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Some formulae similar to Barnes' lemma

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§ 1. Introductory. In a former paper [1] in this journal I obtained a formula which is an extension of Barnes' lemma [2], namely

$$(1)\quad \frac{1}{2\pi i}\int \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\,ds = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)},$$

where the contour of integration is of Barnes' type and is curved, if necessary, to separate the increasing sequences of poles from the decreasing sequences.

In § 2 of this paper some more integrals of this type will be evaluated. These integrals are similar to some other integrals, which are given by Titchmarsh [3], p. 194.

In § 3 of this paper I shall show that the formulae obtained in § 2 can be used to sum certain series of products of generalized hypergeometric functions. It may be noted that the constants and the variables are such that the functions involved do exist.

The following formulae are required in the proofs: (Kummer [4], p. 53):

$$(2)\quad {}_2F_1\left[\begin{matrix}a, b; -1\\ 1+a-b\end{matrix}\right] = 2^{-a}\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{\Gamma(1-b+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}a)},$$

where $1+a-b \neq 0, -1, -2,$

(Whipple [5], p. 114):

$$(3)\quad {}_3F_2\left[\begin{matrix}a, 1-a, c; 1\\ f, 2c+1-f\end{matrix}\right] = \frac{\pi\Gamma(f)\Gamma(2c+1-f)2^{1-2c}}{\Gamma(c+\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}f)\Gamma(\frac{1}{2}a+\frac{1}{2}f)\Gamma(c+\frac{1}{2}+\frac{1}{2}b-\frac{1}{2}f)\Gamma(\frac{1}{2}b+\frac{1}{2}f)},$$

where $b = 1-a$,

(Watson [6], p. 189):

$$(4)\quad {}_3F_2\left[\begin{matrix}a, b, c; 1\\ \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}, 2c\end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2}-\frac{1}{2}a)\Gamma(c+\frac{1}{2}-\frac{1}{2}b)},$$

(Dixon [7], p. 285):

$$(5)\quad {}_3F_2\left[\begin{matrix} a, b, c; 1 \\ 1+a-b, 1+a-c \end{matrix}\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1-b-c+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)},$$

(Whipple [8], p. 251):

$$(6)\qquad {}_4F_3\left[\begin{matrix} a, 1+\frac{1}{2}a, b, c; -1 \\ \frac{1}{2}a, 1+a-b, 1+a-c \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)},$$

(Dougall [9]):

$$(7)\quad {}_5F_4\left[\begin{matrix} a, 1+\frac{1}{2}a, c, d, e; 1 \\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e \end{matrix}\right] = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)},$$

(MacRobert [10], p. 374):

$$(8)\qquad {}_pF_q\left[\begin{matrix} a_1, \dots, a_p; z \\ \varrho_1, \dots, \varrho_q \end{matrix}\right] = \frac{\prod_{s=1}^{q}\Gamma(\varrho_s)}{\prod_{n=1}^{p}\Gamma(a_n)} \cdot \frac{1}{2\pi i}\int \Gamma(\zeta)\frac{\prod_{n=1}^{p}\Gamma(a_n-\zeta)}{\prod_{s=1}^{q}\Gamma(\varrho_s-\zeta)}\left(-\frac{1}{z}\right)^{\zeta} d\zeta$$

where $p \leqslant q+1$ and the contour of integration is of Barnes' type with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $a_1, \dots, a_p$ lie to the rigth of the contour. When $p = q+1$, then $|z| < 1$ and when $p < q+1$ the contour is bent to the left at both ends. Zero and negative integral values of the a's and ϱ's are omitted. Also the following two simple identities are required:

$$(9)\qquad \Gamma(\zeta)\Gamma(1-\zeta) = \pi/\sin\pi\zeta\,,$$

$$(10)\qquad e^{i\pi a}\sin\tfrac{1}{2}\pi p - e^{i\pi p}\sin\pi(a-\tfrac{1}{2}p) = e^{i\pi p/2}\sin\pi(a-p)\,.$$

