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Projective torsion and curvature, axiom of planes and free mobility for almost-product manifolds

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Abstract. We define for almost-product manifolds (M, J) the J-projective torsion, whose nullity characterizes the semi-torsionless J-connections. We then consider Prvanović's J-projective curvature tensor, giving a new treatment which includes a complete study of J-projective flatness. We consider also an "axiom of planes" for almost-product manifolds, and study its consequences and the "free mobility" in the para-Kaehlerian case, giving three characterizations of the spaces of constant J-sectional curvature (see [2], [5]).

1. Introduction

As it is well known, E. Cartan [1] defined for a Riemannian manifold the axiom of the plane and the axiom of free mobility, and proved the equivalence of the property of constant curvature with each of such axioms, and also with the existence of a geodesic representation in the ordinary space. Later, Yano and Mogi [10], Tashiro [9] and Ishihara [3] considered the analogues for the almost-complex and Kaehlerian cases, proving the equivalence of the condition of constant holomorphic sectional curvature with either the axiom of holomorphic planes, the axiom of holomorphic free mobility, or H-projective flatness.

On the other hand, Prvanović [5] and Sinha–Kalpana [7], [8], treated the analogue of H-projective flatness for almost-product and para-Kaehlerian manifolds (for the later see Rasevskii [6], Libermann [4]). However, Prvanović's proof on the equivalence of H-projective flatness with the nullity of the H-projective curvature tensor is partially incorrect, because of the expression (4.7) in [5]. The desired equivalence can only be obtained with some restrictions (see our Theorem 2.5.1), and supplementary conditions must be given in the residual cases. Moreover, Sinha [7] actually gave no proof of the result, and limited the study to the almost paracomplex case.

In the present paper, we give the analogues of the earlier three characterizations for the para-Kaehlerian case, in the context of general almost-product manifolds (M, J), that is, with J being not necessarily almost paracomplex. We introduce the concept of J-projective torsion; give a new treatment of the J-projective curvature of Prvanović, including the complete study of the said equivalence theorem; define the axiom of the plane and prove a related characterization theorem for J-connections; and finally consider the concept of J-free mobility.

2. J-projective torsion and curvature

2.1. Half-torsionless J-connections. Let (M, J) be an almost-product n-manifold. We denote by L_1 and L_2, respectively, the $(+1)$- and (-1)-eigenbundles corresponding to J, and by l_1 and l_2 the corresponding projection operators. We put

$$r_1 = \operatorname{rank} L_1, \quad r_2 = \operatorname{rank} L_2, \quad k = r_1 - r_2.$$

Sinha [7] defines the half-torsionless J-connections as the J-connections whose torsion tensor T satisfies

$$pT = 0,$$

where p is the operator on $(1, 2)$ tensor fields A on M defined by

$$(pA)(X, Y) = A(X, Y) + A(JX, JY) + JA(X, JY) + JA(JX, Y).$$

He also proves several results on such connections. In particular that there always exists such a connection, say $\bar{\nabla}$, obtained form any J-connection ∇ by

$$\bar{\nabla} = \nabla - \tfrac{1}{8} pT.$$

In order to see the geometric significance of the half-torsionless property, we give the following:

PROPOSITION 2.1.1. *Let ∇ be a J-connection on (M, J). Then there exists a unique half-torsionless J-connection $\bar{\nabla}$ on M with the same geodesics as ∇.*

Proof. The half-torsionless J-connection

$$\bar{\nabla} = \nabla - \tfrac{1}{8} pT$$

has the same geodesics as ∇, since the difference tensor $\bar{A} = -\frac{1}{8} pT$ is antisymmetric.

As for uniqueness, if $\tilde{\nabla}$ is a half-torsionless J-connection with the same geodesics as ∇, we have that the difference tensor A of $\tilde{\nabla}$ and $\bar{\nabla}$ satisfies

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + A(X, Y) = \nabla_X Y - \tfrac{1}{8}(pT)(X, Y) + A(X, Y).$$

A is antisymmetric. On the other hand, since $\tilde{\nabla}$ is half-torsionless, we deduce, since $p^2 = 4p$,

$$\begin{aligned} 0 &= (p\tilde{T})(X, Y) \\ &= (pT)(X, Y) - \tfrac{1}{2}(pT)(X, Y) + \tfrac{1}{2}(pT)(Y, X) + (pA)(X, Y) - (pA)(Y, X), \end{aligned}$$

and thus

$$(pA)(X, Y) = (pA)(Y, X).$$

But from this, applying the antisymmetry of A and the fact that $\tilde{\nabla}$ and $\bar{\nabla}$ are J-connections, we obtain

$$A = 0. \quad \text{q.e.d.}$$

2.2. J-projective transformations. J-projective torsion and semi-torsionless J-connections. Let ∇ be a J-connection on the almost-product manifold (M, J), and let us consider a curve $\gamma(t)$ on M such that

$$\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma + \beta(t)J\dot\gamma, \tag{2.2.1}$$

where $\dot\gamma$ denotes the tangent to γ and $\alpha(t)$, $\beta(t)$ are functions of the parameter t.

DEFINITION 2.2.1. ([5], [7]). A curve γ on (M, J) will be said a *J-plane curve* if it satisfies (2.2.1).

Since

$$\nabla_{\dot\gamma}J\dot\gamma = J\nabla_{\dot\gamma}\dot\gamma = \alpha(t)J\dot\gamma + \beta(t)\dot\gamma,$$

it is clear that the geometric significance of such a curve is that the plane generated by $\dot\gamma$ and $J\dot\gamma$ (or the line, if $J\dot\gamma = \lambda\dot\gamma$) is preserved by parallel transport along the curve. We can add:

PROPOSITION 2.2.2. *Let ∇ be a J-connection on (M, J). Then there exists a (not necessarily unique) half-torsionless J-connection $\tilde{\nabla}$ with the same J-plane curves as ∇.*

Proof. As we know, the J-connection $\bar{\nabla}$ defined by

$$\bar{\nabla} = \nabla - \tfrac{1}{8}pT$$

is half-torsionless and has the same geodesics as ∇. Thus, for every vector field X we have

$$\bar{\nabla}_X X = \nabla_X X.$$

In particular, for the field $\dot\gamma$ of tangents to each J-plane curve

$$\bar{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma + \beta(t)J\dot\gamma. \quad \text{q.e.d.}$$

Then we have in a way analogous to [7]:

DEFINITION 2.2.3. Two half-torsionless J-connections ∇ and $\tilde{\nabla}$ on (M, J) will be said *J-projectively related* if they have all the J-plane curves in common.

And:

PROPOSITION 2.2.4. *Two half-torsionless J-connections ∇ and $\tilde{\nabla}$ on (M, J)*

are J-projectively related iff there exist 1-forms ϑ and $\bar{\vartheta}$ on M, such that

$$(2.2.2) \quad \tilde{\nabla}_X Y = \nabla_X Y + \vartheta(X)Y + \vartheta(Y)X + \vartheta(JX)JY + \vartheta(JY)JX + \bar{\vartheta}(X)Y - \bar{\vartheta}(JX)JY.$$

We deduce immediately the:

PROPOSITION 2.2.5 (see also Prvanović [5]). *Two torsionless J-connections ∇ and $\tilde{\nabla}$ are J-projectively related iff there exists a 1-form ϑ such that*

$$\tilde{\nabla}_X Y = \nabla_X Y + \vartheta(X)Y + \vartheta(Y)X + \vartheta(JX)JY + \vartheta(JY)JX.$$

DEFINITION 2.2.6. Let ∇ and $\tilde{\nabla}$ be two half-torsionless J-connections on (M, J). A transformation $\nabla \rightsquigarrow \tilde{\nabla}$ will be said a *J-projective transformation* iff ∇ and $\tilde{\nabla}$ satisfy (2.2.2) for certain 1-forms ϑ and $\bar{\vartheta}$.

