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Unitary dilations of multi-parameter semi-groups of operators

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Abstract. In the present paper we present an elementary proof of the fact that every two-parameter strongly continuous semi-group of contractions has a strongly continuous unitary dilation. We prove also that certain n-parameter (n is finite or not) semi-groups of operators have regular unitary dilations.

1. Preliminaries. Let H be a Hilbert space. $L(H)$ denotes the space of all linear bounded operators in H. If H is a subspace of a Hilbert space K, then P_H denotes the (orthogonal) projection of K onto H. Let $\mathscr{S} \subset L(H)$ be a family of operators. $\langle \mathscr{S} H \rangle$ denotes the closed subspace of K spanned by all vectors Sh, $S \in \mathscr{S}$, $h \in H$.

Let G be a semi-group with unit e. A mapping $T(\cdot)\colon G \to L(H)$ is called a *representation of* G *in* H if $T(e) = I$ and $T(s \cdot t) = T(s)T(t)$ for all $s, t \in G$. A representation $T(\cdot)$ of G in H is called *unitary* if $T(s)$ is a unitary operator in $L(H)$ for all $s \in G$. If G is a topological semi-group and $T(\cdot)$ is continuous in the weak (strong, respectively) operator topology in $L(H)$, then the representation $T(\cdot)$ is called *weakly* (*strongly*, respectively) *continuous*.

Let $T(\cdot)\colon G \to L(H)$ be a mapping and let $U(\cdot)\colon G \to L(K)$ be a unitary representation of G in K. $U(\cdot)$ is called a *unitary dilation of* $T(\cdot)$ if $H \subset K$ and $T(s) = P_H U(s)|_H$ for each $s \in G$. If $K = \langle U(G) H \rangle$, then the dilation $U(\cdot)$ is called *minimal*.

The set of all mappings from a group G into K with finite support is denoted by K^G. If $h \in K^G$, then $\operatorname{supp} h = \{s \in G\colon h(s) \neq 0\}$.

It has been proved in [2], [3] that every two-parameter strongly continuous semi-group of contractions has a strongly continuous unitary dilation but both these proofs used rather advanced techniques. In Section 2 of the present paper we shall present a very elementary proof of this result in the separable case. In Section 3 we prove that certain n-parameter (n finite or not) semi-groups of operators have a regular unitary dilation.

R, R_+, Z, N stand for the sets of real, non-negative real, integer and natural numbers, respectively.

2. Unitary dilation of two-parameter semi-groups of contractions. We begin with the following result which will be of use in the sequel.

PROPOSITION 2.1. *Let G_1, G_2 be topological groups with units e_1, e_2, respectively. Let $U(\cdot,\cdot)$: $G_1 \times G_2 \to L(K)$ be a unitary representation and let $T(\cdot,\cdot)$: $G_1 \times G_2 \to L(H)$ be a mapping such that the mappings $G_1 \to L(H)$, and $G_2 \to L(H)$ given by $s \to T(s, e_2)$, and $t \to T(e_1, t)$ respectively, are continuous in the weak operator topology in $L(H)$. If $U(\cdot,\cdot)$ is a minimal unitary dilation of $T(\cdot,\cdot)$, then $U(\cdot,\cdot)$ is strongly continuous.*

Proof. Define: $U_1(s) = U(s, e_2)$ for $s \in G_1$, $U_2(t) = U(e_1, t)$ for $t \in G_2$. Put $K_1 = \langle U_1(G_1)H\rangle$ and $K_2 = \langle U_2(G_2)H\rangle$. First we prove that $U_1(\cdot)$ is weakly continuous as a mapping from G_1 to K_1. The set $\{\Sigma\, U_1(s) f(s): f \in H^{G_1}\}$ is dense in K_1 and $\|U_1(s)\| = 1$ for $s \in G_1$. Thus it is enough to prove that

$$(2.1) \quad s \to (U_1(s)U_1(s_1)h,\ U_1(s_2)h') \quad \text{is continuous for fixed } h, h' \in H,\ s_1, s_2 \in G_1.$$

