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A Cantor regular set which does not have Markov's property

by W. PLEŚNIAK (Kraków)

Zdzisław Opial in memoriam

Abstract. We construct a Cantor type compact subset of R which is regular in the sense of Green's function of $C\setminus E$ but fails to be "good" regarding Markov's inequality for the derivatives of polynomials.

Introduction. In 1889, Markov [5] proved that for every polynomial p of degree n

$$|p'(x)| \leqslant n^2 \sup_{[-1,1]} |p(x)| \quad \text{for } x \in [-1, 1];$$

this solved a problem posed two years earlier by Mendeleev. During the last century, Markov's inequality has appeared to be of particular significance in approximation theory and its applications. In the one-dimensional case, an excellent introduction to the subject is given in monograph [11], while the series of papers [6]–[10] seems to be an adequate reference regarding the theory of subsets of C^N with the following *Markov property*:

(M) There exist positive constants M and r such that, for each polynomial p: $C^N \to C$ and each multiindex $\alpha \in Z_+^N$,

$$\|D^\alpha p\|_E \leqslant M(\deg p)^{r|\alpha|} \|p\|_E;$$

$\|h\|_E$ denoting the supremum norm of h on E. By Cauchy's integral formula, in order that a compact subset E of C^N have property (M) it is sufficient that *Siciak's extremal function*

$$\Phi_E(x) := \sup\{|p(x)|^{1/\deg p}: p \in P(E)\}, \qquad x \in C^N,$$

where $P(E)$ is the family of non-constant polynomials with $\|p\|_E \leqslant 1$, be Hölder continuous in the sense that

(HCP) $$\Phi_E(x) \leqslant 1 + M\delta^m \quad \text{as} \quad \operatorname{dist}(x, E) \leqslant \delta \leqslant 1,$$

with some positive constants M and m independent of δ.

It was proved in [6] that the collection of (HCP) sets contains a large class of *uniformly polynomially cuspidal sets*, which in particular includes all fat *subanalytic compact subsets* of $\boldsymbol{R}^N$. (Throughout the paper $\boldsymbol{R}^N$ is regarded as the set $\{(z_1, \ldots, z_N) \in \boldsymbol{C}^N\colon \operatorname{Im} z_j = 0,\ j = 1, \ldots, N\}$.) Siciak [12] seems to be the first to construct an (HCP) set which is not uniformly polynomially cuspidal; his example is a Cantor type subset of $\boldsymbol{R}$ obtained from $[0, 1]$ by deleting in each step of the construction sufficiently small subintervals.

So far, it is not known whether the Cantor ternary set has property (M). In this paper, we construct a Cantor type compact subset E of $\boldsymbol{R}$ which is *regular in the sense of the Green's function* of $\boldsymbol{C}\backslash E$ with a pole at ∞ but fails to have Markov's property.

By a result of Baouendi and Goulaouic [1], if E is determining for germs of analytic functions of E, the regularity of E is equivalent to the *Bernstein–Walsh characterization of analytic functions* by means of polynomial approximation. Similarly, by [10], Theorem 3.3, in the class of C^∞ determining sets, Markov's inequality (M) is equivalent to *Bernstein's characterization of C^∞ functions*: f is C^∞ on E iff the sequence of the distances in the supremum norm on E of f from the spaces of polynomials of degree at most k $(k = 0, 1, \ldots)$ is rapidly decreasing. Consequently, we have constructed a compact subset E of $\boldsymbol{R}$ for which the Bernstein–Walsh theorem for analytic functions holds and yet there exist continuous functions on E which are rapidly approximable by polynomials but cannot be extended to C^∞ functions on $\boldsymbol{R}$. We note that in $\boldsymbol{R}^2$ such an example was first constructed by Baouendi and Goulaouic in [1].

The construction of the set E. Let E be a Cantor set constructed as follows. Given a sequence $\{l_n\}$ of positive numbers with $l_{n+1} < l_n/2$ $(n = 1, 2, \ldots)$ and $l_1 < 1/2$, we put $E_1 = I_1^1 \cup I_2^1$, where $I_1^1 = [0, l_1]$ and $I_2^1 = [1-l_1, 1]$. If $E_n = I_1^n \cup \ldots \cup I_{2^n}^n$ is already constructed, with each I_j^n being a closed interval of length $I_j^n = l_n$, then E_{n+1} is obtained by deleting an open concentric subinterval of length $l_n - 2l_{n+1}$ from each I_j^n. We now put

$$E = \bigcap_{n=1}^{\infty} E_n.$$

We have

PROPOSITION 1. *If*

(i) $$\limsup_{n\to\infty} (\log l_n^{-1})/n = \infty,$$

then the set E does not satisfy (M).

