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A note on rational functions of several complex variables

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Let $f(z) = f(z_1, \ldots, z_n)$, $n \geqslant 2$, be holomorphic in a polycylinder $P = \{z: |z_k| \leqslant r, k = 1, \ldots, n\}$. Then

$$(1)\qquad f(z) = \sum_{\nu=0}^{\infty} P_\nu(z), \quad \text{for} \quad z \in P,$$

where $P_\nu(z) = P_\nu(z_1, \ldots, z_n)$ is a homogeneous polynomial of degree ν and the series is uniformly convergent in P. Let

$$(2)\qquad g(t;\, \theta_2, \ldots, \theta_n) = f(t, e^{i\theta_2}t, \ldots, e^{i\theta_n}t) = \sum_{\nu=0}^{\infty} P_\nu(1, e^{i\theta_2}, \ldots, e^{i\theta_n})t^\nu$$

and let E_k, $k = 2, 3, \ldots, n$, denote a subset of the interval $[0, 2\pi]$. The purpose of this note is to prove the following

THEOREM. *If E_i, $i = 2, 3, \ldots, n$, is non-denumerable and $g(t;\, \theta_2, \ldots, \theta_n)$ is rational with respect to t for any $\theta = (\theta_2, \ldots, \theta_n) \in E = E_2 \times \ldots \times E_n$, then $f(z_1, \ldots, z_n)$ is rational with respect to $z = (z_1, \ldots, z_n)$. Moreover, if E_2 is at most denumerable and $E_3, \ldots, E_n$ are arbitrary, then there exists an entire transcendental function $f(z)$ such that $g(t;\, \theta_2, \ldots, \theta_n) = f(t, e^{i\theta_2}t, \ldots, e^{i\theta_n}t)$ is a polynomial in t for each $(\theta_2, \ldots, \theta_n) \in E$.*

Proof. The idea of the proof is based on Kronecker's necessary and sufficient condition for a function of one variable to be rational (see [1], p. 102-103).

Let $\theta = (\theta_2, \ldots, \theta_n)$ be a fixed point of E. The function $g(t; \theta) = g(t;\, \theta_2, \ldots, \theta_n)$ can be written in the form

$$(3)\qquad g(t, \theta) = \frac{a_0 + a_1 t + \ldots + a_{k-1}t^{k-1}}{b_0 + b_1 t + \ldots + b_k t^k},$$

where $a_0, a_1, \ldots, a_{k-1}$, $b_0, b_1, \ldots, b_k$ and k depend on θ. Since $g(t, \theta) = \sum_{\nu=0}^{\infty} P_\nu(1, e^{i\theta_2}, \ldots, e^{i\theta_n})t^\nu$, we have

$$a_0 + a_1 t + \ldots + a_{k-1}t^{k-1} = (P_0 + P_1 t + \ldots)(b_0 + b_1 t + \ldots + b_k t^k),$$

whence

$$
\begin{aligned}
a_0 &= b_0 P_0, \\
a_1 &= b_1 P_0 + b_0 P_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{k-1} &= b_{k-1}P_0 + b_{k-2}P_1 + \ldots + b_0 P_{k-1}, \\
0 &= b_k P_0 + b_{k-1}P_1 + \ldots + b_0 P_k, \\
0 &= b_k P_1 + b_{k-1}P_2 + \ldots + b_0 P_{k+1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{4}
$$

Let

$$
\begin{aligned}
C_{\lambda\mu}(z) &= C_{\lambda\mu}(z_1, \ldots, z_n) \\
&= \begin{vmatrix} P_\lambda(z) & P_{\lambda+1}(z) & \ldots & P_{\lambda+\mu}(z) \\ P_{\lambda+1}(z) & P_{\lambda+2}(z) & \ldots & P_{\lambda+\mu+1}(z) \\ \ldots & \ldots & \ldots & \ldots \\ P_{\lambda+\mu}(z) & P_{\lambda+\mu+1}(z) & \ldots & P_{\lambda+2\mu}(z) \end{vmatrix}, \quad \lambda, \mu = 0, 1, \ldots
\end{aligned}
\tag{5}
$$

