[t, \{u_\gamma\}_{\gamma\in\beta}]$, $\beta\in B$, are defined in the domain $\Omega = \{t\in L_0;\ |u_\gamma|<\infty,\ \gamma\in B\}$ and satisfy the conditions

$$|K_\beta[t, \{u_\gamma\}_{\gamma\in B}]| \leqslant M_K/|t-t^*|^\alpha + M_K \sup_{\gamma\in B}|u_\gamma|, \tag{8}$$

$$\begin{aligned} &|K_\beta[t, \{u_\gamma\}_{\gamma\in B}] - K_\beta[t_1, \{u'_\gamma\}_{\gamma\in B}]| \\ &\qquad \leqslant (M_K/|t-t^*|^\alpha)\varphi(|t-t_1|/|t-t^*|) + M_K\sup_{\gamma\in B}|u_\gamma - u'_\gamma|, \end{aligned} \tag{9}$$

where M_K is a positive constant.

Now we shall investigate whether there exists a solution of system (6). We shall apply a topological method based on the theorem of Schauder–Tichonov ([1]):

In a locally convex Hausdorff space, every continuous transformation of a non-empty convex compact-in-itself set possesses at least one fixed point.

The solution of system (6) in the class $\mathfrak{H}(M, a, \varphi)$. For each $\beta \in B$, let Λ_β be a space whose points

$$U_\beta = [f_\beta(t)] \tag{10}$$

are all complex functions $f_\beta(t)$ continuous in the set L_0 and satisfying the condition

$$\sup_{t \in L_0} [|t-t^*|^{1+a} |f_\beta(t)|] < \infty. \tag{11}$$

We define the sum of two points $U_\beta = [f_\beta(t)]$ and $V_\beta = [g_\beta(t)]$, and the product of a point with a number λ as follows

$$U_\beta + V_\beta = [f_\beta(t) + g_\beta(t)], \qquad \lambda U_\beta = [\lambda f_\beta(t)]; \tag{12}$$

the norm $\|U_\beta\|$ of the point (10) is defined by the formula

$$\|U_\beta\| = \sup_{t \in L_0} [|t-t^*|^{1+a} |f_\beta(t)|]. \tag{13}$$

The distance $\varrho(U_\beta, V_\beta)$ between points U_β, V_β is defined to be the norm of their difference.

It is known (see [3]) that each space Λ_β, $\beta \in B$ defined in this way is a locally convex Hausdorff space.

Consider the linear space $\Lambda = \prod_{\beta \in B} \Lambda_\beta$ whose points are all systems $\{f_\beta(t)\}_{\beta \in B}$ of functions continuous in the set L_0 which satisfy condition (11). Next we define a topology in this cartesian product Λ by choosing as a basis all the sets $\prod_{\beta \in B} G_\beta$, where G_β is an open set in Λ_β.

It is known that the space Λ thus defined is a locally convex Hausdorff space (see [3]).

For each $\beta \in B$, let Z_β denote the set of all points of the set Λ_β satisfying for every $t, t_1 \in L_0$ the conditions

$$\begin{gathered} |t-t^*|^a |f_\beta(t)| \leqslant \varkappa, \\ |t-t^*|^a |f_\beta(t) - f_\beta(t_1)| \leqslant \varkappa \varphi(|t-t_1|/|t-t^*|), \end{gathered} \tag{14}$$

where $\varkappa$ is some positive constant to be determined suitably later.

The set Z_β, $\beta \in B$, is non-empty, convex and compact (see [4]).

In the space Λ consider the set $Z = \prod_{\beta \in B} Z_\beta$. The points of this set are all systems of functions $\{f_\beta(t)\}_{\beta \in B}$ satisfying conditions (14).

The set Z, being the cartesian product of a family of non-empty convex sets, is itself non-empty, convex and compact (see [3] and [7]).

We subject the set Z to the transformation

$$F_\beta(t) = h_\beta(t) + \int_L \frac{K_\beta[\tau, \{f_\gamma(\tau)\}_{\gamma\in B}]}{\tau - t}\, d\tau, \qquad \beta \in B, \tag{15}$$

which with every point $\{f_\beta(t)\}_{\beta\in B}$ of the set Z associates a point $\{F_\beta(t)\}_{\beta\in B}$.