If now we take the antisymmetric part of both sides in (2.2.2) with respect to the covariant indices, we obtain, being T and $\tilde{T}$ the respective torsion tensors of ∇ and $\tilde{\nabla}$:

$$\tilde{T}(X, Y) = T(X, Y) + \bar{\vartheta}(X)Y - \bar{\vartheta}(Y)X - \bar{\vartheta}(JX) + \bar{\vartheta}(JY)JX.$$

Let $\{e_i\}$ be a local frame adapted to J, and let $\{e^i\}$ be its dual. We then write, T being the torsion of ∇,

$$C(X) = e^i(T(X, e_i)),$$

so that

$$\tilde{C}(X) = e^i(\tilde{T}(X, e_i)) = C(X) + n\bar{\vartheta}(X) - k\bar{\vartheta}(JX).$$

DEFINITION 2.2.7. If $n^2 \neq k^2$, the *J-projective torsion* $\mathcal{T}$ of ∇ is defined by

$$\mathcal{T}(X, Y) = T(X, Y) - \frac{1}{n^2 - k^2}[\{nC(X) + kC(JX)\}Y - \{nC(Y) + kC(JY)\}X - \{nC(JX) + kC(X)\}JY + \{nC(JY) + kC(Y)\}JX].$$

And we have, by computation, the following result:

PROPOSITION 2.2.8. *The J-projective torsion $\mathcal{T}$ is a J-projective invariant, that is, it is invariant under J-projective transformations.*

Also, we immediately have:

$$\mathcal{T}(X, Y) = -\mathcal{T}(Y, X)$$

and

$$e^i(\mathcal{T}(X, e_i)) = 0.$$

Then, we have the following generalization of the almost paracomplex case of Sinha:

DEFINITION 2.2.9. A J-connection ∇ on (M, J) will be called *semi-torsionless* if its J-projective torsion $\mathcal{T}$ is zero.

It is easy to prove:

PROPOSITION 2.2.10. *Every semi-torsionless J-connection ∇ on (M, J) is a half-torsionless J-connection.*

And:

PROPOSITION 2.2.11. *There exists a semi-torsionless J-connection on (M, J) iff the Nijenhuis tensor N of J is zero.*

We have also:

PROPOSITION 2.2.12. *A half-torsionless J-connection ∇ on (M, J) is semi-torsionless iff it is J-projectively related to a torsionless J-connection.*

Proof. If ∇ is semi-torsionless, an T denotes its torsion, we consider the J-connection $\tilde{\nabla}$ which is J-projectively related with ∇, being ϑ and $\bar{\vartheta}$, with the notations of Proposition 2.2.4, equal to

$$\begin{aligned} &\vartheta = 0, \\ (2.2.3)\qquad &\bar{\vartheta}(X)\,Y = -\frac{1}{2(n^2-k^2)}[\{nC(X)+kC(JX)\}\,Y-\{nC(JX)+kC(X)\}\,JY]. \end{aligned}$$

Then it is easy to prove that $\tilde{\nabla}$ is a J-connection with torsion $\tilde{T}$ such that

$$\tilde{T}(X, Y) = T(X, Y) = 0.$$

Conversely, if ∇ is J-projectively related to a torsionless J-connection, since $\mathcal{T}$ is invariant by J-projective trnasformations, we deduce $\mathcal{T} = 0$. That is, ∇ is semi-torsionless. q.e.d.

2.3. J-projectively flat J-connections.

DEFINITION 2.3.1 (Prvanović [5]). Let ∇ be a J-connection on the almost-product manifold (M, J). ∇ will be said *J-projectively flat* if, for each $x \in M$, there exists at least a neighbourhood of x on which ∇ is J-projectively related to a torsionless J-connection with zero curvature.

Let ∇ be a half-torsionless J-connection J-projectively flat. Then the connection $\tilde{\nabla}$ given by means of (2.2.3) is torsionless. Indeed, by Proposition 2.2.8, its torsion is zero since ∇ is J-projectively flat. Moreover, the torsionless J-connection $\tilde{\nabla}$ is also J-projectively flat, and thus we have:

PROPOSITION 2.3.2. *A half-torsionless J-connection ∇ is J-projectively flat* *iff there exists a torsionless J-projectively flat J-connection which is J-projectively related with ∇.*

Hence, in account of Proposition 2.2.11, we obtain:

COROLLARY 2.3.3.. *If (M, J) admits a half-torsionless J-connection J-projectively flat, then $N = 0$; that is, (M, J) is locally product.*

2.4. Some J-projective invariants. The J-projective curvature tensor. Let ∇ and $\tilde{\nabla}$ be two torsionless J-connections J-projectively related by

$$\tilde{\nabla}_X Y = \nabla_X Y + \vartheta(X) Y + \vartheta(Y) X + \vartheta(JX) JY + \vartheta(JY) JX, \tag{2.4.1}$$

and let R and $\tilde{R}$ be their respective curvature tensor fields. Then if we write

$$X_1 = l_1 X, \quad X_2 = l_2 X, \quad \text{etc.,}$$

we have from direct computations,

$$\tilde{R}(X, Y)Z_1 - R(X, Y)Z_1 = 2\{(\nabla_X \vartheta)(Y_1) Z_1 - (\nabla_Y \vartheta)(X_1) Z_1 + (\nabla_X \vartheta)(Z_1) Y_1 - \\ - (\nabla_Y \vartheta)(Z_1) X_1 - 2\vartheta(X_1)\vartheta(Z_1) Y_1 + 2\vartheta(Y_1)\vartheta(Z_1) X_1\}$$

and

$$\tilde{R}(X, Y)Z_2 - R(X, Y)Z_2 \\ = 2\{(\nabla_X \vartheta)(Y_2) Z_2 - (\nabla_Y \vartheta)(X_2) Z_2 + (\nabla_X \vartheta)(Z_2) Y_2 - \\ - (\nabla_Y \vartheta)(Z_2) X_2 - 2\vartheta(X_2)\vartheta(Z_2) Y_2 + 2\vartheta(Y_2)\vartheta(Z_2) X_2\},$$

from which if we write

$$\begin{aligned} A(Y, Z) &= -(\nabla_Y \vartheta) Z + \vartheta(Y)\vartheta(Z) + \vartheta(JY)\vartheta(JZ), \\ A_1(Y, Z) &= A(Y, Z_1) = -(\nabla_Y \vartheta) Z_1 + 2\vartheta(Y_1)\vartheta(Z_1), \\ A_2(Y, Z) &= A(Y, Z_2) = -(\nabla_Y \vartheta) Z_2 + 2\vartheta(Y_2)\vartheta(Z_2), \\ B(X, Y)Z &= \tilde{R}(X, Y)Z - R(X, Y)Z, \end{aligned} \tag{2.4.2}$$

we obtain

$$B_1(X, Y)Z = B(X, Y)Z_1 \\ = 2\{A_1(Y, Z) X_1 - A_1(X, Z) Y_1 - A_1(X, Y) Z_1 + A_1(Y, X) Z_1\},$$

and

$$B_2(X, Y)Z = B(X, Y)Z_2 \\ = 2\{A_2(Y, Z) X_2 - A_2(X, Z) Y_2 - A_2(X, Y) Z_2 + A_2(Y, X) Z_2\}.$$

Now let B denote an arbitrary (1.3) tensor field on M; we write, with the usual notations

$$S_B(X, Y) = e^i(B(e_i, X) Y), \quad S_{1_B}(X, Y) = S_B(X, Y_1),$$
$$S_{B_2}(X, Y) = S_B(X, Y_2).$$