Since at least one of the coefficients $b_0, b_1, \ldots, b_k$ is different from zero, it follows from (4) that

$$C_{\lambda\mu}(1, e^{i\theta_2}, \ldots, e^{i\theta_n}) = 0, \quad \text{for} \quad \lambda \geqslant 0 \quad \text{and} \quad \mu = k = k(\theta). \tag{6}$$

Now we shall prove that there exist non-denumerable subsets $E_i^0 \subset E_i$, $i = 2, 3, \ldots, n$, such that the function $k(\theta) = k_0 = \text{const}$ for $\theta \in E^0 = E_2^0 \times \times E_3^0 \times \ldots \times E_n^0$. Indeed, let $(\theta_2^0, \theta_3^0, \ldots, \theta_{n-1}^0)$ be a fixed point of $E_2 \times E_3 \times \times \ldots \times E_{n-1}$. Since the set of points $(\theta_2^0, \theta_3^0, \ldots, \theta_{n-1}^0, \theta_n)$, where $\theta_n \in E_n$, is non-denumerable and $k(\theta_2^0, \theta_3^0, \ldots, \theta_{n-1}^0, \theta_n)$ takes at most denumerably many values, there are a non-denumerable subset E_n^0 of E_n and an integer $k_1(\theta_2^0, \theta_3^0, \ldots, \theta_{n-1}^0)$ such that $k(\theta_2^0, \theta_3^0, \ldots, \theta_{n-1}^0, \theta_n) = k_1(\theta_2^0, \theta_3^0, \ldots, \theta_{n-1}^0)$ for $\theta_n \in E_n^0$. Similarly, there is a non-denumerable subset E_{n-1}^0 of E_{n-1}, such that $k_1(\theta_2^0, \theta_3^0, \ldots, \theta_{n-2}^0, \theta_{n-1}) = k_2(\theta_2^0, \theta_3^0, \ldots, \theta_{n-2}^0)$ for $\theta_{n-1} \in E_{n-1}^0$. Therefore $k(\theta_2^0, \theta_3^0, \ldots, \theta_{n-2}^0, \theta_{n-1}, \theta_n) = k_2(\theta_2^0, \theta_3^0, \ldots, \theta_{n-2}^0)$ for $(\theta_{n-1}, \theta_n) \in E_{n-1}^0 \times E_n^0$. By repeating the procedure, we shall find non-denumerable subsets E_{n-k}^0 of E_{n-k}, $k = 0, 1, \ldots, n-2$, and an integer k_0 such that

$$k(\theta) = k(\theta_2, \ldots, \theta_n) = k_0, \quad \text{for} \quad \theta \in E^0 = E_2^0 \times E_3^0 \times \ldots \times E_n^0.$$

Therefore

$$C_{\lambda\mu}(1, e^{i\theta_2}, \ldots, e^{i\theta_n}) = 0 \quad \text{for} \quad \lambda \geqslant 0,\ \mu = k_0,\ \theta \in E^0. \tag{7}$$

Since

$$P_\lambda(e^{i\varphi}, e^{i(\theta_2+\varphi)}, \ldots, e^{i(\theta_n+\varphi)}) = e^{i\lambda\varphi} P_\lambda(1, e^{i\theta_2}, \ldots, e^{i\theta_n}),$$

we have

$$C_{\lambda\mu}(e^{i\varphi}, e^{i(\theta_2+\varphi)}, \ldots, e^{i(\theta_n+\varphi)}) = e^{i(\mu+1)(\lambda+\mu)\varphi} C_{\lambda\mu}(1, e^{i\theta_2}, \ldots, e^{i\theta_n}).$$