Proof. Let $\{a_0, a_1, \ldots\}$ be a *Leja sequence* of points of E, i.e., $a_0 \in E$ is arbitrarily chosen and then, for $n = 1, 2, \ldots$, we choose $a_n \in E$ so that

$$\max_{x\in E} |(x-a_0) \ldots (x-a_{n-1})| = |(a_n - a_0) \ldots (a_n - a_{n-1})|.$$

Put

$$p_n(x) = [(x-a_0)\dots(x-a_{n-1})]/[(a_n-a_0)\dots(a_n-a_{n-1})], \qquad n = 1, 2, \dots$$

Then obviously $\|p_n\|_E = p_n(a_n) = 1$ and $p_n(a_k) = 0$ for $k < n$. Observe that for each n we can find two indices k_n and m_n such that $0 \leqslant k_n < m_n \leqslant 2^n$ and the points a_{k_n} and a_{m_n} belong to the same interval I_j^n of E_n. For the polynomial p_{m_n}, we then have $\deg p_{m_n} \leqslant 2^n$ and

(1) $$|p'_{m_n}(y_n)| = |p_{m_n}(a_{m_n}) - p_{m_n}(a_{k_n})|/|a_{m_n} - a_{k_n}| = 1/|a_{m_n} - a_{k_n}| \geqslant 1/l_n$$

for some point $y_n \in [a_{k_n}, a_{m_n}]$ (or else $y_n \in [a_{m_n}, a_{k_n}]$), whence in particular

(2) $$\operatorname{dist}(y_n, E) \leqslant 1/l_n.$$

By [10], Theorem 3.3, property (M) is equivalent to the following property:

(P) There exist constants $M_1 > 0$ and $s > 0$ such that for every polynomial p of degree at most n ($n = 1, 2, \dots$) we have

$$|p(x)| \leqslant M_1 \|p\|_E \quad \text{when } \operatorname{dist}(x, E) \leqslant 1/n^s.$$

Now suppose that E has property (M). Then by (1), (2) and (P) we would have

$$1/l_n < |p'_{m_n}(y_n)| \leqslant M_1 \|p'_{m_n}\|_E \leqslant M_1 M 2^{nr},$$

which contradicts (i).

PROPOSITION 2. *The following statements are equivalent*:

(ii) $$\sum_{n=1}^{\infty} 2^{-n}\log(1/l_n) < \infty;$$

(iii) *E has positive logarithmic capacity $c(E)$*;

(iv) *E is regular in the sense of the Green's function of $C \setminus E$ with a pole at ∞.*

Proof. The equivalence (ii)$\Leftrightarrow$(iii) is well known (see e.g. [2], Chapter IV, Theorem 3). Thus, it is sufficient to show that (iii)$\Rightarrow$(iv). To this end we shall use the following

WIENER'S CRITERION (see e.g. [4], Theorem 5.6). *For a sequence $\{\varepsilon_n\}$ of positive numbers with $1 < \alpha \leqslant \varepsilon_n/\varepsilon_{n+1} \leqslant \beta < +\infty$ ($n = 1, 2, \dots$), we put $K_n := E \cap \{x \in C: \varepsilon_{n+1} \leqslant |x-a| \leqslant \varepsilon_n\}$, where $a \in E$ is fixed. Then E is regular at a iff*

$$\sum_{n=l}^{\infty} \log \varepsilon_n / \log c(K_n) = +\infty.$$

Let us pass to the proof of implication (iii)$\Rightarrow$(iv). Fix $a \in E$. Then for each n there is exactly one $j_0 \in \{1, \dots, 2^n\}$ such that $a \in I_{j_0}^n$. We put $I_n := I_{j_0}^n$ and $J_n := (I_{n-1} \setminus I_n) \cap E_n$, $n = 1, 2, \dots$, where we set $I_0 := [0, 1]$. Observe that $a \notin J_n$ and $J_n = I_{j_1}^n$ for some $j_1 \in \{1, \dots, 2^n\}$. We now consider two cases.

(α) There is a constant M such that $l_n/l_{n+1} \leqslant M$ for $n = 1, 2, \ldots$ Define

$$\delta_n := \sup\{|a-y|:\ y \in I_n\}.$$

Clearly,

$$l_n - l_{n+1} \leqslant \delta_n \leqslant l_n. \tag{3}$$

Put $\varepsilon_n := \delta_{2n-1}$, $n = 1, 2, \ldots$ Then by (3) we have

$$2 < 1/(l_{2n+1}/l_{2n}) \leqslant (l_{2n-1} - l_{2n})/l_{2n+1} \leqslant \varepsilon_n/\varepsilon_{n+1}$$
$$= \delta_{2n-1}/\delta_{2n+1} \leqslant l_{2n-1}/(l_{2n+1} - l_{2n-1}) \leqslant M^3.$$

Furthermore, we have

$$E \cap J_{2n+1} \subset K_n \quad \text{as} \quad \operatorname{dist}(a, J_{2n}) < \delta_{2n+1},$$

or else

$$E \cap J_{2n} \subset K_n \quad \text{as} \quad \operatorname{dist}(a, J_{2n}) \geqslant \delta_{2n+1}.$$

It follows from the construction of E that for each n and $j \in \{1, \ldots, 2^n\}$, the set $E \cap I_j^n$ can be obtained from E by translation and a homothety of ratio l_n. Hence $c(E \cap I_j^n) = l_n c(E)$, and we get

$$\sum_{n=1}^{\infty} \log\varepsilon_n/\log c(K_n) \geqslant \sum_{n=1}^{\infty} \log l_{2n-1}/(\log c(E) + \log l_{2n+1}) = +\infty.$$

The result now follows from Wiener's criterion.

