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Dynamical systems with multiplicative perturbations

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Abstract. We give sufficient conditions for asymptotical stability of a Markov operator governing the evolution of densities corresponding to a dynamical system with multiplicative perturbations.

1. Introduction. Consider a random dynamical system whose time evolution is given by the recurrence formula·

$$x_{n+1} = S(x_n)\xi_n \quad \text{for } n = 0, 1, \ldots \tag{1}$$

In (1), S is a given transformation of $\boldsymbol{R}_+$ (or $[0, 1]$) into itself, and ξ_n are independent random variables with the same density. Assuming that the density distribution function f_n of x_n exists, under quite general conditions concerning S and ξ_n, it is easy to find the density f_{n+1} of x_{n+1}. The relation between f_n and f_{n+1} is given by the formula

$$f_{n+1} = Pf_n,$$

where P is a Markov operator.

In the last few years, dynamical systems with stochastic perturbations were intensively studied ([3]–[5], [10], [11]). From the applied point of view, the asymptotical stability of the process. The purpose of the present paper is example, the stochastical model of the brightness of Milky-Way proposed by Chandrasekhar and Münch [1], [2] leads to the same type of kernel operators P as our multiplicative model described by formula (17). Moreover, the multiplicative perturbations in a natural way appear in biological systems when the stimulating factor is described by a deterministic transformation and the restraining factor is stochastic [7].

In studing evolution of densities one of the most interesting problems is the asymptotical stability of the process. The purpose of the present paper is to show sufficient conditions for asymptotical stability of a Markov operator P governing the evolution of the densities of x_n.

The organization of the paper is as follows. In Section 2, we consider the case where the transformation S is defined on the entire halfline $\boldsymbol{R}_+$. The second case where the transformation S is defined only on the interval $[0, 1]$ is studied in Section 3. Section 4 contains some applications and remarks.

2. Notation and results. Let (X, Σ, μ) be a measure space with a nonnegative σ-finite measure μ. Write $L^1(X) = L^1(X, \Sigma, \mu)$. A linear operator P: $L^1(X) \to L^1(X)$ will be called a *Markov operator* if it satisfies the following two conditions:

(a) $$Pf \geqslant 0 \qquad \text{for } f \geqslant 0,\ f \in L^1(X),$$

(b) $$\|Pf\| = \|f\| \qquad \text{for } f \geqslant 0,\ f \in L^1(X),$$

where $\|\cdot\|$ stands for the norm in $L^1(X)$. By $D(X) = D(X, \Sigma, \mu)$ we shall denote the set of all nonnegative elements $f \in L^1(X)$ such that $\|f\| = 1$.

We say that a Markov operator is *asymptotically stable* if there exists a unique $f_* \in D(X)$ such that $Pf_* = f_*$ and

$$\lim_{n \to +\infty} \|P^n f - f_*\| = 0 \qquad \text{for every } f \in D(X).$$

Let S be a nonsingular transformation from the halfline $\boldsymbol{R}_+ = [0, +\infty)$ into itself. Nonsingularity means that $m(S^{-1}(A)) = 0$ whenever $m(A) = 0$ (m is the Lebesgue measure on $\boldsymbol{R}_+$). Assume that ξ_n are independent random variables satisfying $\xi_n \geqslant 0$, $n = 0, 1, \ldots$, with probability one and having the same bounded density g. Finally we assume that the initial condition x_0 is independent of the sequence of perturbation $\{\xi_n\}$.

In order to calculate f_{n+1} from f_n let us denote by h an arbitrary real valued bounded measurable function defined on $\boldsymbol{R}_+$. The mean value $E(h(x_{n+1}))$ of $h(x_{n+1})$ is evidently given by

$$E(h(x_{n+1})) = \int_0^{+\infty} h(x) f_{n+1}(x)\, dx.$$

Since $x_{n+1} = S(x_n)\xi_n$ and the random variables x_n and ξ_n are independent, we also have

$$E(h(x_{n+1})) = \int_0^{+\infty} \int_0^{+\infty} h(S(y)z) f_n(y) g(z)\, dy\, dz.$$

Let P_S be the Frobenius–Perron operator corresponding to S (cf. [8]). Setting $h_z(y) = h(yz)$, we obtain

$$\int_0^{+\infty} h(S(y)z) f_n(y)\, dy = \int_0^{+\infty} h_z(S(y)) f_n(y)\, dy = \int_0^{+\infty} P_S f_n(y) h_z(y)\, dy$$
$$= \int_0^{+\infty} P_S f_n(y) h(yz)\, dy.$$

Using this equality, it is easy to calculate that

$$E\big(h(x_{n+1})\big) = \int_0^{+\infty}\int_0^{+\infty} P_S f_n(y)\, h(yz)\, g(z)\, dy\, dz$$

$$= \int_0^{+\infty} P_S f_n(y) \int_0^{+\infty} h(x)\, g\left(\frac{x}{y}\right)\frac{1}{y}\, dx\, dy$$

$$= \int_0^{+\infty} h(x) \int_0^{+\infty} P_S f_n(y)\, g\left(\frac{x}{y}\right)\frac{1}{y}\, dy\, dx.$$

