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Modulus of continuity of a set function and some of its applications

by W. MATUSZEWSKA and W. ORLICZ (Poznań)

Abstract. Let $(E; \mathcal{E}, \mu)$ be a measure space, where μ is finite σ-additive and let F denote a finitely additive set function on $\mathcal{E}$ and $v_F(e)$ its variation on $e \in \mathcal{E}$. The modulus of continuity ω_F of F is defined for $u > 0$ by

$$v_F(u) = \sup v_F(e),$$

where supremum is taken with respect to $e \in \mathcal{E}$ for which $\mu(e) < u$. With the use of the modulus of continuity of a set function, the sufficient conditions for the set function $F(e) = \int f(t)\mu(dt)$ to have the property $f \in L^{*\varphi}(E, \mu)$ are given in this paper. Here φ denotes a φ-function, not necessarily convex but satisfying condition (∞_1).

1.1. In the present paper E will always denote a non-empty set, $\mathcal{E}$ denotes a σ-algebra of subsets on which a finite σ-additive measure with $\mu(E) > 0$ is defined. F or $F(:)$ designates a real-valued set function defined and additive on $\mathcal{E}$, $v_F(e)$ is its variation on a set $e \in \mathcal{E}$, and v_F denotes its total variation i.e. for $e = E$.

A *modulus of continuity* ω_F of a function F is called a function ω_F defined for $u > 0$ by

$$\sup v_F(e) = \omega_F(u),$$

where sup is taken for sets $e \in \mathcal{E}$ such that $\mu(e) < u$. From the above definition it follows that $\omega_F(u) = v_F$ for $u > \mu(E)$, $0 \leqslant \omega_F(u) \leqslant v_F$, $\lim_{u\to 0+} \omega_F(u) = 0$ if and only if F is absolutely continuous with respect to μ, ω_F is non-decreasing. If μ is a non-atomic measure, then ω_F is subadditive i.e. $\omega_F(u_1+u_2) \leqslant \omega_F(u_1)+\omega_F(u_2)$. To prove it let us assume $e \in \mathcal{E}$, $0 < \mu(e) < u_1+u_2$. Hence for some $\varepsilon > 0$, $\mu(e) = u_1+u_2-\varepsilon$. Since one of the numbers $u_1-\varepsilon/2$, $u_2-\varepsilon/2$ is positive and μ takes every intermediate value on e between 0 and $\mu(e)$, there exist subsets $e_1, e_2 \in e$ such that $e_1 \cup e_2 = e$, $e_1 \cap e_2 = \emptyset$, $\mu(e_1) = u_1-\varepsilon/2$, $\mu(e_2) = u_2-\varepsilon/2$. Consequently

$$v_F(e) = v_F(e_1)+v_F(e_2) \leqslant \omega(u_1)+\omega(u_2)$$

and the subadditivity of ω follows.

Subadditivity implies that if ω_F is finite, then $\omega_F(u_2)-\omega_F(u_1) \leqslant \omega_F(u_2-u_1)$ whenever $u_2 > u_1 > 0$, and it follows that if μ is non-atomic and $F(:)$ is absolutely continuous with respect to μ, then $\omega_F(u)$ is a continuous function.

1.2. By a *φ-function* we shall understand a continuous non-decreasing function $\varphi\colon \langle 0, \infty) \to R_+$, assuming value zero only for $u = 0$, and tending to ∞ when $u \to \infty$.

The following conditions appear often in various problems in which the φ-functions are of importance.

$$(0_1) \qquad \varphi(u)/u \to 0 \quad \text{whenever } u \to 0,$$

$$(\infty_1) \qquad \varphi(u)/u \to \infty \quad \text{whenever } u \to \infty.$$

Let us assume that φ is a convex φ-function satisfying (0_1) and (∞_1). Then we can construct a function complementary to φ defined by

$$\varphi^*(v) = \sup_{u \geqslant 0} \big(uv - \varphi(u)\big) \quad \text{for } v \geqslant 0.$$

It is an easy matter to show that φ^* is also a convex φ-function satisfying (0_1) and (∞_1) and that $(\varphi^*)^* = \varphi$. Let φ be a φ-function. $L^\varphi(E, \mu)$ will denote the set of all μ-measurable functions $x(\cdot)$ for which the integral $\int_E \varphi\big(|x(t)|\big)\mu(dt)$ is finite; and $L^{*\varphi}(E, \mu) = \{x(\cdot)\colon \lambda x \in L^\varphi(E, \mu)$ for some $\lambda > 0\}$.