Thus, by (7), we have

$$C_{\lambda\mu}(e^{i\varphi}, e^{i(\theta_2+\varphi)}, \ldots, e^{i(\theta_n+\varphi)}) = 0, \tag{8}$$

for $\lambda \geqslant 0$, $\mu = k_0$, $\theta \in E^0$ and $\varphi \in [0, 2\pi]$. Therefore the polynomial $C_{\lambda\mu}(z_1, \ldots, z_n)$ vanishes identically if $\lambda \geqslant 0$ and $\mu = k_0$. At any point $z = (z_1, \ldots, z_n)$ the rank $r = r(z)$ of the matrix

$$M_\mu(z) = \begin{pmatrix} P_0(z) & P_1(z) & \ldots & P_\mu(z) \\ P_1(z) & P_2(z) & \ldots & P_{\mu+1}(z) \\ \ldots & \ldots & \ldots & \ldots \\ P_\mu(z) & P_{\mu+1}(z) & \ldots & P_{2\mu}(z) \end{pmatrix} \quad (\mu = k_0)$$

is less than or equal to k_0, $r \leqslant k_0$. Let $r_0 = \max_{z \in C^n} r(z) = r(z^0)$ and let

$$M(z) = \begin{pmatrix} P_{00}(z) & P_{01}(z) & \ldots & P_{0\mu}(z) \\ P_{10}(z) & P_{11}(z) & \ldots & P_{1\mu}(z) \\ \ldots & \ldots & \ldots & \ldots \\ P_{\mu 0}(z) & P_{\mu 1}(z) & \ldots & P_{\mu\mu}(z) \end{pmatrix}$$

be a matrix which differs from $M_\mu(z)$ only by a permutation of its columns or rows. We choose the matrix $M(z)$ in such a way that

$$A_{r_0}(z^0) = \begin{vmatrix} P_{00}(z^0) & P_{01}(z^0) & \ldots & P_{0r_0}(z^0) \\ \ldots & \ldots & \ldots & \ldots \\ P_{r_00}(z^0) & P_{r_01}(z^0) & \ldots & P_{r_0r_0}(z^0) \end{vmatrix} \neq 0 .$$

Therefore $A_{r_0}(z) \neq 0$ in a neighbourhood U of z^0. Given $z \in U$, consider the system of linear equations

(9) $P_{i0}(z)e_0 + P_{i1}(z)e_1 + \ldots + P_{ir_0}(z)e_{r_0} + \ldots + P_{i\mu}(z)e_\mu = 0 , \quad i = 0, 1, \ldots, \mu ,$

with respect to the unknowns e_i, $i = 0, 1, \ldots, \mu$. Putting $e_{r_0+1} = 1$ and $e_{r_0+2} = e_{r_0+3} = \ldots = e_\mu = 0$, we shall solve the system

(10) $P_{i0}(z)e_0 + P_{i1}(z)e_1 + \ldots + P_{ir_0}(z)e_{r_0} = -P_{i,r_0+1}(z) , \quad i = 0, 1, \ldots, r_0 .$

The solution $e_0(z), e_1(z), \ldots, e_{r_0}(z)$, $e_{r_0+1} = 1$, $e_{r_0+2}(z) = e_{r_0+3}(z) = \ldots = e_\mu(z) \equiv 0$, is also a solution of system (9), and, moreover, the functions $e_i(z)$, $i = 0, \ldots, \mu$ are rational with respect to $z = (z_1, \ldots, z_n)$ in the neighbourhood U. Therefore there exists a solution $b_0(z), \ldots, b_\mu(z)$ of the first μ equations of the infinite system

$$\begin{gathered} P_0(z)b_\mu + P_1(z)b_{\mu-1} + \ldots + P_\mu(z)b_0 = 0 , \\ P_1(z)b_\mu + P_2(z)b_{\mu-1} + \ldots + P_{\mu+1}(z)b_0 = 0 , \\ \ldots\ldots\ldots\ldots \end{gathered} \tag{11}$$

and it is a proper permutation of the functions $e_i(z)$, $i = 0, 1, \ldots, \mu$. Since $C_{\lambda\mu}(z) = 0$ for $\lambda \geqslant 0$, $z \in C^n$, it follows that $b_0(z), b_1(z), \ldots, b_\mu(z)$ satisfy all the equations of (11). If we now put