Since h is arbitrary, this implies

(2) $$f_{n+1}(x) = \int_0^{+\infty} P_S f_n(y)\, g(x/y)\frac{1}{y}\, dy.$$

Thus, given an arbitrary initial density f_0, the evolution of densities corresponding to the system (1) is described by the sequence of iterates $\{P^n f_0\}$, where

(3) $$Pf(x) = \int_0^{+\infty} P_S f(y)\, g\left(\frac{x}{y}\right)\frac{1}{y}\, dy = \int_0^{+\infty} f(y)\, g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\, dy.$$

Our first step in the study of the sequence $\{P^n f_0\}$ is to show that the operator P is weakly constrictive. By definition, P is weakly constructive if there exists a weakly precompact set $\mathscr{F} \subset L^1(X)$ such that

(4) $$\lim_{n\to+\infty} \varrho(P^n f, \mathscr{F}) = 0 \quad \text{for } f \in D(X),$$

where $\varrho(f, \mathscr{F})$ denotes the distance between the function f and the set $\mathscr{F}$ in $L^1(X)$ norm.

We shall also use the following two lemmas:

LEMMA 1. *Let $f \in L^1([0, a))$ $(a \leqslant +\infty)$ be given by the equality*

$$f(x) = x^r \omega(x), \qquad x \in [0, a),$$

where ω is a nonnegative, nonincreasing function and r is a nonnegative constant. Then

$$f(x) \leqslant \|f\|(r+1)/x \quad \textit{for } x \in (0, a).$$

Proof of Lemma 1 (cf. [9]):

$$\|f\| = \int_0^a f(y)\, dy = \int_0^a y^r \omega(y)\, dy \geqslant \int_0^x y^r \omega(y)\, dy$$

$$\geqslant \int_0^x y^r \omega(x)\, dy = \frac{\omega(x)\, x^{r+1}}{r+1} = \frac{xf(x)}{r+1}.$$

LEMMA 2. *Let P be a weakly constrictive Markov operator. Assume that there is a set $A \subset X$ of nonzero measure, $\mu(A) > 0$, with the property that for every $f \in D(X)$ there is an integer $n_1(f)$ such that*

$$P^n f(x) > 0$$

for almost all $x \in A$ and all $n \geqslant n_1(f)$. Then P is asymptotically stable.

The proof of this lemma may be found in [6].

Now we are ready to state our first result of this section.

THEOREM 1. *Assume that the transformation $S\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ and the density g satisfy the following conditions:*

(i) *S is differentiable increasing with continuous $dS^{-1}(x)/dx$ and $S(0) = 0$;*

(ii) *$S(x) \leqslant \eta x + \beta$ for $x \geqslant M$, where η, β and M are nonnegative constants and*

$$\eta \int_0^{+\infty} xg(x)\,dx < 1;$$

(iii) *g is bounded and there exists a function $h \in L^1(\boldsymbol{R}_+)$ such that*

$$0 < g(x) \leqslant xh(x) \quad \textit{for } x \in (0, +\infty) \tag{5}$$

and

$$\|h\| < S'(0). \tag{6}$$

Then the operator P given by equation (3) *is asymptotically stable.*

Proof of Theorem 1. Define

$$E(f) = \int_0^{+\infty} xf(x)\,dx$$

and consider the sequence $\{E(P^n f)\}$ for an $f \in D(\boldsymbol{R}_+)$. From equation (3) it follows immediately that

$$\begin{aligned} E(P^{n+1} f) &= \int_0^{+\infty}\int_0^{+\infty} xP^n f(y)\, g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy\,dx \\ &= \int_0^{+\infty} P^n f(y) S(y) \int_0^{+\infty} zg(z)\,dz\,dy. \end{aligned}$$

From assumption (ii) it follows that

$$S(y) \leqslant \max_{0 \leqslant x \leqslant M} S(x) + \eta y + \beta \quad \text{for } y \in \boldsymbol{R}_+.$$

Thus

$$E(P^{n+1}f) \leqslant E(g)\eta \int_0^{+\infty} yP^n f(y)\,dy + (\beta + \max_{0 \leqslant x \leqslant M} S(x))E(g)$$
$$= \eta E(g)E(P^n f) + (\beta + \max_{0 \leqslant x \leqslant M} S(x))E(g).$$

As a consequence

$$E(P^n f) \leqslant \frac{(\beta + \max_{0 \leqslant x \leqslant M} S(x))E(g)}{1-\eta E(g)} + \eta^n E(g)^n E(f).$$

Choose an arbitrary $K_* > (\beta + \max_{0 \leqslant x \leqslant M} S(x))E(g)/(1-\eta E(g))$. If $E(f) < +\infty$ then there is an integer $n_0 = n_0(f)$ such that

