ANNALES
POLONICI MATHEMATICI
XXX (1974)

On some properties of symmetric derivatives

by N. K. Kundu (Hooghly, West Bengal)

Abstract. A sufficient condition for the first symmetric derivative of a real function f is obtained under which the function f is monotone. With the help of this result. certain Mean Value Theorems for the first symmetric derivatives are strengthened and some consequences are studied.

1. Let f be a real function defined in a neighbourhood I of the interval $[a, b]$. For $x \in [a, b]$, let

$$\bar{f}^{(')}(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x-h)}{2h},$$

$$\underline{f}^{(')}(x) = \liminf_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}.$$

Then $\bar{f}^{(')}(x)$ and $\underline{f}^{(')}(x)$ are the upper and the lower symmetric derivatives of f at x. If $\bar{f}^{(')}(x) = \underline{f}^{(')}(x)$, then the common value, denoted by $f^{(')}(x)$, is the symmetric derivative of f at x.

In paper [4] Gal proved that if

(i) $\limsup_{t\to x-0} f(t) \leqslant f(x) \leqslant \limsup_{t\to x+0} f(t)$ for all x,

(ii) $D^+f \geqslant 0$ almost everywhere,

(iii) $D^+f > -\infty$, except of a countable set,

then f is non-decreasing.

In the present paper analogous theorems concerning symmetric derivatives are obtained with the help of the above theorem and certain mean value theorems for symmetric derivatives are sharpened. Some other consequences are also studied.

2. In this section we prove certain theorems which imply the monotonicity of a function.

Lemma 1. *For any function f, the set*

$$Z = \{x : \underline{f}^{(')}(x) > D^+f(x)\}$$

is at most countable.

Proof. For any fixed rational number r, let Z_r denote the set of all points x of Z such that

$$D^+f(x) < r < \underline{f}^{(\prime)}(x). \tag{1}$$

Then

$$Z = \bigcup_r Z_r, \tag{2}$$

where the union extends over the set of all rational numbers r. We shall show that the set Z_r is at most countable for each r and this will by (2) complete the proof. Let r be any fixed rational number. Define

$$\varphi(x) = f(x) - rx.$$

Then by (1), $Z_r = \{x : D^+\varphi(x) < 0 < \underline{\varphi}^{(\prime)}(x)\}$. Let $\xi \in Z_r$. Then since $D^+\varphi(\xi) < 0$, there is δ (> 0) such that

$$\varphi(\xi) > \varphi(x) \quad \text{for all } x,\ \xi < x < \xi + \delta. \tag{3}$$

Also, since $\underline{\varphi}^{(\prime)}(\xi) > 0$, there is δ' (> 0) such that

$$\varphi(\xi + h) > \varphi(\xi - h) \quad \text{for all } h,\ 0 < h < \delta'. \tag{4}$$

Taking $\delta_0 = \min(\delta, \delta')$ we have from (3) and (4)

$$\varphi(\xi + h) < \varphi(\xi) \quad \text{for all } h,\ 0 < |h| < \delta_0.$$

Hence φ assumes a strict maximum at ξ. Since ξ is an arbitrary point of Z_r, φ assumes a strict maximum at each point of Z_r. Since the set of points at which a function assumes a strict maximum is at most countable [9], p. 261, the set Z_r is at most countable.

THEOREM 1. *Let f be such that*

(i) $\limsup\limits_{x \to \xi - 0} f(x) \leqslant f(\xi) \leqslant \limsup\limits_{x \to \xi + 0} f(x)$ *for all* $\xi \in [a, b]$;

(ii) $\bar{f}^{(\prime)}(x) \geqslant 0$ *almost everywhere in* (a, b);

(iii) $\underline{f}^{(\prime)}(x) > -\infty$ *except of a countable set in* (a, b).

Then f is non-decreasing in $[a, b]$.