In particular, for B given by (2.4.2) we have

$$(2.4.3)\quad S_{B_1}(Y, Z)$$
$$= S_{1_B}(Y, Z) = 2\{r_1 A_1(Y, Z) - A_1(Y_1, Z) - A_1(Z_1, Y) + A_1(Y, Z)\}$$
$$= -2A_1(Z_1, Y) + (2r_1+1)A_1(Y, Z) - A_1(JY, Z),$$

and also

$$(2.4.4)\quad S_{B_2}(Y, Z) = 2\{r_2 A_2(Y, Z) - A_2(Y_2, Z) - A_2(Z_2, Y) + A_2(Y, Z)\}$$
$$= -2A_2(Z_2, Y) + (2r_2+1)A_2(Y, Z) + A_2(JY, Z).$$

Solving for A_1 in terms of S_{B_1} and for A_2 in terms of S_{B_2} we put

$$A_1(Y, Z) = \lambda S_{B_1}(Y, Z) + \mu S_{B_1}(JY, Z) + \nu S_{B_1}(Z_1, Y),$$
$$A_2(Y, Z) = \lambda' S_{B_2}(Y, Z) + \mu' S_{B_2}(JY, Z) + \nu' S_{B_2}(Z_2, Y).$$

Substituting the values of S_{B_1} and S_{B_2} given by (2.4.3) and (2.4.4) we obtain:

$$\lambda = \frac{2r_1-1}{4(r_1^2-1)},\quad \mu = \frac{1}{4(r_1^2-1)},\quad \nu = \frac{1}{2(r_1^2-1)},$$
$$\lambda' = \frac{2r_2-1}{4(r_2^2-1)},\quad \mu' = \frac{-1}{4(r_2^2-1)},\quad \nu' = \frac{1}{2(r_2^2-1)}.$$

Consequently,

$$B_1(X, Y)Z = 2[\{\lambda S_{B_1}(Y, Z) + \mu S_{B_1}(JY, Z) + \nu S_{B_1}(Z_1, Y)\} X_1 -$$
$$- \{\lambda S_{B_1}(X, Z) + \mu S_{B_1}(JX, Z) + \nu S_{B_1}(Z_1, X)\} Y_1 -$$
$$- \{\lambda S_{B_1}(X, Y) + \mu S_{B_1}(JX, Y) + \nu S_{B_1}(Y_1, X)\} Z_1 +$$
$$+ \{\lambda S_{B_1}(Y, X) + \mu S_{B_1}(JY, X) + \nu S_{B_1}(X_1, Y)\} Z_1],$$

and since $S_{B_1} = S_{1_{\tilde{R}}} - S_{1_R}$, we have that the tensor field P_1 defined by

$$(2.4.5)\quad P_1(X, Y)Z = R(X, Y)Z_1 -$$
$$-2[\{\lambda S_{1_R}(Y, Z) + \mu S_{1_R}(JY, Z) + \nu S_{1_R}(Z_1, Y)\} X_1 -$$
$$- \{\lambda S_{1_R}(X, Z) + \mu S_{1_R}(JX, Z) + \nu S_{1_R}(Z_1, X)\} Y_1 -$$
$$- \{\lambda S_{1_R}(X, Y) + \mu S_{1_R}(JX, Y) + \nu S_{1_R}(Y_1, X)\} Z_1 +$$
$$+ \{\lambda S_{1_R}(Y, X) + \mu S_{1_R}(JY, X) + \nu S_{1_R}(X_1, Y)\} Z_1]$$

is a J-projective invariant, well defined if $0 \leqslant r_1 \neq 1$. In the same way we obtain the J-projective invariant P_2 whenever $0 \leqslant r_2 \neq 1$.

DEFINITION 2.4.1. We call *J-projective curvature tensor* of a torsionless J-connection ∇ to the tensor field

$$P = P_1 + P_2.$$

It is not difficult to prove that P coincides with the tensor P in Section 3 of Prvanović [5].

Let ∇ be a torsionless J-connection and R its curvature. We will use the following notations:

$$R_1(X, Y)Z = R(X, Y)Z_1;$$

$$S_1(Y, Z) = e^i(R(e_i, Y)Z_1);$$

$$A_1(Y, Z) = \lambda S_1(Y, Z) + \mu S_1(JY, Z) + \nu S_1(Z_1, Y);$$

$$P_1(X, Y)Z = R_1(X, Y)Z - $$
$$-2[A_1(Y, Z)X_1 - A_1(X, Z)Y_1 - A_1(X, Y)Z_1 + A_1(Y, X)Z_1];$$

$$P_1^1(X, Y)Z = P_1(X_1, Y_1)Z = P(X_1, Y_1)Z_1;$$

$$Q(X, Y, Z) = (\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z);$$

$$Q_1(X, Y, Z) = (\nabla_X A_1)(Y, Z) - (\nabla_Y A_1)(X, Z);$$

$$Q_1^1(X, Y, Z) = (\nabla_{X_1} A_1)(Y_1, Z) - (\nabla_{Y_1} A_1)(X_1, Z);$$

and similar expressions for R_2, S_2, etc.

PROPOSITION 2.4.2. (i) *If* $r_1 = 2$, *then* $P_1^1 = 0$ *and* Q_1^1 *is a J-projective invariant.*

(ii) *If* $r_2 = 2$, *then* $P_2^2 = 0$ *and* Q_2^2 *is a J-projective invariant.*

Proof. (i) From (2.4.5) we can compute

$$P_1^1(X, Y)Z = P_1(X_1, Y_1)Z = R(X_1, Y_1)Z_1 - \tag{2.4.6}$$
$$-\frac{1}{r_1+1}\left| \frac{1}{r_1-1}\{r_1 S(Y_1, Z_1) + S(Z_1, Y_1)\} X_1 - \right.$$
$$-\frac{1}{r_1-1}\{r_1 S(X_1, Z_1) + S(Z_1, X_1)\} Y_1 -$$
$$\left. - \{S(X_1, Y_1) - S(Y_1, X_1)\} Z_1 \right|.$$

If ∇^1 denotes the connection induced by ∇ on each leaf of the foliation L_1, then the expression (2.4.6) is precisely that of the classical projective curvature tensor of ∇^1. Hence, for $r_1 = 2$, we have $P_1^1 = 0$.

Now, suppose that ∇ and $\tilde{\nabla}$ are J-projectively related as in (2.4.1). Then

$$\tilde{A}_1(Y_1, Z_1) - A_1(Y_1, Z_1) = -(\nabla_{Y_1} \vartheta)Z_1 + 2\vartheta(Y_1)\vartheta(Z_1).$$

After computation,

$$(\tilde{\nabla}_{X_1}\tilde{A}_1)(Y_1, Z_1)-(\tilde{\nabla}_{Y_1}\tilde{A}_1)(X_1, Z_1)$$
$$=(\nabla_{X_1}A_1)(Y_1, Z_1)-(\nabla_{Y_1}A_1)(X_1, Z_1)+\vartheta\big(P(X_1, Y_1)Z_1\big).$$

And we obtain our claim because the last term is zero when $r_1 = 0$.
(ii) Analogous to (i). q.e.d.

In order to study a symmetry condition for P (see Proposition 2.4.4) that will also become useful in Section 3, we consider the following condition upon R:

$$R(JX, X)X = aX+bJX, \tag{2.4.7}$$

meaning that this relation is fulfilled for every vector $X \in TM$ with $a, b \in \mathbf{R}$ dependent on X.