(7) $$E(P^n f) \leqslant K_* \quad \text{for } n \geqslant n_0(f).$$

Set $L = \sup_{x \in \boldsymbol{R}_+} g(x)$ and $\lambda = S'(0)$. Since $\lambda > \|h\|$, we may choose $\varepsilon > 0$ so small that

$$\|h\|/(\lambda - \varepsilon) < 1.$$

Using the continuity of the function $y \to dS^{-1}(y)/dy$ and the condition $S(0) = 0$, we can find a constant $\delta > 0$ such that

$$S'(S^{-1}(y)) \geqslant \lambda - \varepsilon \quad \text{for } 0 \leqslant y \leqslant \delta.$$

Let $\mathscr{S} \subset D(\boldsymbol{R}_+)$ be the set of all densities f satisfying the following two conditions:

(8) $$\int_{x \geqslant r_0} f(x)\,dx \leqslant K_*/r_0 \quad \text{for } r_0 > 0$$

and

(9) $$f(x) \leqslant c \quad \text{for } x \in \boldsymbol{R}_+,$$

where c is some positive constant. Evidently, $\mathscr{S}$ is a weakly precompact set.

Denote by $D_0(\boldsymbol{R}_+)$ the subset of $D(\boldsymbol{R}_+)$ consisting of all functions f with $E(f) < +\infty$ such that

$$f(x) \leqslant k_0(f) \quad \text{for } x \in \boldsymbol{R}_+,$$

where the constant $k_0(f)$ depends on f. Choose an $f \in D_0(\boldsymbol{R}_+)$. From (7) and the Chebyshev inequality it follows that

$$\int_{x \geqslant r_0} P^n f(x)\,dx \leqslant K_*/r_0 \quad \text{for } r_0 > 0 \text{ and } n \geqslant n_0(f).$$

It is easy to verify that

$$P_S f(y) = f(S^{-1}(y))\frac{1}{S'(S^{-1}(y))}\mathbf{1}_{S(\boldsymbol{R}_+)}(y),$$

where $\mathbf{1}_A$ is the characteristic function of the set A. Thus Pf may be estimated as follows:

$$\begin{aligned} Pf(x) &= \int_0^\delta f(S^{-1}(y))\frac{1}{S'(S^{-1}(y))}\mathbf{1}_{S(\boldsymbol{R}_+)}(y)g\left(\frac{x}{y}\right)\frac{1}{y}dy + \int_\delta^{+\infty} P_S f(y)g\left(\frac{x}{y}\right)\frac{1}{y}dy \\ &\leqslant \frac{k_0(f)}{\lambda-\varepsilon}\int_0^\delta g\left(\frac{x}{y}\right)\frac{1}{y}dy + \frac{L}{\delta}\int_\delta^{+\infty} P_S f(y)\,dy \\ &\leqslant \frac{k_0(f)}{\lambda-\varepsilon}\int_0^\delta h\left(\frac{x}{y}\right)\frac{x}{y^2}dy + \frac{L}{\delta} \leqslant \frac{\|h\|}{\lambda-\varepsilon}k_0(f) + \frac{L}{\delta}. \end{aligned}$$

Since $\|h\|/(\lambda-\varepsilon) < 1$, there exists a real

$$c > \frac{L}{\delta(1-\|h\|/(\lambda-\varepsilon))}$$

such that

$$P^n f(x) \leqslant c$$

for n sufficiently large, say $n \geqslant n_1(f)$. As a consequence, $P^n f \in \mathscr{L}$ for $n \geqslant \max(n_0(f), n_1(f))$. Since $D_0(\boldsymbol{R}_+)$ is dense in $D(\boldsymbol{R}_+)$, this implies (4). We have proved that P is weakly constrictive. Furthermore, since $g(x) > 0$, from (2) it follows that

$$P^n f(x) > 0 \quad \text{for } x \in (0, +\infty);\ n = 1, 2, \ldots$$

According to Lemma 2 the proof of the theorem is complete. □

Up to here we have considered a special case where the transformation S was differentiable and increasing with the continuous derivative $dS^{-1}(x)/dx$. Now we assume that the transformation S: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfies the following conditions:

(2i) There is a partition $0 = a_1 < a_2 < \ldots$ of $\boldsymbol{R}_+$ such that for each integer j the restriction S_j of S to the interval $[a_j, a_{j+1})$ is a C^2 monotonic function;

(2ii) $S(x) \leqslant \eta x + \beta$ for $x \geqslant M$, $\max\limits_{0 \leqslant x \leqslant M} S(x) < +\infty$, where η, β and M

are nonnegative constants and

(10) $$\eta \int_{0}^{+\infty} xg(x)\,dx < 1;$$

(2iii) $S(0) = 0$;

(2iv) There is a constant b such that

$$c_j = \inf\{|S_j'(x)|:\ S_j(x) \leqslant b\} > 0 \quad \text{for } j = 2, 3, \ldots$$

and

$$\sum_{j=2}^{+\infty} \frac{1}{c_j} < +\infty,$$

where $S_j'(a_j)$ denotes the right derivative and $S_1(a_2) = \lim_{x \to a_2^-} S_1(x)$.