If μ-equivalent functions are considered to be equal, then $L^{*\varphi}(E, \mu)$ is a vector space with the natural definitions of vector space operations.

This space can be equipped with a complete F-norm; if φ is a convex φ-function, then in $L^{*\varphi}(E, \mu)$ a complete B-norm can be defined by

$$\|x\|_{(\varphi)} = \inf\Big\{\varepsilon > 0\colon \int_E \varphi\big(|x(t)|/\varepsilon\big)\mu(dt) \leqslant 1\Big\} \qquad (\textit{Luxemburg's norm}).$$

If besides the convexity of φ we assume that φ satisfies conditions (0_1) and (∞_1), then we can introduce in $L^{*\varphi}(E, \mu)$ another norm

$$\|x\|_\varphi = \sup \int_E x(t)y(t)\mu(dt) \qquad (\textit{Orlicz's norm}),$$

where supremum is taken over all measurable functions y such that

$$\int_E \varphi^*\big(|y(t)|\big)\mu(dt) \leqslant 1.$$

It is known that the inequalities

$$\tfrac{1}{2}\|x\|_\varphi \leqslant \|x\|_{(\varphi)} \leqslant \|x\|_\varphi$$

are satisfied and the Hölder inequality

$$\Big|\int_E x(t)y(t)\mu(dt)\Big| \leqslant \|x\|_\varphi \|y\|_{(\varphi^*)}$$

holds (see [2], [4]).

2.1. Let φ be a φ-function, let $\mu(e) = 0$ imply $F(e) = 0$. Denote

$$\sigma_\pi = \sum_{i=1}^{n} \varphi\left(\frac{|F(e_i)|}{\mu(e_i)}\right)\mu(e_i),$$

where π: $e_1, e_2, \ldots, e_n$ is a decomposition of E into disjoint sets $e_i \in \mathcal{E}$. In the above formula and hereinafter in similar situations the term $|F(e_i)|/\mu(e_i)$ is to be replaced by 0 whenever $\mu(e_i) = 0$.

Riesz φ-variation of a set function F is called

$$\sup_\pi \sigma_\pi(F) = \mathrm{Var}_R(F).$$

If $\mathrm{Var}_R(F) < \infty$ and φ satisfies (∞_1), then $F(:)$ is absolutely continuous with respect to μ. This can be proved similarly as in [5], where in addition convexity of φ is assumed. For a given $\varepsilon > 0$ let us take c in such a manner that $\mathrm{Var}_R(F) \leqslant c\varepsilon$. Let for $u \geqslant u_0$, $\varphi(u) \geqslant cu$. If for $e \in \mathcal{E}$ the inequality $|F(e)| \geqslant u_0\mu(e)$ holds, then

$$\mathrm{Var}_R(F) \geqslant \varphi\left(\frac{|F(e)|}{\mu(e)}\right)\mu(e) \geqslant c|F(e)|,$$

and hence $|F(e)| \leqslant \varepsilon$. Therefore if $0 \leqslant \mu(e) < \varepsilon/u_0$, then $|F(e)| \leqslant \varepsilon$.

Moreover, note that if (∞_1) is replaced by $\liminf_{u\to\infty} \varphi(u)/u > 0$, then proceeding in the same way as above we obtain only that $F(:)$ is bounded on $\mathcal{E}$, i.e. $\mathrm{Var}_F < \infty$. This occurs in particular in the limit case $\varphi(u) = u$; then $\mathrm{Var}_R(F) = \mathrm{var}_F$.

2.2. From the above considerations it follows that, assuming (∞_1), if $\mathrm{Var}_R(F) < \infty$, then by the Radon–Nikodym Theorem $F(e) = \int_e x(t)\mu(dt)$, where $x(\cdot)$ is a μ-integrable function on E. Let φ satisfy condition (∞_1). The set of all additive set functions on $\mathcal{E}$ vanishing on sets of μ-measure 0, for which $\mathrm{Var}_R(F) < \infty$, will be denoted by $R^\varphi(E, \mu)$. By $R^{*\varphi}(E, \mu)$ we shall denote the set $R^{*\varphi}(E, \mu) = \{F\colon \lambda F \in R^\varphi(E, \mu)$ for some $\lambda > 0\}$. From the proceeding it follows that these are the sets of indefinite integrals of functions belonging to some subset of μ-integrable functions.