But

$$(U_1(s)U_1(s_1)h,\ U_1(s_2)h') = (U_1(s_2^{-1}ss_1)h,\ h') = (U(s_2^{-1}ss_1, e_2)h,\ h') = (T(s_2^{-1}ss_1, e_2)h,\ h').$$

Hence $U_1(\cdot)$ is weakly continuous. Therefore $U_1(\cdot)$ is strongly continuous (§ 6 appendix [1]). Similarly $U_2(\cdot)$ is strongly continuous as a mapping from G_2 to K_2.

To see that $U(\cdot,\cdot)$ is weakly continuous, it is enough to prove that

$$(2.2) \quad (s, t) \to (U(s, t)U(s_1, t_1)h,\ U(s_2, t_2)h') \quad \text{is continuous for fixed } h, h' \in H,\ s_1, s_2 \in G_1,\ t_1, t_2 \in G_2.$$

We have the following inequalities:

$$\begin{aligned}
&|(U(s, t)U(s_1, t_1)h,\ U(s_2, t_2)h') - (U(s_0, t_0)U(s_1, t_1)h,\ U(s_2, t_2)h')| \\
&= |(U_1(s_2^{-1}ss_1)h,\ U_2(t_1^{-1}t^{-1}t_2)h') - (U_1(s_2^{-1}s_0s_1)h,\ U_2(t_1^{-1}t_0^{-1}t_2)h')| \\
&\leqslant |(U_1(s_2^{-1}ss_1)h,\ U_2(t_1^{-1}t^{-1}t_2)h') - (U_1(s_2^{-1}s_0s_1)h,\ U_2(t_1^{-1}t^{-1}t_2)h')| + \\
&\quad + |(U_1(s_2^{-1}s_0s_1)h,\ U_2(t_1^{-1}t^{-1}t_2)h') - (U_1(s_2^{-1}s_0s_1)h,\ U_2(t_1^{-1}t_0^{-1}t_2)h')| \\
&\leqslant \|U_2(t_1^{-1}t^{-1}t_2)h'\| \cdot \|U_1(s_2^{-1})(U_1(s) - U_1(s_0))U_1(s_1)h\| + \\
&\quad + \|U_1(s_2^{-1}s_0s_1)h\| \cdot \|U_2(t_1^{-1})(U_2(t^{-1}) - U_2(t_1^{-1}))U_2(t_2)h'\| \\
&\leqslant \|h'\| \cdot \|(U_1(s) - U_1(s_0))U_1(s_1)h\| + \|h\| \cdot \|(U_2(t^{-1}) - U_2(t_0^{-1}))U_2(t_2)h'\|.
\end{aligned}$$

This proves that $U(\cdot,\cdot)$ is strongly continuous. q.e.d.

We need the followig result:

LEMMA 2.2. *Let G be a group which is a complete metric space with an invariant metric d and with unit e. Let G_0 be a dense subgroup with the same unit e. Let $T(\cdot)$: $G \to L(H)$ be a strongly continuous mapping and let $U_0(\cdot)$: $G_0 \to L(K)$ be a strongly continuous representation, which is a unitary* *dilation of* $T(\cdot)|_{G_0}$. *Then there is a unique strongly continuous unitary dilation* $U(\cdot)\colon G \to L(K)$ *of* $T(\cdot)$ *such that* $U(\cdot)|_{G_0} = U_0(\cdot)$.

Proof. Take $k \in K$. Since $\|U_0(s)k - U_0(s_0)k\| \leqslant \|U_0(s_0)\| \cdot \|U_0(s_0^{-1}s)k - k\| = \|U_0(s_0^{-1}s)k - k\|$, if follows that $U_0(\cdot)$ is uniformly strongly continuous. We define $U(\cdot)$ as follows:

$$U(s)k = \lim_{G_0 \ni r \to s} U_0(r)k \quad \text{for } s \in G,\ k \in K. \tag{2.3}$$

This is a unique continuous extension of the uniformly strongly continuous mapping $U_0(\cdot)$. It easy to see that $U(\cdot)$ is a unitary representation. Moreover, $U(\cdot)$ is a unitary dilation of $T(\cdot)$ and $U(\cdot)|_{G_0} = U_0(\cdot)$. q.e.d.