An example of transformations satisfying conditions (2i)–(2iv) is given by

$$S(x) = \tfrac{1}{3}\left(\sqrt{1+(x+\pi)^2\,|\sin x|} - 1\right), \qquad x \in \boldsymbol{R}_+$$

and $g(x) = xe^{-x}$.

THEOREM 2. *Let S: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfy conditions* (2i)–(2iv). *Assume that there is a nonincreasing $h \in L^1(\boldsymbol{R}_+)$ such that*

(11) $$0 < g(x) = xh(x) \quad \textit{for } x \in (0, +\infty)$$

and

(12) $$\|h\| < S'(0).$$

Then the operator P given by equation (3) *is asymptotically stable.*

Proof of Theorem 2. Since h is a nonincreasing function, by Lemma 1 we obtain $h(x) \leqslant \|h\|/x$. Thus $g(x) \leqslant \|h\|$ for $x \in \boldsymbol{R}_+$. Set $\lambda = S'(0)$ and choose $\varepsilon > 0$ so small that

$$\frac{\|h\|}{\lambda - \varepsilon} < 1.$$

Since S_1 is a C^2 function and $S_1(0) = 0$, there is a positive constant δ such that $\delta < \min(b, S_1(a_2))$ and

$$S_1'(S_1^{-1}(y)) \geqslant \lambda - \varepsilon \quad \text{for } 0 \leqslant y \leqslant \delta.$$

Let $\mathscr{S} \subset D(\boldsymbol{R}_+)$ be the set of all densities f satisfying the following two conditions:

(13) $$\int_{x \geqslant r_0} f(x)\,dx \leqslant \frac{K_*}{r_0} \quad \text{for } r_0 > 0$$

and

$$(14) \qquad f(x) \leqslant k \quad \text{for } x \in \boldsymbol{R}_+,$$

where k is some positive constant.

Denote by $D_0(\boldsymbol{R}_+)$ the subset of $D(\boldsymbol{R}_+)$ consisting of all functions f with $E(f) < +\infty$ such that

$$f(x) \leqslant k_0(f) \quad \text{for } x \in \boldsymbol{R}_+,$$

where the constant $k_0(f)$ depends on f. Fix $f \in D_0(\boldsymbol{R}_+)$ and set $f_n = P^n f$, $n = 0, 1, 2, \ldots$ It is easy to show, analogously as in the proof of Theorem 1, that

$$\int_{x \geqslant r_0} f_n(x)\,dx \leqslant K_*/r_0 \quad \text{for } n \geqslant n_0(f) \text{ and } r_0 > 0.$$

Since $g(x) = xh(x)$ and h is nonincreasing, it follows from Lemma 1 that $g(x) \leqslant 2/x$ and consequently

$$(15) \qquad P^n f(x) \leqslant \int_0^{+\infty} P_S f_{n-1}(y) \frac{2}{x}\,dy = \frac{2}{x} \quad \text{for } n = 1, 2, \ldots$$

Now our goal is to show that $f_n(x) \leqslant k$ for sufficiently large n. Note that (cf. [8])

$$P_S f(y) = \sum_{j=1}^{+\infty} f(S_j^{-1}(y)) \frac{1}{|S_j'(S_j^{-1}(y))|} \mathbf{1}_{S_j[a_j, a_{j+1})}(y).$$

Thus from equality (3) we obtain

$$(16) \qquad P^n f(x) = \int_0^{\delta} f_{n-1}(S_1^{-1}(y)) \frac{1}{S_1'(S_1^{-1}(y))} \mathbf{1}_{S_1[a_1, a_2)}(y)\, g\left(\frac{x}{y}\right)\frac{dy}{y} +$$
$$+ \sum_{j=2}^{+\infty} \int_0^{\delta} f_{n-1}(S_j^{-1}(y)) \frac{1}{|S_j'(S_j^{-1}(y))|} \mathbf{1}_{S_j[a_j, a_{j+1})}(y)\, g\left(\frac{x}{y}\right)\frac{dy}{y} +$$
$$+ \int_{\delta}^{+\infty} P_S f_{n-1}(y)\, g(x/y)\,dy/y \quad \text{for } n = 1, 2, \ldots$$

Therefore

$$Pf(x) \leqslant \frac{k_0(f)}{\lambda - \varepsilon} \int_0^{\delta} g\left(\frac{x}{y}\right)\frac{dy}{y} + k_0(f) \sum_{j=2}^{+\infty} \frac{1}{c_j} \int_0^{\delta} g\left(\frac{x}{y}\right)\frac{dy}{y} + \frac{\|h\|}{\delta}.$$