It is an easy matter to show that $R^{*\varphi}(E, \mu)$ is a vector space (with the natural definitions of vector space operations) and that $\mathrm{Var}_R(F)$ is in $R^{*\varphi}(E, \mu)$ a modular [1] in the sense of [4].

Consequently, in $R^{*\varphi}(E, \mu)$ a complete norm can be defined by

$$\|F\|_{(\varphi)} = \inf\{\varepsilon > 0\colon \mathrm{Var}_R(F/\varepsilon) \leqslant \varepsilon\}.$$

The proof given in [1] that $\mathrm{Var}_R(F)$ is a modular is based upon the fact that $\mathrm{Var}_R(F, a) = \mathrm{Var}_R(F_0)$, $F_0(e) = F(e \cap a)$ is a set function absolutely continuous with respect to μ.

The proof of absolute continuity given in [1] correct for a convex φ-function, is slightly erroneous in the case when φ is an arbitrary φ-function, since $\mathrm{Var}_R(F, a)$ is not generally an additive set function.

In fact the proof makes use of superadditivity of $\mathrm{Var}_R(F, a)$ only, which occurs for every φ-function.

3.1. We shall now deal with set functions of the form

$$(*) \qquad F(e) = \int_e x(t)\mu(dt), \qquad e \in \mathscr{E},$$

where $x(\cdot)$ is a μ-integrable function.

We seek sufficient conditions in order that $x \in L^{*\varphi}(E, \mu)$ or $x \in L^{\varphi}(E, \mu)$ for some φ-function, expressed with the use of the set function $(*)$. Let us note that (∞_1) implies the existence of $(*)$ for $x \in L^{*\varphi}(E, \mu)$.

3.2. Let φ be an arbitrary φ-function satisfying (∞_1). If $F \in R^{\varphi}(E, \mu)$ and is representable in the form 3.1 $(*)$, then

$$x \in L^{\varphi}(E, \mu), \qquad \text{more precisely},$$

$$(*) \qquad \int_E \varphi(|x(t)|)\mu(dt) \leqslant \mathrm{Var}_R(F).$$

To prove it let us define the following sets, for $a > 1$

$$e^+ = \{t \in E: x(t) > 0\}, \qquad e^- = \{t \in E: x(t) < 0\},$$

$$e^0 = \{t \in E; x(t) = 0\},$$

$$e_0^+ = \{t \in e^+: 1 \leqslant x(t) < a\}, \qquad e_n^+ = \{t \in e^+: a^n \leqslant x(t) < a^{n+1}\} \quad \text{for } n = 1, 2, \ldots,$$

$$e_{-0}^+ = \left\{t \in e^+: \frac{1}{a} \leqslant x(t) < 1\right\}, \qquad e_{-n}^+ = \{t \in e^+: a^{-n-1} \leqslant x(t) < a^{-n}\} \quad \text{for } n = 1, 2, \ldots$$

The following inequalities hold

$$\text{(i)} \qquad \int_{e_n^+} \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant \varphi(a^n)\mu(e_n^+) \qquad \text{for } n = 0, 1, 2, \ldots$$

and

$$\text{(i')} \qquad \int_{e_{-n}^+} \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant \varphi(a^{-n-1})\mu(e_{-n}^+) \qquad \text{for } n = 0, 1, 2, \ldots$$

Moreover, from the definition of e_n^+ and e_{-n}^+ it follows that

$$\text{(ii)} \qquad \varphi(a^n)\mu(e_n^+) \leqslant \varphi\left(\frac{\left|\int_{e_n^+} x(t)\mu(dt)\right|}{\mu(e_n^+)}\right)\mu(e_n^+) \qquad \text{for } n = 0, 1, \ldots$$

and

$$\text{(ii')}\qquad \varphi(a^{-n-1})\mu(e_{-n}^{+}) \leqslant \varphi\left(\frac{\left|\int\limits_{e_{-n}^{+}} x(t)\mu(dt)\right|}{\mu(e_{-n}^{+})}\right)\mu(e_{-n}^{+}) \quad \text{for } n = 0, 1, \ldots$$