PROPOSITION 2.4.3. *Let ∇ be a J-connection. Then, its curvature R satisfies (2.4.7) iff there are 2-covariant symmetric tensor fields a, b on M such that*

(i) $a(X_1, Y_1) = a(X_2, Y_2) = b(X_1, Y_1) = b(X_2, Y_2) = 0;$

(ii) $R(X_1, Y_2)Z_1+R(Z_1, Y_2)X_1 = \{a(X_1, Y_2)+b(X_1, Y_2)\}Z_1+$
$+\{a(Z_1, Y_2)+b(Z_1, Y_2)\}X_1;$

(iii) $R(X_1, Y_2)Z_2+R(X_1, Z_2)Y_2 = \{a(X_1, Y_2)-b(X_1, Y_2)\}Z_2+$
$+\{a(X_1, Z_2)-b(X_1, Z_2)\}Y_2.$

Proof. If (2.4.7) holds it is clear that there are 2-covariant tensor fields a, b on M, which we can always consider symmetric, such that

$$R(JX, X)X = a(X, X)X+b(X, X)JX.$$

Let $\{e_a, e_u\}$ be an adapted frame, that is, $Je_a = e_a$, $Je_u = -e_u$, and let $\{e^a, e^u\}$ be its dual coframe. If $X = X^a e_a+X^u e_u$, substitution on the above formula gives (i), (ii), (iii) by identification of the coefficients of the resulting polynomial in the components of X.

Conversely, if (i), (ii), (iii) are satisfied, we obtain

$$R(JX, X)X = R(X_1-X_2, X_1+X_2)(X_1+X_2)$$
$$= 2a(X_1, X_2)X+2b(X_1, X_2)JX. \quad \text{q.e.d.}$$

PROPOSITION 2.4.4. *A torsionless J-connection ∇ on an almost-product manifold (M, J) with $r_1, r_2 \neq 1$, satisfies condition (2.4.7) iff its J-projective curvature tensor P verifies*

$$P(JX, JY)Z = P(X, Y)Z. \tag{2.4.8}$$

Proof. For every vector field X on M we have, with the usual notations, $X = X_1+X_2$. Then it is immediate, since $P(X, Y) = -P(Y, X)$, that condition (2.4.8) is equivalent to

$$P(X_1, Y_2)Z_1 = P(X_1, Y_2)Z_2 = 0.$$

But we have, in account of (2.4.5), that

$$\begin{aligned} P(X_1, Y_2)Z_1 &= P_1(X_1, Y_2)Z \\ &= R(X_1, Y_2)Z_1 - \frac{1}{r_1+1}\{S(Y_2, Z_1)X_1 + S(Y_2, X_1)Z_1\} \end{aligned}$$

and, similarly,

$$\begin{aligned} P(X_1, Y_2)Z_2 &= P_2(X_1, Y_2)Z \\ &= R(X_1, Y_2)Z_2 + \frac{1}{r_2+1}\{S(X_1, Z_2)Y_2 + S(X_1, Y_2)Z_2\}. \end{aligned}$$

Consequently, if $P(X_1, Y_2)Z_1 = P(X_1, Y_2)Z_2 = 0$,

$$\begin{aligned} R(JX, X)X &= 2[R(X_1, X_2)X_1 + R(X_1, X_2)X_2] \\ &= 2\left[\frac{1}{r_1+1}\{S(X_2, X_1)X_1 + S(X_2, X_1)X_1\} - \right. \\ &\qquad \left. - \frac{1}{r_2+1}\{S(X_1, X_2)X_2 + S(X_1, X_2)X_2\}\right] \\ &= \frac{2}{r_1+1}S(X_2, X_1)(X+JX) - \frac{2}{r_2+1}S(X_1, X_2)(X-JX), \end{aligned}$$

and thus we have condition (2.4.7).

Conversely, if ∇ satisfies this condition, we have, applying the results of Proposition 2.4.3, again with the usual notations, and from the 1st Bianchi identity,

$$\begin{aligned} & e^a(R(e_a, Y_2)Z_1 + R(Z_1, Y_2)e_a) \\ &\qquad = e^a(R(e_a, Y_2)Z_1 - R(Y_2, e_a)Z_1 - R(e_a, Z_1)Y_2) \\ &\qquad = 2S(Y_2, Z_1) = (1+r_1)\{a(Z_1, Y_2) + b(Z_1, Y_2)\}, \end{aligned}$$

that is

$$S(Y_2, Z_1) = \frac{r_1+1}{2}\{a(Z_1, Y_2) + b(Z_1, Y_2)\}. \tag{2.4.9}$$

Analogously we obtain

$$S(Y_1, Z_2) = -\frac{r_2+1}{2}\{a(Y_1, Z_2) - b(Y_1, Z_2)\}. \tag{2.4.10}$$

But, since ∇ is torsionless, we have

$$\begin{aligned} R(X_1, Y_2)Z_1 + R(Z_1, Y_2)X_1 &= R(X_1, Y_2)Z_1 - R(Y_2, X_1)Z_1 - R(X_1, Z_1)Y_2 \\ &= 2R(X_1, Y_2)Z_1 - R(X_1, Z_1)Y_2 \\ &= \{a(X_1, Y_2) + b(X_1, Y_2)\}Z_1 + \\ &\qquad + \{a(Z_1, Y_2) + b(Z_1, Y_2)\}X_1. \end{aligned}$$

Identifying the components on L_1 and L_2, we get

$$\begin{aligned} &2R(X_1, Y_2)Z_1 = \{a(X_1, Y_2) + b(X_1, Y_2)\}Z_1 + \\ &\qquad\qquad + \{a(Z_1, Y_2) + b(Z_1, Y_2)\}X_1, \\ &R(X_1, Z_1)Y_2 = 0. \end{aligned} \tag{2.4.11}$$

Similarly,

$$\begin{aligned} &2R(X_1, Y_2)Z_2 = \{a(X_1, Y_2) - b(X_1, Y_2)\}Z_2 + \\ &\qquad\qquad + \{a(Z_1, Z_2) - b(X_1, Z_2)\}Y_2, \\ &R(X_2, Z_2)Y_1 = 0. \end{aligned} \tag{2.4.12}$$

Substituting (2.4.9)–(2.4.12) in the expressions of $P(X_1, Y_2)Z_1$ and $P(X_1, Y_2)Z_2$, we deduce

$$\begin{aligned} P(X_1, Y_2)Z_1 = R(X_1, Y_2)Z_1 - \frac{1}{r_1+1}\Big[&\frac{r_1+1}{2}\{a(Z_1, Y_2) + b(Z_1, Y_2)\}X_1 + \\ &+ \frac{r_1+1}{2}\{a(X_1, Y_2) + b(X_1, Y_2)\}Z_1\Big] = 0, \end{aligned}$$

and

$$\begin{aligned} P(X_1, Y_2)Z_2 = R(X_1, Y_2)Z_2 + \frac{1}{r_2+1}\Big[&-\frac{r_2+1}{2}\{a(X_1, Z_2) + b(X_1, Z_2)\}Y_2 - \\ &-\frac{r_2+1}{2}\{a(X_1, Y_2) - b(X_1, Y_2)\}Z_2\Big] = 0. \quad \text{q.e.d.} \end{aligned}$$

2.5. Torsionless J-projectively flat J-connections.

THEOREM 2.5.1. *A torsionless J-connection on the almost-product manifold (M, J) is J-projectively flat iff*:

(i) $P = 0$, *when* $r_1, r_2 \notin \{1, 2\}$;
(ii) $P = Q_1^1 = 0$, *when* $r_1 = 2$, $r_2 \notin \{1, 2\}$;
(iii) $P = Q_2^2 = 0$, *when* $r_2 = 2$, $r_1 \notin \{1, 2\}$;
(iv) $P = Q_1^1 = Q_2^2 = 0$, *when* $r_1 = r_2 = 2$;
(v) $P_1 = 0$, *when* $r_2 = 1$, $r_1 \notin \{1, 2\}$;

(vi) $P_2 = 0$, *when* $r_1 = 1$, $r_2 \notin \{1, 2\}$;
(vii) $P_1 = Q_1^1 = 0$, *when* $r_1 = 2$, $r_2 = 1$;
(viii) $P_2 = Q_2^2 = 0$, *when* $r_1 = 1$, $r_2 = 2$.