Setting

$$k_1(f) = k_0(f)\|h\| \left(\frac{1}{\lambda - \varepsilon} + \sum_{j=2}^{+\infty} \frac{1}{c_j}\right) + \frac{\|h\|}{\delta},$$

we obtain

$$Pf(x) \leqslant k_1(f) \quad \text{for } x \in \boldsymbol{R}_+.$$

Using this and inequalities (15) and $S_j^{-1}(y) \geqslant a_j$ ($j \geqslant 2$, $y \in S_j[a_j, a_{j+1})$), we may estimate Pf_1 as follows:

$$Pf_1(x) \leqslant \frac{k_1(f)}{\lambda-\varepsilon}\|h\| + \sum_{j=2}^{+\infty} \frac{1}{c_j} \int_0^{\delta} f_1(S_j^{-1}(y)) \mathbf{1}_{S_j[a_j, a_{j+1})}(y) g\left(\frac{x}{y}\right)\frac{dy}{y} +$$

$$+\frac{\|h\|}{\delta} \leqslant \frac{\|h\|}{\lambda-\varepsilon} k_1(f) + \|h\|\left(2\sum_{j=2}^{+\infty}\frac{1}{a_j c_j} + \frac{1}{\delta}\right).$$

By an induction argument we obtain

$$P^n f(x) \leqslant \left(\frac{\|h\|}{\lambda-\varepsilon}\right)^{n-1} k_1(f) + \|h\|\left(2\sum_{j=2}^{+\infty}\frac{1}{a_j c_j} + \frac{1}{\delta}\right)\sum_{l=0}^{n-2}\left(\frac{\|h\|}{\lambda-\varepsilon}\right)^l$$

for $n = 2, 3, \ldots$ Since $\|h\|/(\lambda-\varepsilon) < 1$, there exists a real

$$k > \frac{\|h\|\left(2\sum_{j=2}^{+\infty} 1/a_j c_j + 1/\delta\right)}{1 - \|h\|/(\lambda-\varepsilon)}$$

such that $f_n(x) \leqslant k$ for sufficiently large n. Further proceeding analogously as in the proof of Theorem 1 and using Lemma 2 we may complete the proof. □

3. Dynamical systems on the interval [0, 1]. In Theorems 1 and 2 of the previous section, we have assumed that the transformation S in (1) is defined on the entire halfline $\boldsymbol{R}_+$. Now was assume that S is a transformation of the unit interval [0, 1] into itself. We assume also that $x_0 \in [0, 1]$ and ξ_n, $n = 0, 1, \ldots$ are [0, 1]-valued independent random variables having the same density g. In formula (3) for the operator P the domain of integration must be altered, so

(17) $$Pf(x) = \int_x^1 P_S f(y) g(x/y)\, dy/y \quad \text{for } f \in L^1([0, 1]).$$

We shall assume that S: $[0, 1] \to [0, 1]$ satisfies the following conditions:

(3i) There is a partition $0 = a_1 < a_2 < \ldots < a_q = 1$ of [0, 1] such that for each integer j the restriction S_j of S to the interval $[a_j, a_{j+1})$ is a C^2 and monotonic function;

(3ii) $S'(0) > 0$ and $S(0) = 0$;

(3iii) There is a constant b such that

$$c_j = \inf\{|S_j'(x)|:\ S_j(x) \leqslant b\} > 0 \quad \text{for } j = 2, 3, \ldots, q-1.$$

Conditions (3i)–(3iii) are quite similar to (2i)–(2iv). Observe that now inequalities (2ii), (10) and (2iv) are automatically satisfied with $\eta = 0$, $M = 0$ and $\beta = 1$. Thus, assuming that there is a nonincreasing function $h \in L^1([0, 1])$ such that

$$(18) \qquad 0 < g(x) = xh(x) \quad \text{for } x \in (0, 1]$$

and

$$(19) \qquad \|h\| < S'(0),$$

we may repeat the proof of Theorem 2 and to obtain the following proposition.

PROPOSITION 1. *If S: $[0, 1] \to [0, 1]$ satisfies conditions* (3i)–(3iii) *and if there is a nonincreasing function $h \in L^1([0, 1])$ satisfying* (18) *and* (19), *then the operator P given by equation* (17) *is asymptotically stable.*

Conditions (18) and (19) are quite restrictive. Consider as an example the parabolic transformation

$$S(x) = 2x(1-x), \qquad x \in [0, 1],$$

and the sequence ξ_n of independent random variables with density $g(x) = 2x$. In this case, $h(x) \equiv 2$ and $S'(0) = \|h\|$. Thus, (19) is not satisfied. However, using another technique, we may prove a theorem where inequalities (18) and (19) are replaced by essentially weaker conditions.