Taking into account that the sets e_m^+ $(m = \pm 0, \pm 1, \ldots)$ are disjoint we get from (i)–(ii') the inequality

$$\text{(iii)}\qquad \int\limits_{e^+\cup e^0} \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant \sum_m \varphi\left(\frac{\left|\int\limits_{e_m^+} x(t)\mu(dt)\right|}{\mu(e_m^+)}\right)\mu(e_m^+),$$

where the summation on the right-hand side is for $m = \pm 0, \pm 1, \ldots$ Define sets e_m^-, $m = 0, \pm 1, \ldots$ similarly as e_m^+ replacing in their definitions $x(t)$ by $|x(t)|$ and e^+ by e^-, then estimating as before we get

$$\text{(iii')}\qquad \int\limits_{e^-} \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant \sum_m \varphi\left(\frac{\left|\int\limits_{e_m^-} x(t)\mu(dt)\right|}{\mu(e_m^-)}\right)\mu(e_m^-),$$

where summation on the right-hand side is for $m = 0, \pm 1, \ldots$ Taking into account that all sets e_n^+, e_m^- are pairwise disjoint and adding the inequalities (iii), (iii') we obtain

$$\int\limits_E \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant \mathrm{Var}_R(F),$$

and hence, applying the Fatou's lemma for $a \to 1$ we get (*).

If φ is a convex φ-function, then by the Jensen inequality, it can be shown that

$$\mathrm{Var}_R(F) \leqslant \int\limits_E \varphi\big(|x(t)|\big)\mu(dt).$$

In this case therefore, finiteness of the Riesz φ-variation is the necessary and sufficient condition for $x \in L^{\varphi}(E, \mu)$, and, moreover,

$$\int\limits_E \varphi\big(|x(t)|\big)\mu(dt) = \mathrm{Var}_R(F).$$

Using the notion of the modulus of continuity of a set function we can formulate another sufficient condition for $x \in L^{*\varphi}(E, \mu)$.

3.3. Let φ be an arbitrary φ-function satisfying (∞_1). Assume that $F(:)$ has the modulus of continuity $\omega = \omega_F$ and that there exists a function ω_0 defined for $u > 0$, with the following properties:

$$\text{(a)}\qquad \omega(u) \leqslant \omega_0(u) \quad \text{for } u > 0;$$

$$\text{(b)}\qquad \gamma(u) = \omega_0(u)/u \quad \text{is non-increasing for } u > 0,$$

$$\gamma(u) \to \infty \quad \text{as } u \to 0, \qquad \gamma(u) \to 0 \quad \text{as } u \to \infty;$$

(c) denote a generalized function inverse to γ by γ_{-1} i.e. denote $\gamma_{-1}(u) = \sup\{t: \gamma(t) \geqslant u\}$, then for some $a > 1$ the series

$$s = \sum_{n=0}^{\infty} \varphi(a^n)\gamma_{-1}(a^n)$$

converges.

Under the above assumptions $x \in L^{*\varphi}(E, \mu)$.

First observe that

$$\mathrm{var}_F(e) = \int_e |x(t)|\,\mu(dt).$$

With the notations of 3.2 denote $e_n = e_n^+ \cup e_n^-$ for $n = 0, 1, 2, \ldots$ From inequality 3.2 (i) for e_n^+ and similarly for e_n^- we obtain

$$\sum_0^{\infty} \int_{e_n} \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant \sum_0^{\infty} \varphi(a^n)\mu(e_n).$$

The inequality

$$a^n \leqslant \frac{\mathrm{var}_F(e_n)}{\mu(e_n)} \leqslant \frac{\omega_0(\mu(e_n))}{\mu(e_n)} = \gamma(\mu(e_n))$$

holds also for $\mu(e_n) > 0$.