Moreover, if $r_1 = r_2 = 1$, *every torsionless* J*-connection is* J*-projectively flat.*

Proof. (i) If such a connection ∇ is J-projectively flat, then there exists at least locally a torsionless J-connection $\tilde{\nabla}$, J-projectively related with ∇, such that $\tilde{R} = 0$; but then $\tilde{S} = 0$, from which $\tilde{A} = 0$ and so $\tilde{P} = 0$, hence, $P = 0$.

Conversely, if $P = 0$., then

$$R(X, Y)Z = 2\{A_1(Y, Z)X - A_1(X, Z)Y - A_1(X, Y)Z + A_1(Y, X)Z + \\ + A_2(Y, Z)X - A_2(X, Z)Y - A_2(X, Y)Z + A_2(Y, X)Z\}.$$

If there exists locally a 1-form ϑ such that

$$A(Y, Z) = (\nabla_Y \vartheta)Z - \vartheta(Y)\vartheta(Z) - \vartheta(JY)\vartheta(JZ), \tag{2.5.1}$$

then ∇ is J-projectively flat. Indeed, if (2.5.1) is satisfied we deduce that B, defined as $B(X, Y)Z = \tilde{R}(X, Y)Z - R(X, Y)Z$ is equal to $-R(X, Y)Z$, from which $\tilde{R}(X, Y) = 0$.

We write (2.5.1) as

$$(\nabla_Y \vartheta)Z = A(Y, Z) + \vartheta(Y)\vartheta(Z) + \vartheta(JY)\vartheta(JZ),$$

from which we obtain, since the torsion of ∇ is zero, the integrability condition

$$R(X, Y)\vartheta = (\nabla_X A)(Y, \ ) - (\nabla_Y A)(X, \ ) + (\nabla_X \vartheta)(Y)\vartheta - (\nabla_Y \vartheta)(X)\vartheta + \\ + \vartheta(Y)\nabla_X \vartheta - \vartheta(X)\nabla_Y \vartheta + (\nabla_X \vartheta)(JY)\vartheta J - \\ - (\nabla_Y \vartheta)(JX)\vartheta J + \vartheta(JY)(\nabla_X \vartheta)J - \vartheta(JX)(\nabla_Y \vartheta)J.$$

Contracting with Z we obtain the integrability condition

$$(\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) = 0 \tag{2.5.2}$$

that is,

$$Q(X, Y, Z) = 0.$$

We note that

$$Q(X, Y, Z) = -Q(Y, X, Z).$$

Substituting in the 2nd Bianchi identity

$$(\nabla_X R)(Y, Z)W + (\nabla_Z R)(X, Y)W + (\nabla_Y R)(Z, X)W = 0 \tag{2.5.3}$$

the expression of R, we get

$$(2.5.4)\quad \begin{aligned} &Q_1(X, Z, W)Y_1+Q_2(X, Z, W)Y_2+Q_1(Y, X, W)Z_1+\\ &+Q_2(Y, X, W)Z_2+Q_1(Y, X, W)W_1+Q_2(Y, X, Z)W_2+\\ &+Q_1(X, Z, Y)W_1+Q_2(X, Z, Y)W_2+Q_1(Z, Y, W)X_1+\\ &+Q_2(Z, Y, W)X_2+Q_1(Z, Y, X)W_1+Q_2(Z, Y, X)W_2=0. \end{aligned}$$

From this we have by contraction, and with the usual notations,

$$(2.5.5)\quad \begin{aligned} 0&=e^i\{Q_1(X, Z, e_i)Y_1+Q_2(X, Z, e_i)Y_2+Q_1(Y, X, e_i)Z_1+\\ &+Q_2(Y, X, e_i)Z_2+Q_1(Y, X, Z)e_a+Q_2(Y, X, Z)e_u+\\ &+Q_1(X, Z, Y)e_a+Q_2(X, Z, Y)e_u+Q_1(Z, Y, e_i)X_1+\\ &+Q_2(Z, Y, e_i)X_2+Q_1(Z, Y, X)e_a+Q_2(Z, Y, X)e_u\\ &=\underset{X,Y,Z}{\mathrm{cicl}}\{(1+r_1)Q_1(X, Z, Y)+(1+r_2)Q_2(X, Z, Y)\}. \end{aligned}$$

Substituting now in (2.5.4) W by JW, and contracting again we get

$$(2.5.6)\quad \begin{aligned} 0&=e^i\{Q_1(X, Z, Je_i)Y_1+Q_2(X, Z, Je_i)Y_2+Q_1(Y, X, Je_i)Z_1\\ &+Q_2(Y, X, Je_i)Z_2+Q_1(Y, X, Z)e_a-Q_2(Y, X, Z)e_u\\ &+Q_1(X, Z, Y)e_a-Q_2(X, Z, Y)e_u+Q_1(Z, Y, Je_i)X_1\\ &+Q_2(Z, Y, Je_i)X_2+Q_1(Z, Y, X)e_a-Q_2(Z, Y, X)e_u\}\\ &=\underset{X,Y,Z}{\mathrm{cicl}}\{(1+r_1)Q_1(X, Z, Y)-(1+r_2)Q_2(X, Z, Y)\}. \end{aligned}$$

From (2.5.5) and (2.5.6) we obtain

$$(2.5.7)\qquad \underset{X,Y,Z}{\mathrm{cicl}}\, Q_1(X, Y, Z)=0, \qquad \underset{X,Y,Z}{\mathrm{cicl}}\, Q_2(X, Y, Z)=0,$$

from which

$$\underset{X,Y,Z}{\mathrm{cicl}}\, Q(X, Y, Z)=0.$$

On the other hand, also from (2.5.4) we obtain, from (2.5.7),

$$(2.5.8)\quad \begin{aligned} 0&=e^i\{Q_1(X, e_i, W)Y_1+Q_2(X, e_i, W)Y_2+Q_1(Y, X, W)e_a+\\ &+Q_2(Y, X, W)e_u+Q_1(Y, X, e_i)W_1+Q_2(Y, X, e_i)W_2+\\ &+Q_1(X, e_i, Y)W_1+Q_2(X, e_i, Y)W_2+ \end{aligned}$$

$$+Q_1(e_i, Y, W)X_1+Q_2(e_i, Y, W)X_2+Q_1(e_i, Y, X)W_1+$$
$$+Q_2(e_i, Y, X)W_2\}$$
$$=Q_1(X, Y_1, W)+Q_1(X_1, Y, W)+r_1 Q_1(Y, X, W)+$$
$$+Q_2(X, Y_2, W)+Q_2(X_2, Y, W)+r_2 Q_2(Y, X, W).$$

Let $W \in L_1$. We then have

$$Q_1(X, Y_1, W)+Q_1(X_1, Y, W)+r_1 Q_1(Y, X, W)=0.$$

For X, $Y \in L_1$ we deduce that, if $r_1 \neq 2$,

$$Q_1(Y_1, X_1, W)=0.$$

If $X \in L_1$, $Y \in L_2$, we obtain from (2.5.8)

$$Q_1(X_1, Y_2, W)+r_1 Q_1(Y_2, X_1, W)=0,$$

that is,

$$Q_1(X_1, Y_2, W)=0.$$

If X, $Y \in L_2$ on (2.5.7) we get

$$Q_1(X_2, Y_2, W)=0.$$

Whence, if $W \in L_1$ we deduce that for $r_1 \notin \{1, 2\}$, we have

$$Q_1(X, Y, W)=0.$$

Similarly, if $W \in L_2$, and $r_2 \notin \{1, 2\}$, we get

$$Q_2(X, Y, W)=0.$$

Thus, if r_1, $r_2 \notin \{1, 2\}$, we have that $P=0$ implies $Q=0$.

(ii) Suppose now $r_1=2$ and $r_2 \notin \{1, 2\}$. If $P=0$, then from the proof of Proposition 2.4.4 we have $A(Y_1, Z_2)=A(Z_2, Y_1)=0$.