§ 2. Formulae and proofs. The integrals to be proved are:

$$(11)\qquad \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(p-\zeta)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-p+\zeta)}(-1)^{\zeta} d\zeta = \frac{1}{2}\frac{\Gamma(\frac{1}{2}p)}{\Gamma(\varrho-\frac{1}{2}p)\Gamma(\varrho-p)},$$

$$(12)\qquad \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\Gamma(p-\zeta)}{\Gamma(\varrho-\zeta)\Gamma(2-\varrho-p+\zeta)}(-1)^{\zeta} d\zeta = 2^{1-p-\varrho}\frac{\Gamma(\frac{1}{2})\Gamma(p)}{\Gamma(2-\varrho-p)\Gamma(\frac{1}{2}+\frac{1}{2}\varrho-\frac{1}{2}p)\Gamma(\frac{1}{2}\varrho+\frac{1}{2}p)},$$

(13) $$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(p-\zeta)\Gamma(a-\zeta)\Gamma(\beta-p+\zeta)}{\Gamma(2a-\zeta)\Gamma(2\beta-p+\zeta)}(-1)^\zeta d\zeta$$
$$=2^{1+2p-2a-2\beta}\frac{\pi\Gamma(\frac{1}{2}p)\Gamma(a+\beta-p)\cdot e^{i\pi p/2}}{\Gamma(a+\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}+a-\frac{1}{2}p)\Gamma(\frac{1}{2}+\beta-\frac{1}{2}p)},$$

(14) $$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(p-\zeta)\Gamma(a-\zeta)\Gamma(a-p+\zeta)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-p+\zeta)}(-1)^\zeta d\zeta$$
$$=e^{i\pi p/2}\frac{\Gamma(a)\Gamma(\frac{1}{2}p)\Gamma(a-\frac{1}{2}p)\Gamma(\varrho-a-\frac{1}{2}p)}{2\Gamma(\varrho-a)\Gamma(\varrho-p)\Gamma(\varrho-\frac{1}{2}p)}.$$

Proof of (11). In virtue of (9) the left side of (11) is equal to

$$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(p-\zeta)}{\Gamma(\varrho-\zeta)}\Gamma(1-\varrho+p-\zeta)(-1)^\zeta\Big\{\sum_{i,-i}\frac{1}{2\pi i}e^{i\pi(\varrho-p+\zeta)}\Big\}d\zeta$$

where $\sum_{i,-i}$ denotes that the expression $\sum_i$ is followed by a similar expression with $-i$ instead of i and the two expressions are added.

Now the last expression is equal to

$$\frac{1}{\Gamma(\varrho-p)\Gamma(1-\varrho+p)}\cdot\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(p-\zeta)\Gamma(1-\varrho+p-\zeta)}{\Gamma(\varrho-\zeta)}d\zeta\,.$$

Here we apply (8) and the last expression becomes

$$\frac{\Gamma(p)}{\Gamma(\varrho)\Gamma(\varrho-p)}\cdot{}_2F_1\begin{bmatrix}p,1-\varrho+p;\,-1\\ \varrho\end{bmatrix}.$$

Here we sum the ${}_2F_1$ using (2) and so obtain the right side of (11).

In the same way, by applying (2), (12) can be proved.

Proof of (13). It follows from the well-known asymptotic expansion of the gamma function that the integral on the left side of (13) is convergent and its value is equal to minus the sum of the residues of the integrand at is poles. By calculating these residues we find that the left side of (13) is equal to

$$\frac{\Gamma(p-a)\Gamma(a)\Gamma(a+\beta-p)}{\Gamma(a)\Gamma(2\beta+a-p)}e^{i\pi a}{}_3F_2\begin{bmatrix}a,1-a,a+\beta-p;\,1\\ 1+a-p,2\beta+a-p\end{bmatrix}+$$
$$+\frac{\Gamma(a-p)\Gamma(p)\Gamma(\beta)}{\Gamma(2a-p)\Gamma(\beta)}e^{i\pi p}{}_3F_2\begin{bmatrix}\beta,p,1+p-2a;\,1\\ 2\beta,1+p-a\end{bmatrix}.$$