But γ_{-1} is non-increasing such that $\gamma_{-1}(\gamma(u)) \geqslant u$, therefore $\gamma_{-1}(a^n) \geqslant \mu(e_n)$ hence

$$\sum_{n=0}^{\infty} \varphi(a^n)\mu(e_n) \leqslant s.$$

Which means that $\int_E \varphi\left(\frac{|x(t)|}{a}\right)\mu(dt) \leqslant s + \varphi\left(\frac{1}{a}\right)\mu(E)$, consequently $x \in L^{*\varphi}(E, \mu)$.

3.4. We shall now consider a criterion given in 3.3 assuming that a φ-function φ satisfying (0_1) and (∞_1) is given and the function $x(\cdot)$ in the integral representation $F(e) = \int_E x(t)\mu(dt)$ belongs to $L^{\varphi}(E, \mu)$. Assume that besides φ, a convex φ-function ψ also satisfying (0_1) and (∞_1) is given. In order to present some condition on modulus of continuity $\omega_F(u)$ we shall assume for simplicity that $\|x\|_{(\varphi)} \leqslant 1$ in the first case and $\|x\|_{\varphi} \leqslant 1$ in the second, and apply the Hölder inequality

$$\mathrm{var}_F(e) = \int_e |x(t)|\,\mu(dt) \leqslant \|x\|_{(\varphi)}\|\chi_e\|_{\varphi^*} \leqslant \|\chi_e\|_{\varphi^*},$$

$$\mathrm{var}_F(e) = \int_e |x(t)|\,\mu(dt) \leqslant \|x\|_{\varphi}\|\chi_e\|_{(\varphi^*)} \leqslant \|\chi_e\|_{(\varphi^*)}.$$

Here χ_e denotes the characteristic function of e.
But as is known [2] for $\mu(e) > 0$

$$\|\chi_e\|_{\psi*} = \mu(e)\psi_{-1}\left(\frac{1}{\mu(e)}\right), \quad \|\chi_e\|_{(\psi*)} = \frac{1}{\psi^*_{-1}\left(\frac{1}{\mu(e)}\right)}.$$

Hence we get the following estimations of moduli of continuity

$$\omega(u) \leqslant \omega_1(u), \quad \text{where } \omega_1(u) = u\psi_{-1}\left(\frac{1}{u}\right) \quad \text{for } u > 0$$

and

$$\omega(u) \leqslant \omega_2(u), \quad \text{where } \omega_2(u) = \frac{1}{\psi^*_{-1}\left(\frac{1}{u}\right)} \quad \text{for } u > 0.$$

Both these majorants are closely connected to each other, namely

$$\omega_2(u) \leqslant \omega_1(u) \leqslant 2\omega_2(u),$$

since the following inequalities hold

$$u \leqslant \psi_{-1}(u)\varphi^*_{-1}(u) \leqslant 2u.$$

With no regard to the origin of the majorants ω_1, ω_2 we can use them as majorants of the modulus of continuity in the criterion of 3.3 (the assumption of convexity was needed only to establish the formulae for majorants). To this end let us note that if

$$\gamma(u) = \frac{\omega_1(u)}{u} = \psi_{-1}\left(\frac{1}{u}\right), \quad \text{then } \gamma_{-1}(u) = \frac{1}{\psi(u)}.$$

We can therefore formulate the following criterion: If for some φ-function φ satisfying (∞_1) the series

$$(+) \qquad \sum_{n=1}^{\infty}\varphi(a^n)[\psi(a^n)]^{-1}$$

converges for some $a > 1$, and

$$\omega_F(u) \leqslant u\psi_{-1}\left(\frac{1}{u}\right) \quad \text{for } u > 0,$$

where ψ denotes a φ-function, $\psi(0) = 0$, then $x \in L^{*\varphi}(E, \mu)$.

3.4.1. Under the assumption that $\varphi(u)/u\psi(u)\downarrow 0$ as $u\to\infty$, the convergence of the series $(+)$ for each a is equivalent to the convergence of the integral

$$(++) \qquad \int_1^\infty \frac{\varphi(u)}{u\psi(u)}\,du.$$

3.5. Let $\varphi(u) = u^a$, $a > 1$ and let $\beta > a$, $\psi(u) = u^\beta$. As a majorant we take $\omega_0(u) = u\psi_{-1}\left(\frac{1}{u}\right) = u^{1/\beta'}$, where β' is the exponent conjugated to β, i.e. $1/\beta + 1/\beta' = 1$.