THEOREM 3. *Let S: $[0, 1] \to [0, 1]$ be a transformation satisfying conditions* (3i)–(3iii). *Assume that the density g satisfies*

$$(20) \qquad 0 < g(x) \leqslant Kx^r, \qquad x \in (0, 1],$$

with positive constants r and K, and that there exists a constant $\alpha \in (0, 1)$ such that

$$(21) \qquad K\lambda^{\alpha-1}/(\alpha+r) < 1, \quad \text{where } \lambda = S'(0).$$

Then the Markov operator P defined by (17) *is asymptotically stable.*

Proof of Theorem 3. Fix an $\alpha \in (0, 1)$ for which (21) holds true and denote by D_0 the subset of $D([0, 1])$ consisting of all functions f which satisfy the inequality

$$(22) \qquad f(x) \leqslant M_0(f)/x^\alpha \quad \text{for } x \in (0, 1],$$

where the constant $M_0(f)$ depends on f. It is evident that D_0 is dense in

$D([0, 1])$. Fix $f \in D_0$ and set $f_n = P^n f$, $n = 0, 1, \ldots$ Then

(23) $$f_{n+1}(x) = \int_x^1 P_S f_n(y) g\left(\frac{x}{y}\right)\frac{1}{y} dy.$$

Using the inequality $g(x) \leqslant Kx^r$, we obtain

(24) $$f_{n+1}(x) \leqslant x^r K \int_x^1 P_S f_n(y) \frac{1}{y^{r+1}} dy.$$

Denoting the second factor in the right-hand side of (24) by $\omega(x)$ and applying Lemma 1, we obtain

(25) $$f_n(x) \leqslant K/x \quad \text{for } x \in (0, 1] \text{ and } n = 1, 2, \ldots$$

We may derive an explicit formula for the function f_n. First, note that (cf. [8])

$$P_S f(y) = \sum_{j=1}^{q-1} \frac{1}{|S_j'(S_j^{-1}(y))|} f(S_j^{-1}(y)) \mathbf{1}_{S_j[a_j, a_{j+1})}(y).$$

Thus from equality (23) we have

(26) $$f_{n+1}(x) = \sum_{j=1}^{q-1} \int_x^1 \frac{1}{|S_j'(S_j^{-1}(y))|} f_n(S_j^{-1}(y)) \mathbf{1}_{S_j[a_j, a_{j+1})}(y) g\left(\frac{x}{y}\right)\frac{dy}{y}.$$

Now by an induction argument we are going to show that the functions f_n satisfy the inequalities

(27) $$f_n(x) \leqslant M_n(f)/x^\alpha \quad \text{for } n = 1, 2, \ldots,$$

where $M_{n+1}(f) = \gamma M_n(f) + \beta$ with some constants γ, β, $\gamma < 1$ which do not depend on f. Thus assume (27) for a given $n \geqslant 1$. Since

$$\frac{K}{\lambda (1/\lambda)^\alpha (\alpha + r)} < 1,$$

we may choose $\varepsilon > 0$ so small that

$$\gamma := \frac{K}{(\lambda - \varepsilon)(1/\lambda - \varepsilon)^\alpha (\alpha + r)} < 1.$$

From the continuity of the functions $y \to S_1'(S_1^{-1}(y))$ and $y \to [S_1^{-1}(y)]'$ follows the existence of a positive constant $\delta \leqslant S_1(a_2)$ such that

(28) $$S_1'(S_1^{-1}(y)) \geqslant S_1'(S_1^{-1}(0)) - \varepsilon = \lambda - \varepsilon \quad \text{for } 0 \leqslant y \leqslant \delta$$

and

(29) $$[S_1^{-1}(y)]' \geqslant [S_1^{-1}(0)]' - \varepsilon = 1/\lambda - \varepsilon \quad \text{for } 0 \leqslant y \leqslant \delta.$$

Further, from (29) and $S_1^{-1}(0) = 0$ it follows that

$$S_1^{-1}(y) \geqslant (1/\lambda - \varepsilon)\, y \quad \text{for } 0 \leqslant y \leqslant \delta. \tag{30}$$

The first term in the sum (26) may be estimated as follows

$$\begin{aligned} B_1 := \int_x^1 \frac{1}{|S_1'(S_1^{-1}(y))|} f_n(S_1^{-1}(y))\mathbf{1}_{S_1[a_1,a_2)}(y)\, g\left(\frac{x}{y}\right)\frac{dy}{y} &\leqslant Kx^r \int_{[x,1]\cap(0,\delta]} \frac{1}{S_1'(S_1^{-1}(y))} f_n(S_1^{-1}(y))\frac{dy}{y^{r+1}} + \\ &\quad + Kx^r \int_{[x,1]\cap(\delta, S_1(a_2))} \frac{1}{S_1'(S_1^{-1}(y))} f_n(S_1^{-1}(y))\frac{dy}{y^{r+1}}. \end{aligned} \tag{31}$$