Proof. Condition (i) ensures that the function f is upper semicontinuous at all points off a countable set and hence f is measurable. Since f is measurable and $\underline{f}^{(\prime)}(x) > -\infty$ off a countable set, $f'(x)$ exists and is finite almost everywhere in (a, b) (cf. [5]). Since $\bar{f}^{(\prime)}(x) \geqslant 0$ almost everywhere in (a, b), we conclude that $f'(x) \geqslant 0$ almost everywhere in (a, b). Further, by Lemma 1, $D^+f(x) \geqslant \underline{f}^{(\prime)}(x)$, off a countable set, hence condition (iii) ensures that $D^+f(x) > -\infty$ off a countable set. So by the theorem of Gal [4] we conclude that f is non-decreasing in $[a, b]$. This completes the proof.

Note 1. If $\bar{f}^{(\prime)}$ is replaced by $\underline{f}^{(\prime)}$ in condition (ii) above, then we get a result of Mukhopadhyay [7].

THEOREM 2. *Let f be such that*

(i) $\liminf\limits_{x\to\xi-0} f(x) \geqslant f(\xi) \geqslant \liminf\limits_{x\to\xi+0} f(x)$ *for all* $\xi \in [a, b]$;

(ii) $\underline{f}^{(\prime)}(x) \leqslant 0$ *almost everywhere in* (a, b);

(iii) $\bar{f}^{(\prime)}(x) < \infty$ *except of a countable set in* (a, b).

Then f is non-increasing in $[a, b]$.

Proof. This can be proved by putting $f(x) = -g(x)$ and applying the result of Theorem 1.

THEOREM 3. *Let f be such that*

(i) $\lim\limits_{x\to\xi-0} f(x) = f(\xi), \liminf\limits_{x\to\xi+0} f(x) \leqslant f(\xi) \leqslant \limsup\limits_{x\to\xi+0} f(x)$

for all $\xi \in [a, b]$;

(ii) $-\infty < \underline{f}^{(\prime)}(x) \leqslant \bar{f}^{(\prime)}(x) < +\infty$ *except of a countable set in* (a, b);

(iii) *for almost all points x of* $[a, b]$ *either* $\bar{f}^{(\prime)} = 0$ *or* $\underline{f}^{(\prime)}(x) = 0$.

Then f is constant in $[a, b]$.

Proof. The proof follows from Theorem 1 and 2.

3. Mean value theorems for symmetric derivatives

LEMMA 2. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3 *and let* $f(b) > f(a)$ [*or* $f(b) < f(a)$]. *Then there is a set of points x, $a < x < b$, of positive measure such that*

$$\underline{f}^{(\prime)}(x) > 0 \qquad [\textit{or } \bar{f}^{(\prime)}(x) < 0].$$

Proof. As in Theorem 1 we conclude that, under the hypotheses, f is measurable and f' exists and is finite almost everywhere; hence $\bar{f}^{(\prime)}$ and $\underline{f}^{(\prime)}$ are measurable. Suppose that the set $\{x : \underline{f}^{(\prime)}(x) > 0\}$ is of measure zero. Then by Theorem 2, f is non-increasing in $[a, b]$ and hence $f(a) \geqslant f(b)$ which is a contradiction. The other case follows similarly.

LEMMA 3. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3 *and let* $f(a) = f(b)$. *Then there exist two sets C and D of positive measure such that* $C \cup D \subset [a, b]$ *and*

$$\underline{f}^{(\prime)}(c) \geqslant 0 \quad \textit{for all } c \in C,$$

$$\bar{f}^{(\prime)}(d) \leqslant 0 \quad \textit{for all } d \in D.$$

Proof. If f is constant in $[a, b]$, then $\underline{f}^{(\prime)}(x) = \bar{f}^{(\prime)}(x) = 0$ in (a, b) and we may take $C = D = (a, b)$.