Now we proceed to the proof of main theorem of this section.

THEOREM 2.3. *Let H be a separable complex Hilbert space and $T(\cdot,\cdot)\colon R_+^2 \to L(H)$ be a strongly continuous representation. If $T(s, t)$ is a contraction for each $s, t \geqslant 0$, then $T(\cdot,\cdot)$ has a strongly continuous unitary dilation $U(\cdot,\cdot)\colon R_+^2 \to L(K)$.*

Proof. Let $Q = \{s = k \cdot 2^{-n}\colon n \in N,\ k \in Z\}$ and put $Q_+ = Q \cap R_+$. By the Ando theorem (Theorem 6.4 of Chapter I, [4]), there are commuting unitary operators $U_{1,n}$, $U_{2,n}$, which are unitary dilations of $T(2^{-n}, 0)$, $T(0, 2^{-n})$, respectively.

For every $(s, t) \in Q^2$ we define a sequence of operators $a_n(s, t)$, $n \in N$, as follows: If $(s, t) \in Q^2$, then there is $n_{s,t}$ such that if $n \geqslant n_{s,t}$, then the pair (s, t) has the form $(k_s 2^{-n}, k_t \cdot 2^{-n})$, where $k_s, k_t \in Z$. For $n < n_{s,t}$ we define $a_n(s, t) = I_H$. If $n \geqslant n_{s,t}$, then

$$a_n(s, t) = P_H U_{1,n}^{k_s} U_{2,n}^{k_t}|_H, \quad \text{where } (s, t) = (k_s \cdot 2^{-n}, k_t \cdot 2^{-n}). \tag{2.4}$$

If n is big enough, then

$$a_n(s, t) = T(s, t) \quad \text{for } (s, t) \in Q_+^2,\ n \in N. \tag{2.5}$$

It is easy to see that

$$a_n(-s, -t) = a_n(s, t)^* \quad \text{for } (s, t) \in Q^2,\ n \in N. \tag{2.6}$$

Let $\{e_i\}_{i=0}^{\infty}$ be an orthogonal base of H. Then $|(a_n(s, t)e_i, e_j)| \leqslant 1$ for all $n, i, j \in N$, $s, t \in Q$. By the diagonalization procedure (in four dimensions), we can extract a subsequence of $(a_n(s, t)e_i, e_j)$, also indexed by n, converging for all s, t, i, j (because $N^2 \times Q^2$ is countable).

Now we define the function $\tilde{T}\colon Q^2 \to L(H)$. It is enough to define the quadratic form $(\tilde{T}(s, t)h, h')$ for $h, h' \in H$. If $h = e_i$ and $h' = e_j$ we put $(\tilde{T}(s, t)e_i, e_j) = \lim_{n\to\infty} (a_n(s, t)e_i, e_j)$ and then we extend it by linearity and continuity. It occurs that

$$(\tilde{T}(s, t)h, h') = \lim_{n\to\infty} (a_n(s, t)h, h') \quad \text{for all } h, h' \in H. \tag{2.7}$$

Now, by the polarization formula the operator $\bar{T}(s, t)$ is well defined. It is clear that $\|\bar{T}(s, t)\| \leq 1$. By (2.5)

$$\bar{T}(s, t) = T(s, t) \quad \text{for } (s, t) \in Q_+^2 \tag{2.8}$$

and by (2.6)

$$\bar{T}(-s, -t) = \bar{T}(s, t)^* \quad \text{for } (s, t) \in Q^2. \tag{2.9}$$

Now we prove that:

$$\sum_{(s,t)\in Q^2} \sum_{(s',t')\in Q^2} \big(T(s-s', t-t')h(s, t), h(s', t')\big) \geq 0 \quad \text{for each } h \in H^{Q^2}. \tag{2.10}$$