Now we find sufficient conditions in order that operation (15) transform the set Z into itself. In view of (8), (9) and (14) for every $\beta\in B$ we have

$$|t-t^*|^\alpha |K_\beta[t, \{f_\gamma(t)\}_{\gamma\in B}]| \leqslant M_K(1+\varkappa) \tag{16}$$

and

$$|t-t^*|^\alpha |K_\beta[t, \{f_\gamma(t)\}_{\gamma\in B}] - K_\beta[t_1, \{f_\gamma(t_1)\}_{\gamma\in B}]| \leqslant M_K(1+\varkappa)\varphi(|t-t_1|/|t-t^*|). \tag{17}$$

From this, applying the theorem proved in [6] to the integral on the right-hand side of formula (15) and using (7), we obtain, for all $t, t_1 \in L_0$ and every $\beta\in B$, the inequalities

$$|t-t^*|^\alpha |F_\beta(t)| \leqslant M_h + cM_K(1+\varkappa) \tag{18}$$

and

$$|t-t^*|^\alpha |F_\beta(t) - F_\beta(t_1)| \leqslant [M_h + cM_K(1+\varkappa)]\varphi(|t-t_1|/|t-t^*|), \tag{19}$$

where c is a positive constant independent of the functions h_β and K_β, $\beta\in B$, and independent of the point $\{f_\beta(t)\}_{\beta\in B}$ in the set Z.

Hence operation (15) transforms the set Z into itself if the following inequality holds:

$$M_h + cM_K(1+\varkappa) \leqslant \varkappa. \tag{20}$$

We easily observe that if

$$cM_K < 1, \tag{21}$$

then, assuming

$$\varkappa = M = \frac{M_h + cM_K}{1 - cM_K},$$

we can assert that operation (15) transforms the set Z into itself, since inequality (20) holds.

We now prove that operation (15) is continuous.

Let D be a directed set (see [7]), and $\vdash$ the relation ordering this set. We consider an arbitrary generalized sequence $\{f_\beta^{(m)}(t)\}_{\beta\in B}$, $m\in D$, of points of the set Z convergent in the generalized sense (see [2]) to the point $\{f_\beta(t)\}_{\beta\in B}$. We must prove that the generalized sequence $\{F_\beta^{(m)}(t)\}_{\beta\in B}$ of the transformed points is convergent in the generalized sense to the point $\{F_\beta(t)\}_{\beta\in B}$, which is the image of the point $\{f_\beta(t)\}_{\beta\in B}$ under transformation (15). Accordingly, we assume that for every $\varepsilon > 0$ there exists an $m_0 \epsilon D$ such that for each $\beta \epsilon B$

$$\sup_{t \epsilon L_0} [|t-t^*|^{1+\alpha} |f_\beta^{(m)}(t) - f_\beta(t)|] < \varepsilon \tag{22}$$

whenever $m \succeq m_0$. On the other hand, we are going to show that for every $\varepsilon > 0$ there exists an $m_* \epsilon D$ such that for every $\beta \epsilon B$

$$\sup_{t \epsilon L_0} [|t-t^*|^{1+\alpha} |F_\beta^{m}(t) - F_\beta(t)|] < \varepsilon \tag{23}$$

whenever $m \succeq m_*$.

In view of (15) we have for every $\beta \epsilon B$ and every $t \epsilon L_0$

$$F_\beta^{(m)}(t) - F_\beta(t) = \int_L \frac{K_\beta[\tau, \{f_\gamma^{(m)}(\tau)\}_{\gamma\epsilon B}] - K_\beta[\tau, \{f_\gamma(\tau)\}_{\gamma\epsilon B}]}{\tau - t} d\tau.$$