If f is not constant in $[a, b]$, then suppose that there is a point x_0, $a < x_0 < b$, such that $f(a) < f(x_0)$. Then, by Lemma 2, there exist two sets C and D of positive measure such that $C \subset [a, x_0]$, $D \subset [x_0, b]$ and

$$\underline{f}^{(\prime)}(c) > 0 \quad \text{for all } c \in C,$$

$$\bar{f}^{(\prime)}(d) < 0 \quad \text{for all } d \in D.$$

If there is no x_0, $a < x_0 < b$, for which $f(a) < f(x_0)$, then there is x_1, $a < x_1 < b$, such that $f(a) > f(x_1)$ and by the same argument we obtain $D \subset [a, x_1]$ and $C \subset [x_1, b]$. This completes the proof.

THEOREM 4 (Quasi-mean-value theorem). *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3. *Then there exist two sets C and D of positive measure such that $C \cup D \subset [a, b]$ and*

$$\bar{f}^{(\prime)}(d) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant \underline{f}^{(\prime)}(c)$$

for all $c \in C$ and $d \in D$.

If, further, $f^{(\prime)}$ exists and has the Darboux property in (a, b), then there is $\xi \in (a, b)$ such that

$$f^{(\prime)}(\xi) = \frac{f(b)-f(a)}{b-a}.$$

Proof. The proof of the first part follows by applying Lemma 3 to the function

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x.$$

To complete the proof we see that if the equality holds for at least one $c \in C \cap (a, b)$ or one $d \in D \cap (a, b)$, then we are done. So we suppose that

$$f^{(\prime)}(d) < \frac{f(b)-f(a)}{b-a} < f^{(\prime)}(c)$$

for all $c \in C \cap (a, b)$ and $d \in D \cap (a, b)$. Since $f^{(\prime)}$ has the Darboux property there is $\xi \in (a, b)$ such that

$$f^{(\prime)}(\xi) = \frac{f(b)-f(a)}{b-a}.$$

Note 2. The above theorem sharpens a result of Aull [1] who proved that if f is continuous in $[a, b]$ and if $f^{(\prime)}$ exists in (a, b), then there are points x_1 and x_2 in (a, b) such that

$$f^{(\prime)}(x_2) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant f^{(\prime)}(x_1).$$

Since by Lemma 1 the set $\{x : D_+f(x) > \bar{f}^{(')}(x)\} \cup \{x : D^+f(x) < \underline{f}^{(')}(x)\}$, is at most countable, we get

COROLLARY 1. *If f satisfies conditions* (i) *and* (ii) *of Theorem* 3, *then there are sets C and D of positive measure such that $C \cup D \subset [a, b]$ and*

$$D_+f(d) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant D^+f(c)$$

for all $c \in C$ and $d \in D$.

THEOREM 5. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3. *Then there exist two sets C and D of positive measure such that $C \cup D \subset [a, b]$ and*

$$\underline{f}^{(')}(c) \geqslant \frac{f(c)-f(a)}{b-c} \quad \textit{and} \quad \bar{f}^{(')}(d) \leqslant \frac{f(d)-f(a)}{b-d}$$

for all $c \in C$ and $d \in D$. If, further, f is continuous in $[a, b]$, $f^{(')}$ exists and satisfies the Darboux property in (a, b), then there is $\xi \in (a, b)$ such that

$$f^{(')}(\xi) = \frac{f(\xi)-f(a)}{b-\xi}.$$

Proof. Applying the result of Lemma 3 to the function $g(x) = [f(x)-f(a)](b-x)$ the proof of the first part is completed.

If the equality holds for at least one $c \in C \cap (a, b)$ or one $d \in D \cap (a, b)$, there is nothing to prove. So we suppose

$$f^{(')}(c) > \frac{f(c)-f(a)}{b-c} \quad \text{and} \quad f^{(')}(d) < \frac{f(d)-f(a)}{b-d} \tag{1}$$

for all $c \in C \cap (a, b)$ and $d \in D \cap (a, b)$. Let $g(x) = [f(x)-f(a)](b-x)$. Then g is continuous in $[a, b]$ and

$$g^{(')}(x) = (b-x)f^{(')}(x) - [f(x)-f(a)]. \tag{2}$$

Now, $f^{(')}$ being the limit of a sequence of continuous functions, it is of Baire Class 1. Since it is known that $\varphi + \psi$ and $\varphi \cdot \psi$ are Darboux function of Baire Class 1 whenever φ is a Darboux function of Baire Class 1 and ψ is continuous [2], we conclude from (2) that $g^{(')}$ possesses the Darboux property in (a, b). Thus, since by (1)