Using the inequality

$$f_n(S_1^{-1}(y)) \leqslant M_n(f)/[S_1^{-1}(y)]^\alpha$$

and (28), (30), we obtain

$$\begin{aligned} Kx^r \int_{[x,1]\cap(0,\delta]} \frac{1}{S_1'(S_1^{-1}(y))} f_n(S_1^{-1}(y))\frac{dy}{y^{r+1}} \\ \leqslant \frac{KM_n(f)}{(\lambda-\varepsilon)(1/\lambda-\varepsilon)^\alpha} x^r \int_{[x,1]\cap(0,\delta]} \frac{dy}{y^{\alpha+r+1}} \leqslant \gamma\frac{M_n(f)}{x^\alpha}. \end{aligned} \tag{32}$$

It is obvious that

$$\int_{S_j[a_j,a_{j+1})} \frac{dy}{|S_j'(S_j^{-1}(y))|} \leqslant \sum_{i=1}^{q-1} \int_{S_i[a_i,a_{i+1})} \frac{dy}{|S_i'(S_i^{-1}(y))|} = 1 \tag{33}$$

for $j = 1, 2, \ldots, q-1$. For $y \geqslant \delta$, inequalities $S_1^{-1}(y) \geqslant S_1^{-1}(\delta)$ and (25) imply

$$f_n(S_1^{-1}(y)) \leqslant K/S_1^{-1}(\delta).$$

Therefore,

$$\begin{aligned} Kx^r \int_{[x,1]\cap(\delta,S_1(a_2))} \frac{1}{S_1'(S_1^{-1}(y))} f_n(S_1^{-1}(y))\frac{dy}{y^{r+1}} \\ \leqslant \frac{K^2 x^r}{S_1^{-1}(\delta)\,\delta^{r+1}} \int_\delta^{S_1(a_2)} \frac{1}{S_1'(S_1^{-1}(y))}\, dy \leqslant \frac{K^2}{S_1^{-1}(\delta)\,\delta^{r+1}}. \end{aligned}$$

Consequently, we obtain

$$Kx^r \int_{[x,1]\cap(\delta,S_1(a_2))} \frac{1}{S_1'(S_1^{-1}(y))} f_n(S_1^{-1}(y))\frac{dy}{y^{r+1}} \leqslant \frac{c}{x^\alpha}, \tag{34}$$

where $c = K^2/S_1^{-1}(\delta)\delta^{r+1}$. Inequalities (31), (32) and (34) imply

(35) $$B_1 \leqslant \frac{\gamma M_n(f)+c}{x^\alpha} \quad \text{for every } x\in(0, 1].$$

Now define

$$B_2 := \sum_{j=2}^{q-1} \int_x^1 \frac{1}{|S_j'(S_j^{-1}(y))|} f_n(S_j^{-1}(y))\, 1_{S_j[a_j,a_{j+1})}(y)\, g\left(\frac{x}{y}\right)\frac{dy}{y}.$$

From (25) we obtain immediately

$$f_n(S_j^{-1}(y)) \leqslant \frac{K}{a_2} \quad \text{for } j \geqslant 2,\ y\in S_j[a_j, a_{j+1}).$$

Using this and inequalities (20) and (33), we may evaluate B_2 as follows:

$$\begin{aligned} B_2 &\leqslant \frac{K^2 x^r}{a_2} \sum_{j=2}^{q-1} \int_{[x,1]\cap(0,b]} \frac{1}{|S_j'(S_j^{-1}(y))|} 1_{S_j[a_j,a_{j+1})}(y) \frac{dy}{y^{r+1}} + \\ &\qquad + \frac{K^2 x^r}{a_2} \sum_{j=2}^{q-1} \int_{[x,1]\cap(b,1]} \frac{1}{|S_j'(S_j^{-1}(y))|} 1_{S_j[a_j,a_{j+1})}(y) \frac{dy}{y^{r+1}} \\ &\leqslant \frac{K^2 x^r}{a_2}\left(\sum_{j=2}^{q-1} \frac{1}{c_j}\int_x^1 \frac{dy}{y^{r+1}} + \sum_{j=2}^{q-1} \frac{1}{b^{r+1}} \int_{S_j[a_j,a_{j+1})} \frac{dy}{|S_j'(S_j^{-1}(y))|}\right) \\ &\leqslant \frac{K^2}{a_2 r}\sum_{j=2}^{q-1}\frac{1}{c_j} + \frac{K^2}{a_2 b^{r+1}}. \end{aligned}$$

Setting

$$d = \frac{K^2}{a_2}\left(\frac{1}{r}\sum_{j=2}^{q-1}\frac{1}{c_j} + \frac{1}{b^{r+1}}\right),$$

we have

(36) $$B_2 \leqslant d/x^\alpha \quad \text{for } x\in(0, 1].$$

Combining (35) and (36) with equality (26), we immediately obtain

$$f_{n+1}(x) \leqslant M_{n+1}(f)/x^\alpha \quad \text{for } x\in(0, 1],$$

where $M_{n+1}(f) = \gamma M_n(f)+c+d$.