Indeed, from $P(X_1, Y_2)Z_1=0$ we have (see (2.4.11))

$$\begin{aligned}R(X_1, Y_2)Z_1 &= \tfrac{1}{2}[\{a(X_1, Y_2)+b(X_1, Y_2)\}Z_1+\{a(Z_1, Y_2)+b(Z_1, Y_2)\}X_1] \\ &= -R(Y_2, X_1)Z_1 \\ &= -\tfrac{1}{2}[\{a(X_1, Y_2)+b(X_1, Y_2)\}Z_1+\{a(Z_1, X_1)+b(Z_1, X_1)\}Y_2] \\ &= -\tfrac{1}{2}\{a(X_1, Y_2)+b(X_1, Y_2)\}Z_1.\end{aligned}$$

Taking now X_1, Z_1 independent, we deduce

$$a(X_1, Y_2)+b(X_1, Y_2)=0.$$

Similarly, from $P(X_1, Y_2)Z_2=0$, we obtain (see (2.4.12))

$$a(X_1, Y_2)-b(X_1, Y_2)=0$$

and thus $a = b = 0$. Hence, from (2.4.9) and (2.4.10) we have $S(X_1, Y_2) = S(Y_2, X_1) = 0$, and thus $A(X_1, Y_2) = A(Y_2, X_1) = 0$, for every X_1, Y_2.

Since by hypothesis $Q_1^1 = 0$, it only rests to prove that

$$Q(X_2, Y_1, Z_1) = Q(X_1, Y_2, Z_2) = Q(X_2, Y_2, Z_2) = 0,$$

that is,

$$Q(X_2, Y_2, Z_2) = 0, \quad (\nabla_{X_2} A)(Y_1, Z_1) = 0 \quad \text{and} \quad (\nabla_{X_1} A)(Y_2, Z_2) = 0.$$

As for the first one, we consider that since $P(X_2, Y_2)Z_2 = 0$, from (2.5.4) it follows

$$\begin{aligned}Q(X_2, Z_2, W_2)Y_2 - Q(X_2, Y_2, W_2)Z_2 - Q(X_2, Y_2, Z_2)W_2 + \\ + Q(X_2, Z_2, Y_2)W_2 + Q(Z_2, Y_2, W_2)X_2 + Q(Z_2, Y_2, X_2)W_2 = 0.\end{aligned}$$

Putting $W_2 = Z_2$, and being X_2, Y_2, Z_2 independent we deduce

$$\begin{aligned}Q(X_2, Z_2, Z_2)Y_2 - Q(X_2, Y_2, Z_2)Z_2 - Q(X_2, Y_2, Z_2)Z_2 + \\ + Q(X_2, Z_2, Y_2)Z_2 + Q(Z_2, Y_2, Y_2)X_2 + Q(Z_2, Y_2, X_2)Z_2 = 0.\end{aligned} \tag{2.5.9}$$

In particular

$$Q(X_2, Z_2, Z_2) = 0,$$

that is, for arbitrary fields,

$$Q(X_2, Y_2, Z_2) = -Q(X_2, Z_2, Y_2).$$

Since, moreover,

$$Q(X_2, Y_2, Z_2) = -Q(Y_2, X_2, Z_2),$$

we obtain, from the coefficient of Z_2 in (2.5.9),

$$\begin{aligned}0 &= -2Q(X_2, Y_2, Z_2) + Q(X_2, Z_2, Y_2) - Q(Y_2, Z_2, X_2) \\ &= 3Q(X_2, Z_2, Y_2) + Q(Y_2, X_2, Z_2) = 4Q(X_2, Y_2, Z_2).\end{aligned}$$

As for the second one, we have from $a = b = 0$ that

$$R(X, Y) = R(X_1, Y_1)Z_1 + R(X_2, Y_2)Z_2$$

and thus we obtain

$$\begin{aligned}0 &= (\nabla_{W_2} R)(X_1, Y_1)Z_1 + (\nabla_{X_1} R)(Y_1, W_2)Z_1 + (\nabla_{Y_1} R)(W_2, X_1)Z_1 \\ &= (\nabla_{W_2} R)(X_1, Y_1)Z_1.\end{aligned}$$

But

$$(\nabla_{W_2} R)(X_1, Y_1)Z_1 = 2\{(\nabla_{W_2} A)(Y_1, Z_1)X_1 - (\nabla_{W_2} A)(X_1, Z_1)Y_1 - \\ -(\nabla_{W_2} A)(X_1, Y_1)Z_1 + (\nabla_{W_2} A)(Y_1, X_1)Z_1\} = 0.$$

If we take $Z = Y$, we deduce

$$(\nabla_{W_2} A)(Y_1, Y_1)X_1 - (\nabla_{W_2} A)(X_1, Y_1)Y_1 - \\ -(\nabla_{W_2} A)(X_1, Y_1)Y_1 + (\nabla_{W_2} A)(Y_1, X_1)Y_1 = 0,$$

from which we obtain

$$(2.5.10) \qquad (\nabla_{W_2} A)(Y_1, Y_1) = 0,$$

and also

$$(2.5.11) \qquad 2(\nabla_{W_2} A)(X_1, Y_1) - (\nabla_{W_2} A)(Y_1, X_1) = 0,$$

because $r_2 \neq 1$.

From (2.5.10) we deduce that $\nabla_{W_2} A$ is antisymmetric, and so, from (2.5.11) we get

$$(\nabla_{W_2} A)(X_1, Y_1) = 0.$$

The proof of the 3rd one is similar.

(iii) The proof is analogous to that of case (ii).

(iv) This case follows from (ii) and (iii), since we can apply the previous proofs, in account of the fact that $r_1, r_2 \neq 1$.

(v) If $r_2 = 1$ and $r_1 \notin \{1, 2\}$, then P_2 is not defined, but P_1 is well defined. Before considering the condition $P_1 = 0$, we note that there exists a local nonvanishing vector field U of L_2 which is basic, that is, which verifies

$$[X_1, U] \in L_1$$

for each $X_1 \in L_1$.

For an arbitrary X we have

$$\begin{aligned} l_2(\nabla_X U - \nabla_U X - [X, U]) &= l_2(\nabla_{X_1} U + \nabla_{fU} U - \nabla_U fU - [fU, U]) \\ &= \nabla_{X_1} U + l_2(f\nabla_U U - (Uf)U - f\nabla_U U - \\ &\quad - f[U, U] + (Uf)U) = \nabla_{X_1} U = 0. \end{aligned}$$

Accordingly, there exists a 1-form β_2 such that $\beta_2(X) = \beta_2(l_2 X)$ and

$$\nabla_X U = \beta_2(X)U.$$

Moreover, it is immediate that

$$R(X, Y)U = (d\beta_2)(X, Y)U.$$

So, in general we have

$$(2.5.12) \qquad R(X, Y)Z_2 = (d\beta_2)(X, Y)Z_2.$$

Now, if $P_1 = 0$, then

$$(2.5.13) \qquad R_1(X, Y)Z$$
$$= 2\{A_1(Y, Z)X_1 - A_1(X, Z)Y_1 - A_1(X, Y)Z_1 + A_1(Y, X)Z_1\}.$$

If there exists locally a 1-form ϑ such that

$$(2.5.14) \qquad A_1(Y, Z) = (\nabla_Y \vartheta)Z_1 - 2\vartheta(Y_1)\vartheta(Z_1),$$

then there exists a torsionless J-connection $\tilde{\nabla}$ J-projectively related with ∇ such that $\tilde{R}_1(X, Y)Z = 0$. But then, in a way parallel to case (i), we have that the integrability condition of (2.5.14) is

$$(\nabla_X A_1)(Y, Z) - (\nabla_Y A_1)(X, Z) = 0,$$

that is,

$$Q_1(X, Y, Z) = 0.$$

But, since $P_1 = 0$, by using the same method as in (i), that is, applying the expression (2.5.13) of R_1 in (2.5.3) for $W = W_1 \in L_1$, we deduce the Q_1 part of (2.5.4)–(2.5.8), from which we have $Q_1 = 0$ for $r_1 \notin \{1, 2\}$.