$$g^{(')}(c) > 0 > g^{(')}(d) \quad \text{for all } c \in C \cap (a, b) \text{ and } d \in D \cap (a, b),$$

there is $\xi \in (a, b)$ such that

$$g^{(')}(\xi) = 0, \quad \text{i.e.,} \quad f^{(')}(\xi) = \frac{f(\xi)-f(a)}{b-\xi}.$$

Note 3. The above theorem generalizes a result of Simeon Reich [8].

Similarly to Corollary 1 we get

COROLLARY 2. *If f satisfies conditions* (i) *and* (ii) *of Theorem* 3, *then there are sets C and D of positive measure such that $C \cup D \subset [a, b]$ and*

$$D^+f(c) \geqslant \frac{f(c)-f(a)}{b-c} \quad \text{and} \quad D_+f(d) \leqslant \frac{f(d)-f(a)}{b-d}$$

for all $c \in C$ and $d \in D$.

THEOREM 6. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3. *Let $D^+f(a) = D_+f(a) = D^-f(b) = D_-f(b)$. Then there exist two sets C and D of positive measure such that $C \cup D \subset [a, b]$ and*

$$\underline{f}^{(\prime)}(c) \geqslant \frac{f(c)-f(a)}{c-a} \quad \text{and} \quad \bar{f}^{(\prime)}(d) \leqslant \frac{f(d)-f(a)}{d-a}$$

for all $c \in C$ and $d \in D$.

If, moreover, f is continuous, $f^{(\prime)}$ exists and has the Darboux property in (a, b), then there is $\xi \in (a, b)$ such that

$$f^{(\prime)}(\xi) = \frac{f(\xi)-f(a)}{\xi-a}.$$

Proof. Let

$$(1) \qquad \begin{aligned} g(x) &= \frac{f(x)-f(a)}{x-a}, \quad x \neq a, \\ g(a) &= D^+f(a). \end{aligned}$$

Then for $x \in (a, b)$

$$(2) \qquad \begin{aligned} \bar{g}^{(\prime)}(x) &= \frac{\bar{f}^{(\prime)}(x)}{x-a} - \frac{f(x)-f(a)}{(x-a)^2}, \\ \underline{g}^{(\prime)}(x) &= \frac{\underline{f}^{(\prime)}(x)}{x-a} - \frac{f(x)-f(a)}{(x-a)^2}. \end{aligned}$$

Thus g satisfies conditions (i) and (ii) of Theorem 3. Now, if

$$D^-f(b) = \frac{f(b)-f(a)}{b-a},$$

then

$$g(a) = D^+f(a) = D^-f(b) = g(b)$$

and so by Lemma 3 there are sets C and D of positive measure such that $C \cup D \subset [a, b]$ and

$$(3) \qquad \underline{g}^{(\prime)}(c) \geqslant 0 \quad \text{and} \quad \bar{g}^{(\prime)}(d) \leqslant 0$$

for all $c \in C$ and $d \in D$.

If $D^-f(b) \neq \frac{f(b)-f(a)}{b-a}$, then $g(a) \neq g(b)$. Let $g(a) < g(b)$. Then by (1)

$$D^-g(b) = \frac{D^-f(b)}{b-a} - \frac{f(b)-f(a)}{(b-a)^2} = \frac{D^+f(a)-g(b)}{b-a}$$
$$= \frac{g(a)-g(b)}{b-a} < 0.$$

So, there is x_0, $a < x_0 < b$, such that $g(a) < g(b) < g(x_0)$. Hence, by Lemma 2, there are sets C and D of positive measure such that $C \cup D \subset [a, b]$ and

$$\text{(4)} \qquad \underline{g}^{(')}(c) > 0 \quad \text{and} \quad \bar{g}^{(')}(d) < 0$$

for all $c \in C$ and $d \in D$.