Thus we have proved that $f_n(x) \leqslant M_n(f)/x^\alpha$ implies $f_{n+1}(x) \leqslant M_{n+1}(f)/x^\alpha$ for $n = 1, 2, \ldots$ Now, we are going to show that

(37) $$f_1(x) \leqslant M_1(f)/x^\alpha.$$

However, in this case we cannot use inequality $f_0(x) \leqslant K/x$. Instead of that we may use inequality (22) to obtain

$$f_1(x) \leqslant \left[\gamma M_0(f) + \frac{K M_0(f)}{(S_1^{-1}(\delta))^\alpha \delta^{r+1}} + \frac{K M_0(f)}{a_2^\alpha}\left(\frac{1}{r}\sum_{j=2}^{q-1}\frac{1}{c_j} + \frac{1}{b^{r+1}}\right)\right]\frac{1}{x^\alpha}.$$

This implies (37) if we denote by M_1 the term in brackets and completes the proof of inequality (27). Now recall that $\gamma < 1$. Thus for sufficiently large n, say $n > n_0(f)$, there exists a real $M_* > (c+d)/(1-\gamma)$ such that

$$f_n(x) \leqslant M_*/x^\alpha \quad \text{for } x \in (0, 1].$$

Since M_*/x^α is integrable on $[0, 1]$, this proves that P is weakly constrictive. Finally, from (20) it follows the existence of $u > 0$ such that

$$f_n(x) > 0 \quad \text{for } x \in (0, u);\ n > n_0(f).$$

Thus all of the conditions of Lemma 2 are satisfied and the proof is complete. □

4. Final remarks.

EXAMPLE 1. Let $S_\lambda(x) = \lambda x(1-x)$, $x \in [0, 1]$, with a constant $\lambda \in (1, 4]$. Then there is a sequence ξ_n of independent random variables such that the Markov operator P corresponding the random dynamical system

$$x_{n+1} = S_\lambda(x_n)\,\xi_n$$

is asymptotically stable.

The proof is immediate. First note that S_λ satisfies conditions (3i)–(3iii) and that $S'_\lambda(0) = \lambda$. Pick a constant

$$r > \frac{1 - \frac{1}{2}\sqrt{\lambda}}{\sqrt{\lambda} - 1}.$$

If we define the density g by the equality

$$g(x) = (r+1)\,x^r,$$

then condition (21) of Theorem 3 becomes

$$\frac{r+1}{\sqrt{\lambda}\,(r+\frac{1}{2})} < 1.$$

Consequently, all the assumptions of Theorem 3 are satisfied.

As illustrated in the following example, the condition $K\lambda^{\alpha-1}/(\alpha+r) < 1$ in Theorem 3 is quite essential.

EXAMPLE 2. Consider the random dynamical system

$$x_{n+1} = S(x_n)\xi_n,$$

where $S(x) = x$, $x \in [0, 1]$, and the random variables ξ_n have the density $g(x) = 2x$. Since $P_S f(y) = f(y)$, from (17) it follows

$$Pf(x) = 2x \int_x^1 f(y) \frac{1}{y^2} dy. \tag{38}$$

Set $f_n = P^n f_0$ and pick the initial density $f_0 \equiv 1$. From (38) by an elementary calculation we obtain

$$f_n(x) = 2^n \left[1 - x \sum_{j=0}^{n-1} \frac{(-1)^j (\ln x)^j}{j!} \right], \qquad x \in (0, 1].$$

Now take an arbitrary constant $\varepsilon > 0$. Then

$$f_n(x) \leqslant 2^n \left[1 - \varepsilon \sum_{j=0}^{n-1} \frac{(\ln 1/\varepsilon)^j}{j!} \right] \quad \text{for } x \in [\varepsilon, 1].$$

Further from the obvious equality

$$\frac{1}{\varepsilon} = e^{\ln 1/\varepsilon} = \sum_{j=0}^{n-1} \frac{(\ln 1/\varepsilon)^j}{j!} + \frac{e^{\theta \ln 1/\varepsilon}}{n!} \left(\ln \frac{1}{\varepsilon} \right)^n, \qquad 0 < \theta < 1,$$

it follows that

$$f_n(x) \leqslant \frac{\varepsilon (1/\varepsilon)^\theta 2^n (\ln 1/\varepsilon)^n}{n!} \leqslant \frac{(\ln 1/\varepsilon^2)^n}{n!} \quad \text{for } x \in [\varepsilon, 1]$$

and implies that the sequence $f_n = P^n \mathbf{1}$ converges uniformly to zero on $[\varepsilon, 1]$. Since P is a Markov operator, this implies in turn [8] that the sequence $\{P^n f\}$ converges to zero in $L^1([\varepsilon, 1])$ norm for every $\varepsilon > 0$. Consequently, the equation $Pf = f$ has no solution in $L^1([0, 1])$ except $f \equiv 0$.

Remark 1. Observe that if $S(0) > 0$, then we may omit assumptions (12) and (21) in Theorems 2 and 3, respectively.

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