Here we sum the first ${}_3F_2$ using (3) and the second ${}_3F_2$ by (4), add the two results applying (9) and get

$$\frac{\pi^2\cdot 2^{1+2p-2a-2\beta}\Gamma(a+\beta-p)}{\sin\pi(p-a)\Gamma(a+\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}+\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}+a-\frac{1}{2}p)}\times$$
$$\times\left[\frac{e^{i\pi a}}{\Gamma(1-\frac{1}{2}p)}-\frac{e^{i\pi p}\Gamma(\frac{1}{2}p)}{\Gamma(a-\frac{1}{2}p)\Gamma(1-a+\frac{1}{2}p)}\right]\times$$
$$\times\frac{\pi\cdot 2^{1+2p-2a-2\beta}\Gamma(\frac{1}{2}p)\Gamma(a+\beta-p)}{\sin\pi(p-a)\Gamma(a+\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}+\beta-\frac{1}{2}p)\Gamma(\frac{1}{2}+a-\frac{1}{2}p)}\cdot A\,,$$

where

(15) $$A = e^{i\pi a}\sin\tfrac{1}{2}\pi p - e^{i\pi p}\sin\pi(a-\tfrac{1}{2}p)\,.$$

Now we substitute the values of A from (10) and so obtain the right side of (13).

Proof of (14). Similarly by calculating the residues of the integral on the left side of (14), we find that the integral is equal to

$$\frac{\Gamma(p-a)\Gamma(a)\Gamma(2a-p)}{\Gamma(a+\varrho-p)\Gamma(\varrho-a)}e^{i\pi a}{}_3F_2\left[\begin{matrix}a, 2a-p, 1+a-\varrho; 1\\ 1+a-p, a+\varrho-p\end{matrix}\right]+$$
$$+\frac{\Gamma(a-p)\Gamma(p)\Gamma(a)}{\Gamma(\varrho)\Gamma(\varrho-p)}e^{i\pi p}{}_3F_2\left[\begin{matrix}a, p, 1+p-\varrho; 1\\ 1+p-a, \varrho\end{matrix}\right].$$

Here we sum each ${}_3F_2$ using (5), add the results applying (9), and the last expression becomes

$$\frac{\pi\Gamma(a)\Gamma(\varrho-a-\frac{1}{2}p)}{2\sin\pi(p-a)\Gamma(\varrho-a)\Gamma(\varrho-p)\Gamma(\varrho-\frac{1}{2}p)}\cdot\left[e^{i\pi a}\frac{\Gamma(a-\frac{1}{2}p)}{\Gamma(1-\frac{1}{2}p)}-e^{i\pi p}\frac{\Gamma(\frac{1}{2}p)}{\Gamma(1-a+\frac{1}{2}p)}\right]$$
$$=\frac{\Gamma(a)\Gamma(\varrho-a-\frac{1}{2}p)\Gamma(\frac{1}{2}p)\Gamma(a-\frac{1}{2}p)}{2\sin\pi(p-a)\Gamma(\varrho-a)\Gamma(\varrho-p)\Gamma(\varrho-\frac{1}{2}p)}A\,,$$

where A is given by (15). We substitute the value of A as before from (10) and so obtain the right side of (14). Thus (14) is proved.

§ 3. Series of products of generalized hypergeometric functions. It will be shown that the formulae (11), (12), (13) and (14) can be applied to sum new infinite series of products of generalized hypergeometric functions of the same type from an extensive list of similar series which have been given by Burchnall and Chaundy [11], [12] and [13]. Their method is purely symbolic and they introduce a certain type of differential operators and then deduce their results by applying these operators.

Thus, as an example I prove the following expansions:

(16) $$\sum_{r=0}^{\infty}\frac{1}{r!(c+r-1;r)(c;2r)\{(\varrho;r)\}^2}x^{2r}{}_0F_2\left[\begin{matrix};x\\ \varrho+r, c+2r\end{matrix}\right]{}_0F_2\left[\begin{matrix};-x\\ \varrho+r, c+2r\end{matrix}\right]$$
$$={}_0F_5[;\tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, \varrho, \tfrac{1}{2}\varrho, \tfrac{1}{2}\varrho+\tfrac{1}{2}; -x^2/16]\,,$$