The case $g(a) > g(b)$ can be similarly treated. Thus in any case there are sets C and D of positive measure such that $C \cup D \subset [a, b]$ and

$$\underline{f}^{(')}(c) \geqslant \frac{f(c)-f(a)}{c-a} \quad \text{and} \quad \bar{f}^{(')}(d) \leqslant \frac{f(d)-f(a)}{d-a}$$

for all $c \in C$ and $d \in D$.

If, further, f is continuous, $f^{(')}$ exists and has the Darboux property, then from (2)

$$g^{(')}(x) = \frac{f^{(')}(x)}{x-a} - \frac{f(x)-f(a)}{(x-a)^2}.$$

Now if the equality holds for at least one $c \in C \cap (a, b)$ or one $d \in D \cap (a, b)$, the proof is complete. So suppose that

$$g^{(')}(c) > 0 \quad \text{and} \quad g^{(')}(d) < 0$$

for all $c \in C \cap (a, b)$ and $d \in D \cap (a, b)$.

By the argument applied in the proof of Theorem 5, $g^{(')}(x)$ possesses the Darboux property and hence there is $\xi \in (a, b)$ such that

$$g^{(')}(\xi) = 0, \quad \text{i.e.,} \quad f^{(')}(\xi) = \frac{f(\xi)-f(a)}{\xi - a}.$$

This completes the proof.

Note 4. The above theorem gives a generalization of the analogue of a theorem due to Flett [3] concerning ordinary derivatives which states that if f is differentiable in $[a, b]$ and if $f'(a) = f'(b)$, then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(\xi)-f(a)}{\xi - a}.$$

Again as in Corollary 1 we get

COROLLARY 3. *If f satisfies conditions* (i) *and* (ii) *of Theorem* 3 *and if $D^+f(a) = D_+f(a) = D^-f(b) = D_-f(b)$, then there are sets C and D of positive measure such that $C \cup D \subset [a, b]$ and*

$$D^+f(c) \geqslant \frac{f(c)-f(a)}{c-a} \quad \textit{and} \quad D_+f(d) \leqslant \frac{f(d)-f(a)}{d-a}$$

for all $c \in C$ and $d \in D$.

Note 5. Flett's theorem is a consequence of the above corrollary.

4. Some other consoquences

THEOREM 7. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3. *Then the functions $\bar{f}^{(')}$, $\underline{f}^{(')}$, D^+f, D_+f, D^-f and D_-f have the same bounds in (a, b).*

Proof. To prove the theorem we shall show that the upper and the lower bounds of each of the functions $\bar{f}^{(')}$, $\underline{f}^{(')}$, D^+f, D_+f, D^-f and D_-f are respectively equal to the upper and the lower bounds of the set

$$\left\{\frac{f(x_2)-f(x_1)}{x_2-x_1};\ x_1, x_2 \in (a, b),\ x_1 \neq x_2\right\}.$$

We shall consider $\bar{f}^{(')}$ and D^+f; the other cases follow similarly. Let $M = \sup\{\bar{f}^{(')}(x);\ x \in (a, b)\}$ and suppose that $M < +\infty$. Choose M', $M < M' < +\infty$. Then $\bar{\varphi}^{(')}(x) < 0$ for all $x \in (a, b)$, where $\varphi(x) = f(x) - M'x$ and so by Theorem 2, φ is non-increasing in (a, b). So, for every pair of points $x_1, x_2 \in (a, b)$, $x_1 < x_2$, we have $\varphi(x_1) \geqslant \varphi(x_2)$, i.e. $\frac{f(x_2-f(x_1)}{x_2-x_1} \leqslant M'$. Since M' is any number greater than M, we have $\frac{f(x_2)-f(x_1)}{x_2-x_1} \leqslant M$. Also for any $\varepsilon > 0$ there is $\xi \in (a, b)$ such that $f^{(')}(\xi) > M - \varepsilon$ and hence there is $h > 0$ such that

$$\frac{f(\xi+h)-f(\xi-h)}{2h} > M-\varepsilon, \quad \xi \pm h \in (a, b).$$

This shows that

$$M = \sup\left\{\frac{f(x_2)-f(x_1)}{x_2-x_1};\ x_1, x_2 \in (a, b),\ x_1 \neq x_2\right\}.$$

If $M = +\infty$, the above equality is trivial.