Let $h \in H^{Q^2}$. There is only a finite number, say m, of components of the form: $\big(\bar{T}(s-s', t-t')h(s, t), h(s', t')\big)$ in (2.10) which are not equal to 0. By (2.7), for any $\varepsilon > 0$ there is n such that:

$$\big|\big(\bar{T}(s-s', t-t')h(s, t), h(s', t')\big) - \big(a_n(s-s, t-t)h(s, t), h(s', t')\big)\big| \leq \varepsilon \cdot m^{-1}.$$

We take n so big that $a_n(s-s', t-t')$ is defined by (2.4).

$U_{1,n}$, $U_{2,n}$ are unitary dilations of $T(2^{-n}, 0)$, $T(0, 2^{-n})$, respectively. Thus, by Theorem 7.1 of Chapter I, [4]

$$\sum_{p,q\in N} \sum_{p',q'\in N} \big(a_n((p-p')2^{-n}, (q-q')2^{-n})h(p\cdot 2^{-n}, q\cdot 2^{-n}), h(p'\cdot 2^{-n}, q'\cdot 2^{-n})\big) \geq 0.$$

Hence $\sum_{(s,t)\in Q^2} \sum_{(s',t')\in Q^2} \big(\bar{T}(s-s', t-t')h(s, t), h(s', t')\big) \geq -\varepsilon$ and (2.10) holds. By the above quoted theorem, there is a minimal unitary dilation $U(\cdot,\cdot)\colon Q^2 \to L(K)$ of $\bar{T}(\cdot,\cdot)$, because (2.10) and (2.9) hold. By (2.8), $T(s, t) = P_H U(s, t)|_H$ for $(s, t) \in Q_+^2$. Proposition 2.1 and Lemma 2.2 finish the proof. q.e.d.

3. Regular unitary dilations of n-parameter semi-groups of operators. We shall first introduce some notation. Let Ω be a set. We denote by S the group R^{Ω}. S is a topological group with pointwise convergence topology. By S_+ we denote the set of all non-negative function in S. For $s \in S$ we put $s^+(\omega) = \max(s(\omega), 0)$, $s^-(\omega) = -\min(s(\omega), 0)$, $\omega \in \Omega$. For $v \subset \Omega$ we define a function $e_v \in S$ by $e_v(\omega) = 1$ if $\omega \in v$, otherwise $e_v(\omega) = 0$. Let $|v|$ denote the cardinality of v.

Let $T(\cdot)\colon S_+ \to L(H)$ be a representation of S_+ in H and let $U(\cdot)\colon S \to L(K)$ be a unitary representation of S in K. $U(\cdot)$ is called a *regular unitary dilation of* $T(\cdot)$ if $H \subset K$ and

$$T(s^-)^* T(s^+) = P_H U(s)|_H \quad \text{for each } s \in S. \tag{3.1}$$

We need the following lemma:

LEMMA 3.1. *Let $s_0 \in S$ and let $\{s_k\}$ be a sequence of elements of S. Let $T(\cdot)\colon S_+ \to L(H)$ be a mapping such that $T(s_k^-)$ is a contraction for all $k \in N$. If the sequence $T(s_k^-)$ converges to $T(s_0^-)$ and $T(s_k^+)$ converges to $T(s_0^+)$*

whenever $k \to \infty$ *in the strong operator topology in* $L(H)$, *then the sequence* $T(s_k^-)^* T(s_k^+)$ *converges to* $T(s_0^-)^* T(s_0^+)$ *whenever* $k \to \infty$ *in weak operator topology in* $L(H)$.