(17) $$\sum_{r=0}^{\infty}\frac{(-1)^r(\gamma;r)(c-\gamma;r)}{r!(c+r-1;r)(c;2r)\{(\varrho;r)\}^2}x^{2r}{}_1F_2\left[\begin{matrix}\gamma+r;x\\ \varrho+r, c+2r\end{matrix}\right]{}_1F_2\left[\begin{matrix}\gamma+r;-x\\ \varrho+r, c+2r\end{matrix}\right]$$
$$={}_2F_5\left[\begin{matrix}\frac{1}{2}\gamma, \frac{1}{2}\gamma+\frac{1}{2}; -x^2/4\\ \frac{1}{2}\varrho, \frac{1}{2}\varrho+\frac{1}{2}, \varrho, \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}\end{matrix}\right],$$

(18) $$\sum_{r=0}^{\infty}(-1)^r\frac{(\gamma;r)}{r!\{(\varrho;r)\}^2}x^{2r}{}_1F_1\left[\begin{matrix}\gamma+r;x\\ \varrho+r\end{matrix}\right]{}_1F_1\left[\begin{matrix}\gamma+r;-x\\ \varrho+r\end{matrix}\right]$$
$$={}_2F_3\left[\begin{matrix}\frac{1}{2}\gamma, \frac{1}{2}\gamma+\frac{1}{2}; -x^2\\ \frac{1}{2}\varrho, \frac{1}{2}\varrho+\frac{1}{2}, \varrho\end{matrix}\right],$$

(19) $$\sum_{r=0}^{\infty}(-1)^r \frac{(\gamma;r)(c-\gamma;r)}{r!(c+r-1;r)(c;2r)(\varrho;r)(2-\varrho;r)}x^{2r}\times$$
$$\times {}_1F_2\begin{bmatrix}\gamma+r;\,x\\ \varrho+r,\,c+2r\end{bmatrix}{}_1F_2\begin{bmatrix}\gamma+r;\,-x\\ 2-\varrho+r;,\,c+2r\end{bmatrix}$$
$$={}_2F_5\begin{bmatrix}\tfrac{1}{2}\gamma,\,\tfrac{1}{2}\gamma+\tfrac{1}{2};\,-x^2/4\\ \tfrac{1}{2},\,\tfrac{1}{2}c,\,\tfrac{1}{2}c+\tfrac{1}{2},\,\tfrac{3}{2}-\tfrac{1}{2}\varrho,\,\tfrac{1}{2}+\tfrac{1}{2}\varrho\end{bmatrix}+$$
$$+\frac{2(1-\varrho)\gamma x}{\varrho(2-\varrho)c}\,{}_2F_5\begin{bmatrix}\tfrac{1}{2}\gamma+\tfrac{1}{2},\,\tfrac{1}{2}\gamma+1;\,-x^2/4\\ \tfrac{3}{2},\,\tfrac{1}{2}c+\tfrac{1}{2},\,1+\tfrac{1}{2}c,\,2-\tfrac{1}{2}\varrho,\,1+\tfrac{1}{2}\varrho\end{bmatrix},$$

(20) $$\sum_{r=0}^{\infty}\frac{1}{r!(c+r-1;r)(c;2r)(\varrho;r)(2-\varrho;r)}x^{2r}\times$$
$$\times {}_0F_2[;\,\varrho+r,\,c+2r;\,x]\,{}_0F_2[;\,2-\varrho+r,\,c+2r;\,-x]$$
$$={}_0F_5[;\,\tfrac{1}{2},\,\tfrac{1}{2}c+\tfrac{1}{2},\,\tfrac{1}{2}c,\,\tfrac{3}{2}-\tfrac{1}{2}\varrho,\,\tfrac{1}{2}+\tfrac{1}{2}\varrho;\,-x^2/16]+$$
$$+\frac{2(1-\varrho)x}{\varrho(2-\varrho)}\,{}_0F_5[;\,\tfrac{3}{2},\,\tfrac{1}{2}c+\tfrac{1}{2},\,1+\tfrac{1}{2}c,\,2-\tfrac{1}{2}\varrho,\,1+\tfrac{1}{2}\varrho;\,-x^2/16],$$