Hence, if we define the 1-form ϑ_1 by

$$\vartheta_1(X) = \vartheta(l_1 X),$$

then the connection $\tilde{\nabla}$ given by

$$\begin{aligned}\tilde{\nabla}_X Y = {} & \nabla_X Y + \vartheta_1(X)Y + \vartheta_1(Y)X + \vartheta_1(JX)JY + \vartheta_1(JY)JX - \\ & -\tfrac{1}{4}\{\beta_2(X)Y + \beta_2(Y)X + \beta_2(JX)JY + \beta_2(JY)JX\} \\ = {} & \nabla_X Y + 2\{\vartheta_1(X)Y + \vartheta_1(Y)X\} - \beta_2(X)Y_2\end{aligned}$$

satisfies

$$\tilde{R}(X, Y)Z = 0,$$

as desired, taking account of (2.5.12).

(vi), (vii), (viii) We need only combine the previous techniques.

Finally, if $r_1 = r_2 = 1$, we have as before the 1-forms β_1 and β_2. If we consider

$$\beta = \beta_1 + \beta_2,$$

then the torsionless J-connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{4}\{\beta(X)Y + \beta(Y)X + \beta(JX)JY + \beta(JY)JX\}$$
$$= \nabla_X Y - \beta_1(X)Y_1 - \beta_2(X)Y_2$$

has zero curvature, q.e.d.

2.6. The para-Kaehlerian case. Now, we suppose that (M, J, g) is para-Kaehlerian, i.e., $J^2 = 1$, g is symmetric and nondegenerate, $g(JX, Y) + g(JY, X) = 0$, and $\nabla J = 0$ if ∇ stands for the Levi-Civita connection of g. We call ∇ the *para-Kaehlerian connection.*

Let S be the Ricci tensor of the para-Kaehlerian connection ∇. Then it is immediate to prove $S(JX, Y) + S(X, JY) = 0$. If $\dim M > 2$, then we deduce from the expressions of A_1 and A_2 in terms of S, and having in mind that $r_1 = r_2$ $(= r)$:

$$A(X, Y) = \frac{1}{n+2} S(X, Y).$$

Consequently,

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n+2}\{S(Y, Z)X - S(X, Z)Y + S(Y, JZ)JX - S(X, JZ)JY - 2S(X, JY)JZ\}, \tag{2.6.1}$$

and

$$P(JX, Y)Z + P(X, JY)Z = 0. \tag{2.6.2}$$

From (2.6.1), if $P = 0$, we obtain after calculation

$$S(X, Y) = -\frac{\varrho}{2r} g(X, Y),$$

where ϱ denotes the scalar curvature. Thus, since $\dim M > 3$, ϱ is constant. Hence, if we put $\varrho = cr(1+r)$, the Riemann–Christoffel tensor has the expression

$$R(X, Y, Z, W) = \tfrac{1}{4}c\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W) + g(X, JZ)g(JY, W) - g(Y, JZ)g(JX, W) + 2g(X, JY)g(JZ, W)\},$$

that is, (M, J, g) is a space of constant J-sectional curvature (see [2]). Moreover, since then $A(X, Y) = \tfrac{1}{4}cg(X, Y)$, we have $Q_1^1 = Q_2^2 = 0$, even in the case $n = 4$. Therefore we have proved:

PROPOSITION 2.6.1. *A para-Kaehlerian manifold (M, J, g) with $\dim M > 2$ is J-projectively flat iff $P = 0$ or, equivalently, if it is a space of constant J-sectional curvature.*

3. J-planes and axiom of J-planes

3.1. The axiom of J-planes. The structure J on an almost-product manifold (M, J) induces the automorphism J_x in $T_x M$ for each $x \in M$. We will say that a submanifold W of M is *invariant* if $T_x W$ is J-invariant for each $x \in W$.

Definition 3.1.1. Let W be an invariant 2-submanifold of (M, J) and ∇ a J-connection. We will say that W is a *J-plane* if it is totally geodesic.

Then, if $\{e_1, e_2\}$ is a local frame of tangent vectors in W, we have

$$\nabla_{e_i} e_j = \alpha_{ij}^k e_k, \quad i, j, k = 1, 2, \tag{3.1.1}$$

for certain functions α_{ij}^k, and also

$$J e_i = \beta_i^j e_j, \quad i, j = 1, 2. \tag{3.1.2}$$

Recall that if ∇ is a J-connection, then

$$\tilde{\nabla} = \nabla - \tfrac{1}{8} pT$$

is a half-torsionless J-connection. If the invariant 2-submanifold W is a solution to (3.1.1), then it also verifies the equation for the associated connection $\tilde{\nabla}$. Thus, $\tilde{\nabla}$ has all the J-planes in common ∇ with ∇, and we obtain the:

Proposition 3.1.2. *Let ∇ be an arbitrary J-connection on the almost-product manifold (M, J). Then there exists a half-torsionless J-connection which has all the J-planes in common with the given ∇.*

Furthermore, from Definition 2.2.3 we have:

Proposition 3.1.3. *If two half-torsionless J-connections are J-projectively related, then they have all the J-planes in common.*

Definition 3.1.4. We will say that a J-connection ∇ on the almost-product manifold (M, J) satisfies the *axiom of J-planes* if there exists, for each $x \in M$, and each J-invariant 2-subspace E of $T_x M$, a J-plane W such that $x \in W$ and $T_x W = E$.

From the Proposition 3.1.2 we deduce:

Proposition 3.1.5. *If the J-connection ∇ on (M, J) satisfies the axiom of J-planes, then there exists a half-torsionless J-connection which satisfies the axiom of J-planes and has all the J-planes in common with ∇.*

If we apply also the Proposition 3.1.3, it follows:

Proposition 3.1.6. *If a half-torsionless J-connection ∇ on (M, J) satisfies the axiom of J-planes, then any half-torsionless J-connection J-projectively related with ∇ also satisfies the axiom of J-planes.*

Now, we prove the following characterization:

Proposition 3.1.7. *A J-connection ∇ on (M, J) satisfies the axiom of J-*

planes iff its torsion and curvature tensors T and R satisfy, for every X, the conditions

$$T(JX, X) = aX + bJX, \tag{3.1.3}$$

$$R(JX, X)X = cX + eJX, \tag{3.1.4}$$

for certain a, b, c, e which are functions of $x \in M$ and $X \in T_x M$. Moreover, the associated half-torsionless J-connection $\tilde{\nabla} = \nabla - \frac{1}{8} pT$ also satisfies the conditions with exactly the same functions.

Proof. If ∇ satisfies the axiom, then it is immediate to obtain (3.1.3) and (3.1.4). If $JX = \pm X$, it suffices to take $a = b = c = e = 0$.