Proof. Let $h, h' \in H$. Then

$$\begin{aligned}
&|(T(s_k^-)^* T(s_k^+)h - T(s_0^-)^* T(s_0^+)h, h')| \\
&\quad \leqslant |(T(s_k^-)^* T(s_k^+)h - T(s_k^-)^* T(s_0^+)h, h')| + |(T(s_k^-)^* T(s_0^+)h - \\
&\qquad\qquad - T(s_0^-)^* T(s_0^+)h, h')| \\
&\quad \leqslant |(T(s_k^+)h - T(s_0^+)h, T(s_k^-)h')| + |(T(s_0^+)h, T(s_k^-)h' - T(s_0^-)h')| \\
&\quad \leqslant \|T(s_k^-)\| \cdot \|h'\| \cdot \|T(s_k^+)h - T(s_0^+)h\| + \|T(s_0^+)h\| \cdot \|T(s_k^-)h' - T(s_0^-)h'\| \\
&\quad \leqslant \|h'\| \cdot \|T(s_k^+)h - T(s_0^+)h\| + \|T(s_0^+)h\| \cdot \|T(s_k^-)h' - T(s_0^-)h'\|
\end{aligned}$$

This proves the desired convergence. q.e.d.

Now we prove the following theorem:

THEOREM 3.2. *Let* $T(\cdot)\colon S_+ \to L(H)$ *be a strongly continuous representation. Then the following conditions are equivalent*:

(a) $T(\cdot)$ *has a regular unitary dilation* $U(\cdot)\colon S \to L(K)$,

(b) *there is a sequence* $d_k > 0$ *converging to* 0 *such that for all* k *and all finite subsets* $u \subset \Omega$

$$A_{d_k} \overset{\mathrm{df}}{=} \sum_{v \subset u} (-1)^{|v|} T(d_k e_v)^* T(d_k e_v) \geqslant 0. \tag{3.2}$$

Moreover, such a dilation $U(\cdot)$ *can be chosen minimal. In this case the regular unitary dilation* $U(\cdot)$ *is strongly continuous and it is determined up to unitary isomorphism.*

Proof. (a)$\Rightarrow$(b). Take $d > 0$. It easy to see that $A_d(u) \geqslant 0$ for every finite subset $u \subset \Omega$, by Theorem 9.1 of Chapter I, [4].

(b)$\Rightarrow$(a). The representation $T(\cdot)\colon S_+ \to L(H)$ can be extended to a function on the whole S as follows:

$$T(s) = T(s^-)^* T(s^+) \quad \text{for } s \in S. \tag{3.3}$$

This extension is also denoted by $T(\cdot)$. It is then obvious that

$$T(-s) = T(s)^* \quad \text{for } s \in S. \tag{3.4}$$

Now we shaw that

$$\sum_{s \in S} \sum_{s' \in S} (T(s-s')h(s), h(s')) \geqslant 0 \quad \text{for all } h \in H^S. \tag{3.5}$$

Let $h \in H^S$. We can assume that d_k converges decreasingly to 0. Let us observe that if $r \in R$ and $k \in N$, then there is $n_k \in Z$ such that $|d_k \cdot n_k| \leqslant |r|$ and $|d_k \cdot n_k - r|$ has the smallest value. Then $d_k \cdot n_k \to r$ whenever $k \to \infty$, because $d_k \to 0$. If $s \in \operatorname{supp} h$ and $\omega \in \Omega$ then there is a sequence $n_k^{s(\omega)}$ of elements of Z

such that $d_k \cdot n_k^{s(\omega)} \to s(\omega)$ whenever $k \to \infty$, by the above remark. We define $n_k^s \in Z^\Omega$ as follows: $n_k^s(\omega) = n_k^{s(\omega)}$, $\omega \in \Omega$. Then $d_k \cdot n_k^s \to s$ whenever $k \to \infty$ in the topology of S. Sequences n_k^s have the following properties:

(3.6) if $s(\omega) = s'(\omega)$, then $n_k^s(\omega) = n_k^{s'}(\omega)$,

(3.7) if $s \neq s'$ and k is big enough, then $n_k^s \neq n_k^{s'}$.

Since supp h is finite, it is enough to take the tail of the sequence n_k^s; therefore we may assume that $n_k^s \neq n_k^{s'}$ for each k if $s, s' \in \operatorname{supp} h$ and $s \neq s'$.