(21) $$\sum_{r=0}^{\infty}(-1)^r\frac{(\gamma;r)}{r!(\varrho;r)(2-\varrho;r)}x^{2r}{}_1F_1\begin{bmatrix}\gamma+r;\,x\\ \varrho+r\end{bmatrix}{}_1F_1\begin{bmatrix}\gamma+r;\,-x\\ 2-\varrho+r\end{bmatrix}$$
$$={}_2F_3\begin{bmatrix}\tfrac{1}{2}\gamma,\,\tfrac{1}{2}\gamma+\tfrac{1}{2};\,-x^2\\ \tfrac{1}{2},\,\tfrac{3}{2}-\tfrac{1}{2}\varrho,\,\tfrac{1}{2}+\tfrac{1}{2}\varrho\end{bmatrix}+\left[\frac{(1-\varrho)\gamma x}{\varrho(2-\varrho)}\right]{}_2F_3\begin{bmatrix}\tfrac{1}{2}\gamma+\tfrac{1}{2},\,\tfrac{1}{2}\gamma+1;\,-x^2\\ \tfrac{3}{2},\,2-\tfrac{1}{2}\varrho,\,1+\tfrac{1}{2}\varrho\end{bmatrix},$$

(22) $$\sum_{r=0}^{\infty}(-1)^r\frac{(\gamma;r)(c-\gamma;r)(a;r)(\beta;r)}{r!(c+r-1;r)(c;2r)(2a;r)(2\beta;r)}x^{2r}\times$$
$$\times {}_2F_2\begin{bmatrix}a+r,\,\gamma+r;\,x\\ 2a+r,\,c+2r\end{bmatrix}{}_2F_2\begin{bmatrix}\beta+r,\,\gamma+r;\,-x\\ 2\beta+r,\,c+2r\end{bmatrix}$$
$$={}_4F_5\begin{bmatrix}\tfrac{1}{2}\gamma,\,\tfrac{1}{2}\gamma+\tfrac{1}{2},\,\tfrac{1}{2}a+\tfrac{1}{2}\beta,\,\tfrac{1}{2}a+\tfrac{1}{2}\beta+\tfrac{1}{2};\,x^2/4\\ \tfrac{1}{2}c,\,\tfrac{1}{2}c+\tfrac{1}{2},\,a+\beta,\,\tfrac{1}{2}+a,\,\tfrac{1}{2}+\beta\end{bmatrix},$$

(23) $$\sum_{r=0}^{\infty}\frac{(a;r)(\beta;r)}{r!(c+r-1;r)(c;2r)(2a;r)(2\beta;r)}x^{2r}\times$$
$$\times {}_1F_2\begin{bmatrix}a+r;\,x\\ 2a+r,\,c+2r\end{bmatrix}{}_1F_2\begin{bmatrix}\beta+r;\,-x\\ 2\beta+r,\,c+2r\end{bmatrix}$$
$$={}_2F_5\begin{bmatrix}\tfrac{1}{2}a+\tfrac{1}{2}\beta,\,\tfrac{1}{2}a+\tfrac{1}{2}\beta+\tfrac{1}{2};\,x^2/16\\ \tfrac{1}{2}c,\,\tfrac{1}{2}c+\tfrac{1}{2},\,a+\beta,\,\tfrac{1}{2}+a,\,\tfrac{1}{2}+\beta\end{bmatrix},$$

(24) $$\sum_{r=0}^{\infty}(-1)^r\frac{(\gamma;r)(a;r)(\beta;r)}{r!(2a;r)(2\beta;r)}x^{2r}\times$$
$$\times {}_2F_1\begin{bmatrix}\gamma+r,\,a+r;\,x\\ 2a+r\end{bmatrix}{}_2F_1\begin{bmatrix}\gamma+r,\,\beta+r;\,-x\\ 2\beta+r\end{bmatrix}$$
$$={}_4F_3\begin{bmatrix}\tfrac{1}{2}\gamma,\,\tfrac{1}{2}\gamma+\tfrac{1}{2},\,\tfrac{1}{2}a+\tfrac{1}{2}\beta,\,\tfrac{1}{2}a+\tfrac{1}{2}\beta+\tfrac{1}{2};\,x^2\\ a+\beta,\,\tfrac{1}{2}+a,\,\tfrac{1}{2}+\beta\end{bmatrix},$$