The integral $(++)$ is convergent and the criterion 3.4 leads us to the following conclusion:

If $\omega_F(u) \leqslant u^{1/\beta'}$, then $x \in L^a(E, \mu)$. A more general condition $\omega_F(u) \leqslant ku^{1/\beta'}$ obviously leads to the same conclusion. Let us note, moreover, that if $x \in L^a(E, \mu)$, then from the Hölder inequality we get the estimation $\omega_F(u) \leqslant lu^{1/a'}$, where $1/a + 1/a' = 1$. Since $\beta' < a'$, then $u^{1/\beta'} < u^{1/a'}$ for $0 < u \leqslant 1$ which proves that $\omega_F(u) \leqslant lu^{1/a'}$ gives a weaker estimation of the modulus of continuity than $\omega_F(u) \leqslant ku^{1/\beta'}$.

3.5′. Let us now consider the characterization of the integral representation $F(e) = \int_e x(t)\mu(dt)$, where $x \in L^{a-0}(E, \mu) = \bigcap_{1<\beta<a} L^\beta(E, \mu)$, $a > 1$.

To give the necessary and sufficient conditions for this class of set functions $F(e)$ it is possible with the use of modulus of continuity, whereas to characterize this in the case $x \in L^a(E, \mu)$ the corresponding Riesz variations were needed.

3.5.1. The necessary and sufficient condition for $x \in L^{a-0}(E, \mu)$, $a > 1$ is that for every β, $1 < \beta < a$ there exists a constant k_β, such that $\omega_F(u) \leqslant k_\beta u^{1/\beta'}$.

To prove it let us take an increasing sequence of exponents $1 < \beta_k < a$, $\beta_k \to a$. Consider a pair of these exponents, then by 3.4.1 the condition $\omega_F(u) \leqslant k_{\beta+1} u^{\beta_{k+1}}$ for $k = 1, 2, \ldots$ implies that $x \in L^{\beta_k}(E, \mu)$. Since $L^{a-0}(E, \mu) = \bigcap_{k=1}^{\infty} L^{\beta_k}(E, \mu)$, $x \in L^{a-0}(E, \mu)$.

If $x \in L^{a-0}(E, \mu)$, then for every $1 < \beta < a$, $x \in L^\beta(E, \mu)$ hence $\omega_F(u) \leqslant k_\beta u^{1/\beta'}$. The above proven theorem was given in a slightly different form by J. Marcinkiewicz [3] for the particular case of $E = \langle 0, 1\rangle$ and μ being a Lebesgue measure.

3.5.2. Given a sequence of convex φ-functions $\varphi_1, \varphi_2, \ldots$ satisfying conditions (0_1) and (∞_1), with the following property:

The series

$$\sum_{n=1}^{\infty} \varphi_k(a_k^n)\big(\varphi_{k+1}(a_k^n)\big)^{-1}, \qquad a_k > 1,$$

converges for $k = 1, 2, \ldots$

The necessary and sufficient condition for $x \in \bigcap_{k=1}^{\infty} L^{\varphi_k}(E, \mu)$ is that for every k there exists a constant $l_k > 0$ such that

$$(+) \qquad \omega_F(u) \leqslant l_k u (\varphi_k)_{-1}\left(\frac{1}{u}\right) \quad \text{for } k = 1, 2, \ldots$$

The necessity of $(+)$ follows from the Hölder inequality as in 3.4. Sufficiency is a consequence of the criterion given in 3.4.

References

[1] J. Albrycht and W. Orlicz, *A note on modular spaces*. II, Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys. 10 (1962), p. 99–106.

[2] M. A. Krasnosel'skii and Y. B. Rutickii, *Convex functions and Orlicz spaces*, Groningen 1961.

[3] J. Marcinkiewicz and S. Kaczmarz, *Sur les multiplicateurs des séries orthogonales*, Studia Math. 7 (1938), p. 73–81.

[4] J. Musielak and W. Orlicz, *On modulared spaces*, ibidem 18 (1959), p. 49–65.

[5] W. Orlicz, *Absolute continuity of set functions with respect to a finitely subadditive measure*, Comm. Math. 14 (1970), p. 101–118.

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