$$\begin{aligned} a_0(z) &= b_0(z)P_0(z) , \\ a_1(z) &= b_1(z)P_0(z) + b_0(z)P_1(z) , \\ &\ldots\ldots\ldots\ldots \\ a_{\mu-1}(z) &= b_{\mu-1}(z)P_0(z) + b_{\mu-2}(z)P_1(z) + \ldots + b_0(z)P_{\mu-1}(z) , \end{aligned} \tag{12}$$

then $a_0(z), \ldots, a_{\mu-1}(z), b_0(z), \ldots, b_\mu(z)$ are rational and satisfy equations (11) and (12) for any $z \in C^n$. Therefore

$$\sum_{\nu=0}^{\infty} P_\nu(z) t^\nu = \frac{a_0(z) + a_1(z)t + \ldots + a_{\mu-1}(z)t^{\mu-1}}{b_0(z) + b_1(z)t + \ldots + b_\mu(z)t^\mu} \quad (\mu = k_0)$$

for $|t| \leqslant 2$ and $|z_k| \leqslant \varrho$, ϱ being sufficiently small. We can check that

$$\begin{aligned} &a_0(z) + a_1(z)t + \ldots + a_{\mu-1}(z)t^{\mu-1} \\ &\qquad = A_0(z) + A_1(z)(t-1) + \ldots + A_{\mu-1}(z)(t-1)^{\mu-1}, \\ &b_0(z) + b_1(z)t + \ldots + b_\mu(z)t^\mu \\ &\qquad = B_0(z) + B_1(z)(t-1) + \ldots + B_\mu(z)(t-1)^\mu, \end{aligned} \tag{13}$$

where $A_i(z)$, $i = 0, \ldots, \mu-1$, and $B_k(z)$, $k = 0, 1, \ldots, \mu$, are polynomials in $a_0, \ldots, a_{\mu-1}$ and $b_0, \ldots, b_\mu$, respectively. (Thus $A_i(z)$ and $B_k(z)$ are rational functions in $z = (z_1, \ldots, z_n)$). Suppose that $B_0(z) = B_1(z) = \ldots = B_{l-1}(z) \equiv 0$ and. $B_l(z) \not\equiv 0$. Such a $B_l(z)$ certainly exists because at least one of the $b_0(z), b_1(z), \ldots, b_\mu(z)$ does not vanish identically. Therefore, since $\lim_{t \to 1} \sum_{\nu=0}^{\infty} P_\nu(z) t^\nu = \sum_{\nu=0}^{\infty} P_\nu(z) = f(z)$, we have by (13)

$$f(z) = A_l(z)/B_l(z),$$

i.e. $f(z)$ is rational.

To prove the second part of the theorem, let $E_2 = \{e^{i\theta_2^{(\nu)}}\}$, $\nu = 1, 2, \ldots$, and put

$$f(z_1, \ldots, z_n) = \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \prod_{k=1}^{\nu} (z_1 - e^{-\theta_2^{(k)}} z_2).$$

The function $f(z)$ is obviously entire and transcendental. Moreover,

$$f(t, e^{i\theta_2^{(l)}}t, e^{i\theta_3}t, \ldots, e^{i\theta_n}t) \equiv \sum_{\nu=0}^{l-1} \frac{t^\nu}{\nu!} \prod_{k=1}^{\nu} (1 - e^{i(\theta_2^{(l)} - \theta_2^{(k)})})$$

for arbitrary $\theta_3, \theta_4, \ldots, \theta_n$.

COROLLARY. Weierstrass-Hurwitz's theorem ([2], p. 236). *If a function $f(z)$ is meromorphic at any point $z \in C^n$ and at the point* $(\infty, \infty, \ldots, \infty)$ *(i.e. if $f(1/z_1, 1/z_2, \ldots, 1/z_n)$ is meromorphic in a neighbourhood of* $(0, \ldots, 0)$), *then $f(z)$ is rational.*

References

[1] G. Pólya und G. Szegö, *Aufgaben und Lehrsätze II*, Berlin 1925.

[2] Б. А. Фукс, *Теория аналитических функций многих комплексных переменных*, Москва 1948.

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