This implies

$$\begin{aligned} F_\beta^{(m)}(t) - F_\beta(t) \\ = \int_l \frac{K_\beta[\tau, \{f_\gamma^{(m)}(\tau)\}_{\gamma\epsilon B}] - K_\beta[t, \{f_\gamma^{(m)}(t)\}_{\gamma\epsilon B}]}{\tau - t} d\tau + \\ + \int_l \frac{K_\beta[t, \{f_\gamma(t)\}_{\gamma\epsilon B}] - K_\beta[\tau, \{f_\gamma(\tau)\}_{\gamma\epsilon B}]}{\tau - t} d\tau + \\ + (K_\beta[t, \{f_\gamma^{(m)}(t)\}_{\gamma\epsilon B}] - K_\beta[t, \{f_\gamma(t)\}_{\gamma\epsilon B}]) \int_l \frac{d\tau}{\tau - t} + \\ + \int_{L-l} \frac{K_\beta[\tau, \{f_\gamma^{(m)}(\tau)\}_{\gamma\epsilon B}] - K_\beta[\tau, \{f_\gamma(\tau)\}_{\gamma\epsilon B}]}{\tau - t} d\tau, \end{aligned}$$

where l is an arc that is a part of the arc L containing in its interior the point t. We assume that the length of the arc l does not exceed half of the arc L, and, moreover, we assume that t divides the arc l into two parts of equal length if l together with its end-points belongs to L_0. If, on the other hand, one of the end-points of the arc l coincides with the point t^*, then we assume that the length of the arc joining the points t and t^* does not exceed the length of the remaining part of the arc l.

The set Z, being the cartesian product of a family of closed sets, is itself closed. Hence the limit point $\{f_\beta(t)\}_{\beta\epsilon B}$ of the generalized sequence $\{f_\beta^{(m)}(t)\}_{\beta\epsilon B}$, $m \epsilon D$, satisfies for each $\beta \epsilon B$ conditions (14).

The integral $\int_l \frac{d\tau}{\tau - t}$ has the property that it tends to 0 if the length of the arc l, lying together with its end-points inside L, tends to 0. The modulus of this integral is unbounded similarly as $|\log|t-t^*||$ if the length of the l tends to 0 and one of the end-points of this arc coincides with t^*.

Making use of (1)–(3), (8), (9), (14) and considering the facts mentioned above, one can prove that the least upper bound of the product whose first factor is $|t-t^*|^{1+\alpha}$ and the second factor is the sum of the moduli of the first three components on the right-hand side of equality (24), is less than $\varepsilon/2$ whenever the length of the arc l (which does not depend on $t \in L_0$) is sufficiently small.

If the length of the arc l has been fixed, then the integral appearing in the fourth component of the right-hand side of equality (24) is weakly singular. From (9) and (22) we infer that there exists an $m_0 \in D$ such that the least upper bound of the product of $|t-t^*|^{1+\alpha}$ with the modulus of the fourth component on the right-hand side of equality (24) is less than $\varepsilon/2$ whenever $m \succ m_0$.

Hence according to (24) for every $\varepsilon > 0$ there exists an $m_* = m_0 \in D$ such that for each $\beta \in B$

$$\sup_{t \in L_0} [|t-t^*|^{1+\alpha} |F_\beta^{(m)}(t) - F_\beta(t)|] < \varepsilon$$

if $m \succ m_*$. This establishes the continuity of transformation (15).

Therefore we have proved that all the hypotheses of the theorem of Schauder–Tichonov hold. Consequently we can state the following theorem:

THEOREM. *If assumptions* 1°–3° *and condition* (21) *hold, then system* (6) *has at least one solution belonging to the class* $\mathfrak{H}(M, \alpha, \varphi)$.

References

[1] Р. Эдвардс, *Функциональный анализ*, Москва 1969.

[2] К. Иосида, *Функциональный анализ*, Москва 1967.

[3] K. Kuratowski, *Wstęp do teorii mnogości i topologii*, Warszawa 1966.

[4] W. Leksiński, *On some class* $\mathfrak{H}(M, \alpha, \varphi)$ *of functions defined on a non-closed arc*, Demonstratio Math. 4 (1972), p. 51–61.

[5] — and W. Żakowski, *On some classes of non-bounded functions*, ibidem 4 (1972), p. 125–131.

[6] — *A theorem on preserving the class* $\mathfrak{H}(M, \alpha, \varphi)$ *by a singular integral in the sense of Cauchy's principal value*, ibidem 2 (1973), p. 139–145.

[7] W. Żelazko, *Algebry Banacha*, Warszawa 1968.

Reçu par la Rédaction le 24. 7. 1973