$$(25)\quad \sum_{r=0}^{\infty}(-1)^r \frac{(c-\gamma;\,r)(\gamma;\,r)\{(a;\,r)\}^2}{r!(c+r-1;\,r)(c;\,2r)\{(\varrho;\,r)\}^2}x^{2r}\times$$
$$\times {}_2F_2\left[\begin{matrix}a+r,\,\gamma+r;\,x\\ \varrho+r,\,c+2r\end{matrix}\right]{}_2F_2\left[\begin{matrix}a+r,\,\gamma+r;\,-x\\ \varrho+r,\,c+2r\end{matrix}\right]$$
$$={}_4F_6\left[\begin{matrix}\tfrac12\gamma,\,\tfrac12\gamma+\tfrac12,\,a,\,\varrho-a;\,x^2/4\\ \varrho,\,\tfrac12\varrho,\,\tfrac12\varrho+\tfrac12,\,\tfrac12c,\,\tfrac12c+\tfrac12\end{matrix}\right],$$

$$(26)\quad \sum_{r=0}^{\infty}\frac{\{(a;\,r)\}^2}{r!\{(\varrho;\,r)\}^2}x^{2r}{}_1F_2\left[\begin{matrix}a+r;\,x\\ \varrho+r,\,c+2r\end{matrix}\right]{}_1F_2\left[\begin{matrix}a+r;\,-x\\ \varrho+r,\,c+2r\end{matrix}\right]$$
$$={}_2F_5\left[\begin{matrix}a,\,\varrho-a;\,x^2/16\\ \varrho,\,\tfrac12\varrho,\,\tfrac12\varrho+\tfrac12,\,\tfrac12c,\,\tfrac12c+\tfrac12\end{matrix}\right],$$

$$(27)\quad \sum_{r=0}^{\infty}(-1)^r\frac{(\gamma;\,r)\{(a;\,r)\}^2}{r!\{(\varrho;\,r)\}^2}x^{2r}{}_2F_1\left[\begin{matrix}a+r,\,\gamma+r;\,x\\ \varrho+r\end{matrix}\right]{}_2F_1\left[\begin{matrix}a+r,\,\gamma+r;\,-x\\ \varrho+r\end{matrix}\right]$$
$$={}_4F_3\left[\begin{matrix}\tfrac12\gamma,\,\tfrac12\gamma+\tfrac12,\,a,\,\varrho-a;\,x^2\\ \varrho,\,\tfrac12\varrho,\,\tfrac12\varrho+\tfrac12\end{matrix}\right].$$

Proofs of (17), (19), (22) and (25). To prove (17), we substitute for the two ${}_1F_2$ on the left-hand side of (17) the corresponding expressions obtained from (8); then the left side of (17) becomes

$$\sum_{r=0}^{\infty}(-1)^r\frac{\{\Gamma(c)\}^2(c;\,2r)(c-\gamma;\,r)}{r!(c+r-1;\,r)(\gamma;\,r)\{\Gamma(\gamma)\}^2}x^{2r}\times$$
$$\times\frac{1}{2\pi i}\int\Gamma(\zeta)\frac{\Gamma(\gamma+r-\zeta)}{\Gamma(\varrho+r-\zeta)\Gamma(c+2r-\zeta)}(-x)^{-\zeta}d\zeta\times$$
$$\times\frac{1}{2\pi i}\int\Gamma(\xi)\frac{\Gamma(\gamma+r-\xi)}{\Gamma(\varrho+r-\xi)\Gamma(c+2r-\xi)}(x)^{-\xi}d\xi\,.$$

Here we write $\zeta+r$, $\xi+r$ for ζ and ξ respectively, and change the order of integration and summation; then the last expression becomes

$$\left(\frac{1}{2\pi i}\right)^2\int\int(-1)^{\xi}\frac{\{\Gamma(c)\}^2\Gamma(\zeta)\Gamma(\xi)\Gamma(\gamma-\zeta)\Gamma(\gamma-\xi)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-\xi)\Gamma(c-\zeta)\Gamma(c-\xi)}(-x)^{-(\zeta+\xi)}d\zeta d\xi\times$$
$$\times\frac{1}{\{\Gamma(\gamma)\}^2}\,{}_5F_4\left[\begin{matrix}c-1,\,\tfrac12+\tfrac12c,\,\zeta,\,\xi,\,c-\gamma;\,1\\ \tfrac12c-\tfrac12,\,c-\zeta,\,c-\xi,\,\gamma\end{matrix}\right].$$