Conversely, if we have (3.1.3) and (3.1.4), let $x \in M$ and $X_0 \in T_x M$ be such that $JX_0 \neq X_0$. Consider the geodesic

$$\alpha\colon s \rightsquigarrow \alpha(s); \quad \alpha(0) = x, \quad \dot{\alpha}(0) = JX_0.$$

For each s, consider the geodesic

$$\sigma\colon t \rightsquigarrow \sigma(s, t), \quad \sigma(s, 0) = \alpha(s), \quad \dot{\sigma}(s, 0) = J\dot{\alpha}(s).$$

Thus, we have a 2-submanifold in a neighbourhood of x, with equations

$$(s, t) \rightsquigarrow \sigma(s, t).$$

Let us denote

$$u = \sigma_*(\partial/\partial t), \quad v = \sigma_*(\partial/\partial s).$$

We have

$$\nabla_u u = 0.$$

Let f and h be two smooth functions defined in the same domain as $\sigma(s, t)$. By applying (3.1.3) and (3.1.4) we have

$$\nabla_u \nabla_u (fu + hJu) + \nabla_u (T(fu + hJu, u)) + R(fu + hJu, u)u$$
$$= \left(\frac{\partial^2 f}{\partial t^2} + \frac{\partial h}{\partial t} a + h\frac{\partial a}{\partial t} + hc\right)u + \left(\frac{\partial^2 h}{\partial t^2} + \frac{\partial h}{\partial t} b + h\frac{\partial b}{\partial t} + he\right)Ju.$$

Consider the differential equations

$$\frac{\partial^2 f}{\partial t^2}(s, t) + \frac{\partial h}{\partial t}(s, t)a(u)(s, t) + h(s, t)\frac{\partial a(u)}{\partial t}(s, t) + h(s, t)c(u)(s, t) = 0,$$

and

$$\frac{\partial^2 h}{\partial t^2}(s, t) + \frac{\partial h}{\partial t}(s, t)b(u)(s, t) + h(s, t)\frac{\partial b(u)}{\partial t}(s, t) + h(s, t)e(u)(s, t) = 0.$$

If we fix s, we have two ordinary differential equations, which have a unique solution determined by

$$f(s,0)=0,\quad h(s,0)=1,\quad \frac{\partial f}{\partial t}(s,0)=-a(u)(s,0)$$

$$\text{and}\quad \frac{\partial h}{\partial t}(s,0)=-b(u)(s,0).$$

From (3.1.3), we obtain

$$(\nabla_u v)_{(s,0)}=\big(\nabla_v u-a(u)u-b(u)v\big)_{(s,0)}.$$

Also, since $v=Ju$ along the curve $\alpha(s)$, we have

$$(\nabla_v v)_{(s,0)}=(J\nabla_v u)_{(s,0)}=0.$$

Hence

$$(\nabla_v u)_{(s,0)}=0,$$

and

$$(\nabla_u v)_{(s,0)}=\big(-a(u)u-b(u)v\big)_{(s,0)}.$$

On the other hand,

$$\big(\nabla_u(fu+hJu)\big)_{(s,0)}=\left(\frac{\partial f}{\partial t}u+\frac{\partial h}{\partial t}v\right)_{(s,0)}=\big(-a(u)u-b(u)v\big)_{(s,0)}.$$

We deduce that

$$w=fu+hJu$$

is a Jacobi field, such that

$$w_{(s,0)}=v_{(s,0)}\quad\text{and}\quad(\nabla_u w)_{(s,0)}=(\nabla_u v)_{(s,0)}$$

and thus

$$w=v.$$

Hence σ is J-invariant, and it follows that

$$\nabla_u v=\nabla_u(fu+hJu)=u(f)u+u(h)Ju.$$

On the other hand, we have $\nabla_u u=0$, and from $[u,v]=0$, we deduce easily

$$T(u,v)=-h\big(a(u)u+b(u)Ju\big)=\nabla_u v-\nabla_v u,$$

whence

$$\nabla_v u=\big(u(f)+ha(u)\big)u+\big(u(h)+hb(u)\big)Ju,$$

and finally

$$\nabla_v v = \{v(f)+fu(f)+fha(u)+hu(h)+h^2 b(u)\}\, u+ \\ +\{v(h)+fu(h)+fhb(u)+hu(f)+h^2 a(u)\}\, Ju.$$

That is, σ is a J-plane.

The last claim follows by direct computation. q.e.d.

With respect to condition (3.1.3) we have also:

PROPOSITION 3.1.8. *A half-torsionless J-connection ∇ on (M, J) satisfies condition* (3.1.3) *iff*

$$T(X, Y) = -\tfrac{1}{4} N(X, Y)+\vartheta(X)Y-\vartheta(Y)X-\vartheta(JX)JY+\vartheta(JY)JX, \tag{3.1.5}$$

for certain 1-*form* ϑ.

Proof. If T satisfies (3.1.3), then by polarization we have

$$T(X, Y)-T(JX, JY) = \omega(JX)Y+\omega(Y)JX+\bar{\omega}(JX)Y+\bar{\omega}(Y)X,$$

for some 1-forms ω and $\bar{\omega}$. By the antisymmetry of T one deduces $\omega+\bar{\omega}J = 0$. Thus, if $\vartheta = -\frac{1}{2}\bar{\omega}$, we have

$$T(X, Y)-T(JX, JY) = 2\{\vartheta(X)Y-\vartheta(Y)X-\vartheta(JX)JY+\vartheta(JY)JX\}. \tag{3.1.6}$$

But if we consider the operator q on (1, 2) tensor fields defined by

$$(qA)(X, Y) = A(X, Y)-JA(X, JY)-JA(JX, Y)+A(JX, JY),$$

it is easy to prove that $N = -qT$; but, since $pT = 0$, we can write

$$N(X, Y) = -((p+q)T)(X, Y),$$

from which and from the identity

$$T(X, Y) = \tfrac{1}{4}((p+q)T)(X, Y)+\tfrac{1}{2}(T(X, Y)-T(JX, JY))$$

we obtain, substituting in (3.1.6), the desired expression (3.1.5). The converse is immediate. q.e.d.

COROLLARY 3.1.9. *If a half-torsionless J-connection ∇ on an almost-product manifold (M, J) satisfies the axiom of J-planes, then there exists a half-torsionless J-connection $\tilde{\nabla}$ on M, J-projectively related with ∇, which also satisfies the axiom of J-planes and whose torsion is*

$$\tilde{T} = -\tfrac{1}{4}N.$$

Finally, according to Proposition 2.4.4 we can state:

PROPOSITION 3.1.10. *A torsionless J-connection ∇ on an almost-product manifold (M, J), with $r_1, r_2 \neq 1$, satisfies the axiom of J-planes iff its J-*

projective curvature tensor P satisfies

$$P(JX, JY)Z = P(X, Y)Z.$$

4. Para-Kaehlerian manifolds with J-free mobility

Let π: $TM \to M$ be the tangent bundle of the manifold M.

DEFINITION 4.1. We say that the para-Kaehlerian manifold (M, J, g) admits *J-free mobility* if for each pair of vectors X, $Y \in TM$ with the same length, there is a neighbourhood U of $\pi(X)$, a neighbourhood V of $\pi(Y)$ and a J-preserving isometry of U into V which sends X into Y.

That is, M admits locally a transitive group of J-isometries. From this, if it is immediate that the J-sectional curvature is constant at a fixed point, and thus the Riemann–Christoffel curvature tensor has the expression (see [2], [5]).

$$(4.1)\quad R(X, Y, Z, W) = \tfrac{1}{4}c\,\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) - \\ - g(X, JZ)g(Y, JW) + g(X, JW)g(Y, JZ) - 2g(X, JY)g(Z, JW)\},$$

where c is a constant function on M if $\dim M > 2$.

Conversely, if R has that expression, then M admits J-free mobility. The proof can be given in a way similar to Yano [11], by using now the symmetry properties of the Riemann–Christoffel curvature tensor of the para-Kaehlerian connection. Alternatively, we can use the results in [2]: We have the model $(P_r(B), J, g)$ of spaces of constant J-sectional curvature, which is an analogue of the models of constant holomorphic sectional curvature. The para-unitarian group $U(B; r+1)$ acts transitively on $P_r(B)$ preserving J and g. Then we have locally the same property for every space of constant J-sectional curvature.

Whence, applying also the earlier results, we can state:

THEOREM 4.2. *Let (M, J, g) be a para-Kaehlerian manifold with* $\dim M > 2$. *Then the following properties are equivalent*:

(a) *M is a space of constant J-sectional curvature;*

(b) *The Riemann–Christoffel tensor field R has the expression* (4.1);

(c) *M admits J-free mobility;*

(d) *M is J-projectively flat;*

(e) *M satisfies the axiom of J-planes.*

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