(β) Suppose that $\limsup_{n\to\infty} l_n/l_{n+1} = +\infty$. It is seen from the construction of E that we may assume $l_n/l_{n+1} \uparrow +\infty$. Observe that for each $n \in N$ there is exactly one $k_n \in N$ such that $k_n - 1 < \log_{2^{-1}} l_n \leqslant k_n$. We now put

$$m_n := \begin{cases} k_n, & \text{as} \quad k_n - \log_{2^{-1}} l_n \leqslant \log_{2^{-1}} l_n - (k_n - 1), \\ k_n - 1 & \text{as} \quad k_n - \log_{2^{-1}} l_n > \log_{2^{-1}} l_n - (k_n - 1). \end{cases}$$

Then we have

$$|m_n - \log_{2^{-1}} l_n| \leqslant 1/2. \tag{4}$$

Futhermore, observe that if $m \in N$ and $|m - m_n| = 1$ then there is no $p \in N$ such that $m = m_p$. Now define

$$\varepsilon_n := \begin{cases} 2^{-n} & \text{if } n \notin \{m_1, m_2, \ldots\}, \\ \delta_k & \text{if } n = m_k \text{ for some } k. \end{cases}$$

We want to show that there are constants α, β such that $l < \alpha \leqslant \varepsilon_n/\varepsilon_{n+1} \leqslant \beta$ for almost all n. To this end, consider three cases that can occur:

(a) Neither n nor $n+1$ belongs to $m_1, m_2, \ldots$ Then evidently $\varepsilon_n/\varepsilon_{n+1} = 2$.

(b) $n = m_k$ for some k. Then $n+1 \notin \{m_1, m_2, \ldots\}$. From (4) we get

$$\sqrt{2}\, l_n \leqslant 2^{-m_n+1} \leqslant 2\sqrt{2}\, l_n. \tag{5}$$

Hence, by (3),

$$7/5 < (l_k - l_{k+1})/(l_k/\sqrt{2}) \leqslant \varepsilon_n/\varepsilon_{n+1} = \delta_k/2^{-m_k-1} \leqslant l_k/(l_k/2\sqrt{2}) = 2\sqrt{2}$$

for n sufficiently large, say, $n \geqslant n_0$.

(c) $n+1 = m_k$ for some k. Then by (3) and (5) we get

$$\sqrt{2} = \sqrt{2}l_k/l_k \leqslant \varepsilon_n/\varepsilon_{n+1} = 2^{-m_k+1}/\delta_k \leqslant 2\sqrt{2}l_k/(l_k - l_{k+1}) < 4\sqrt{2}.$$

Now, if n is sufficiently large we have $K_{m_n} \supset E \cap J_{n+1}$. For, if $x \in J_{n+1}$, then $|x-a| \leqslant \delta_n = \varepsilon_{m_n}$, and by (5),

$$|x-a| \geqslant \delta_n - l_{n+1} \geqslant l_n - 2l_{n+1} \geqslant l_n/\sqrt{2} \geqslant 2^{-m_n-1} = \varepsilon_{m_n+1} \quad \text{for } n \geqslant n_1$$

with some $n_1 \geqslant n_0$. Hence $c(K_{m_n}) \geqslant c(E \cap J_{n+1}) = l_{n+1}c(E)$, whence we get

$$\sum_{k=1}^{\infty} \log\varepsilon_k/\log c(K_k) \geqslant \sum_{n=n_1}^{\infty} \log\varepsilon_{m_n}/\log c(K_{m_n}) \geqslant \sum_{n=n_1}^{\infty} \log\delta_n/\log[l_{n+1}c(E)]$$
$$\geqslant \sum_{n=n_1}^{\infty} \log l_n/(\log c(E) + \log l_{n+1}) = +\infty,$$

since, by (ii), $(\log l_n/\log l_{n+1})$ cannot tend to 0. Thus, by Wiener's criterion, E is regular at a.

Remark. I wish to thank Professor Siciak for calling my attention to the equivalence (ii)$\Leftrightarrow$(iii). In order to prove that (ii) implies (iii) we can also repeat a potential-theoretic argument used e.g. by Diederich and Fornaess in [3].

The equivalence (iii)$\Leftrightarrow$(iv) seems to be known. We do not know, however, any bibliography to refer to.

COROLLARY. *Assume that the sequence* $\{l_n\}$ *satisfies* (i) *and* (ii) (*take e.g,* $l_n = 1/(n+2)!$). *Then the corresponding set E is regular but does not have Markov's property* (M).

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DEPARTMENT OF MATHEMATICS, JAGIELLONIAN UNIVERSITY
KRAKÓW, POLAND

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