Here we sum the ${}_5F_4$ by Dougall's theorem (7), and get

$$\frac{\Gamma(c)}{\Gamma(\gamma)}\left(\frac{1}{2\pi i}\right)^2\int\int(-1)^{\xi}\frac{\Gamma(\zeta)\Gamma(\xi)\Gamma(\gamma-\zeta-\xi)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-\xi)\Gamma(c-\zeta-\xi)}(-x)^{-(\zeta+\xi)}d\zeta d\xi\,.$$

Now we put $\xi=p-\zeta$ where p is the new variable of integration and get

$$\frac{\Gamma(c)}{\Gamma(\gamma)}\cdot\frac{1}{2\pi i}\int e^{i\pi p}\frac{\Gamma(\gamma-p)}{\Gamma(c-p)}(-x)^{-p}dp\times$$
$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(p-\zeta)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-p+\zeta)}(-1)^{\zeta}d\zeta\,.$$

We again evaluate the last integral by means of (11), apply the duplication formula for the gamma function; then finally from (8) the result follows.

In the same way we apply in the last proof (12), (13) and (14) instead of (11); then formulae (19), (22) and (25) can be deduced.

Proof of (16), (20), (23) and (26). These four formulae can be proved in the same way as before but by using Whipple's formula (6) instead of (7). They can also be deduced from (17), (19), (22) and (25) respectively by writing x/γ for x and then letting γ tend to ∞.

Proofs of (18), (21), (24) and (27). For example, to prove 27) we substitute for the two Gauss hypergeometric functions on the left side of (27) the corresponding expressions obtained from (8) so getting

$$\frac{1}{\{\Gamma(\gamma)\}^2}\sum_{r=0}^{\infty}(-1)^r x^{2r}\cdot\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(a+r-\zeta)\Gamma(\gamma+r-\zeta)}{\Gamma(\varrho+r-\zeta)}(-x)^{-\zeta}d\zeta\times$$

$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\Gamma(a+r-\xi)(\Gamma(\gamma+r-\xi)}{\Gamma(\varrho+r-\xi)}(-x)^{\xi}d\xi\,.$$

Here wre write $\zeta+r$, $\xi+r$ for ζ and ξ respectively, change the order of integration and summation; then the last expression becomes

$$\frac{1}{\{\Gamma(\gamma)\}^2}\cdot\left(\frac{1}{2\pi i}\right)^2\int\int(-1)^{\xi}\frac{\Gamma(\zeta)\Gamma(\xi)\Gamma(a-\zeta)\Gamma(a-\xi)\Gamma(\gamma-\zeta)\Gamma(\gamma-\xi)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-\xi)}(-x)^{-(\zeta+\xi)}\times$$

$$\times\,{}_2F_1\left[\begin{matrix}\zeta,\ \xi;\ 1\\ \gamma\end{matrix}\right]d\zeta d\xi\,.$$

Now we sum the ${}_2F_1$ by Gauss's theorem and get

$$\frac{1}{\Gamma(\gamma)}\left(\frac{1}{2\pi i}\right)^2\int\int(-1)^{\xi}\frac{\Gamma(\zeta)\Gamma(\xi)\Gamma(a-\zeta)\Gamma(a-\xi)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-\xi)}\Gamma(\gamma-\zeta-\xi)\times$$

$$\times\,(-x)^{-(\zeta+\xi)}d\zeta d\xi\,.$$

We write $\xi=p-\zeta$ where p is the new variable of integration and get

$$\frac{1}{\Gamma(\gamma)}\cdot\frac{1}{2\pi i}\int e^{i\pi p}\frac{\Gamma(\gamma-p)}{1}(-x)^{-p}dp\times$$

$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Gamma(p-\zeta)\Gamma(a-\zeta)\Gamma(a-p+\zeta)}{\Gamma(\varrho-\zeta)\Gamma(\varrho-p+\zeta)}(-1)^{\zeta}d\zeta\,.$$

Here we evaluate the last integral by means of (14), apply (8) again and so obtain the right side of (24).

In the same way, when (11), (12) and (13) are used instead of (14), formulae (18), (21) and (24) can be deduced respectively.

References

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