For each k there is an operator function $T_{d_k}(\cdot)$: $Z^\Omega \ni n \to T(d_k n) \in L(H)$, and by Theorem 9.1 of Chapter I, [4] we have $\sum_n \sum_{n'} (T(d_k \cdot n - d_k \cdot n') g(n), g(n')) \geqslant 0$ for all $g \in H^{Z^\Omega}$. If $g(n_k^s) = h(s)$ for $s \in \operatorname{supp} h$ and $g(n) = 0$ for remaining n, then

$$\sum_{n_k^s} \sum_{n_k^{s'}} (T(d_k \cdot n_k^s - d_k \cdot n_k^{s'}) g(n_k^s), g(n_k^{s'})) \geqslant 0,$$

that is,

$$\sum_s \sum_{s'} (T(d_k \cdot n_k^s - d_k \cdot n_k^{s'}) h(s), h(s')) \geqslant 0. \tag{3.8}$$

Let $s_k = d_k \cdot n_k^s - d_k \cdot n_k^{s'}$. Now we show that $T(s_k^-)$ is a contraction, $k \in N$. By (3.2), $0 \leqslant A_{d_k}(\{\omega\}) = I - T(d_k \cdot e_{\{\omega\}})^* T(d_k \cdot e_{\{\omega\}})$. This means that $T(d_k \cdot e_{\{\omega\}})$ is a contraction. Then $T(s_k^-)$ is a contraction because $T(s_k^-) = T(d_k \cdot (n_k^s - n_k^{s'})^-) = \prod_{\omega \in \Omega} T(d_k \cdot e_{\{\omega\}})^{(n_k^s - n_k^{s'})^-}$

We know that $d_k \cdot n_k^s \to s$ and $d_k \cdot n_k^{s'} \to s'$; hence $s_k \to s - s' \stackrel{\text{df}}{=} s_0$, and thus $s_k^+ \to s_0^+$ and $s_k^- \to s_0^-$. By the strong continuity of $T(\cdot)$: $S_+ \to L(H)$, the sequence $T(s_k^+)$ converges to $T(s_0^+)$ and $T(s_k^-)$ converges to $T(s_0^-)$ in the strong operator topology. Now, by Lemma 3.1 and by (3.8) we get (3.5). By (3.4) and (3.5), the assumptions of Theorem 7.1 of Chapter I, [4] are satisfied. Thus there is a unitary representation $U(\cdot)$: $S \to L(K)$ (K containing H) fulfilling (3.1), and (a) holds.

By the above quoted theorem, $U(\cdot)$ can be chosen minimal. By the proof of implication (a) $\Rightarrow$ (b), $T(s)$ is a contraction for all $s \in S_+$. Now, Lemma 3.1 and the above quoted theorem imply that $U(\cdot)$ is strongly continuous on the minimal space K. In order to get the regular dilation we must extend $T(\cdot)$ to the whole S by (3.3). Therefore the unitary isomorphism between minimal dilations follows from Theorem 7.1 of Chapter I, [4].

From the above theorem and the remarks to Theorem 9.1 of Chapter I, [4] we obtain the following result:

Corollary 3.3. *Let $T(\cdot)$: $S_+ \to L(H)$ be a strongly continuous representation and suppose that there is sequence $0 < d_k \to 0$ ($k \to \infty$) satisfying one of the conditions:*

(a) $\{T(d_k e_{\{\omega\}})\}_{\omega\in\Omega}$ *is a family of isometries,*

(b) *for any finite subset* $u \subset \Omega$, $\sum_{\omega\in u} \|T(d_k e_{\{\omega\}})\|^2 \leqslant 1$,

(c) $\{T(d_k e_{\{\omega\}})\}_{\omega\in\Omega}$ *is a set of doubly commuting contractions.*

Then $T(\cdot)$ *has a regular unitary dilation* $U(\cdot)\colon S \to L(K)$.

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