It can be shown similarly that

$$m = \inf\left\{\frac{f(x_2)-f(x_1)}{x_2-x_1};\ x_1, x_2 \in (a, b),\ x_1 \neq x_2\right\},$$

where

$$m = \inf\{\bar{f}^{(')}(x);\ x \in (a, b)\}.$$

Again since $-\infty < \underline{f}^{(')}(x) \leqslant \bar{f}^{(')}(x) < +\infty$ off a countable set, we conclude from Lemma 1 that $D^+f(x) > -\infty$ and $D_+f(x) < +\infty$ holds off a countable set. Hence by applying the result of Gal [4] it can similarly be shown that the upper and the lower bounds of D^+f in (a, b) are respectively equal to the upper and the lower bounds of the set

$$\left\{\frac{f(x_2)-f(x_1)}{x_2-x_1};\ x_1, x_2 \in (a, b),\ x_1 \neq x_2\right\}.$$

Note 6. A similar but less general result is obtained in [6].

THEOREM 8. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3. *Then the continuity of any of the functions $\bar{f}^{(')}$, $\underline{f}^{(')}$, D^+f, D_+f, D^-f and D_-f at a point $\xi \in (a, b)$ implies the continuity of the other functions at ξ and $f'(\xi)$ exists.*

Proof. Suppose that $\bar{f}^{(')}(x)$ is continuous at a point $\xi \in (a, b)$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that $\bar{f}^{(')}(\xi) - \varepsilon < \bar{f}^{(')}(x) < \bar{f}^{(')}(\xi) + \varepsilon$ whenever $x \in (\xi - \delta, \xi + \delta)$.

If M and m are the upper and the lower bounds of $\bar{f}^{(')}$ in $(\xi - \delta, \xi + \delta)$, then

$$\bar{f}^{(')}(\xi) - \varepsilon \leqslant m \leqslant \bar{f}^{(')}(x) \leqslant M \leqslant \bar{f}^{(')}(\xi) + \varepsilon.$$

Thus, by Theorem 7,

$$\underline{f}^{(')}(\xi) - \varepsilon \leqslant m \leqslant \underline{f}^{(')}(x) \leqslant M \leqslant \bar{f}^{(')}(\xi) + \varepsilon \quad \text{for all } x \in (\xi - \delta, \xi + \delta). \tag{1}$$

Hence

$$|\underline{f}^{(')}(x) - \underline{f}^{(')}(\xi)| \leqslant 2\varepsilon \quad \text{for all } x \in (\xi - \delta, \xi + \delta),$$

showing that $\underline{f}^{(')}(x)$ is continuous at ξ. Since in inequalities (1) $\underline{f}^{(')}$ may be replaced by any one of the four Dini derivatives, the continuity of the Dini derivatives and the existence of $f'(\xi)$ follows.

THEOREM 9. *Let f satisfy conditions* (i) *and* (ii) *of Theorem* 3. *If $\bar{f}^{(')}$ (or $\underline{f}^{(')}$) is bounded in (a, b), then f satisfies the Lipschitz condition in (a, b).*

Proof. We shall prove the assertion for $\bar{f}^{(')}$. The case is similar for $\underline{f}^{(')}$. Let M be the upper bound of $|\bar{f}^{(')}|$ in (a, b). Then, by Theorem 7, for every pair of points $x_1, x_2 \in (a, b)$ we have

$$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right| \leqslant M$$

and hence $|f(x_2) - f(x_1)| \leqslant M|x_2 - x_1|$, proving our assertion.

Note 7. The result again sharpens a result of Aull [1].

In conclusion, I offer my grateful thanks to Dr. S. N. Mukhopadhyay for his kind help and suggestions in the preparation of this paper.

References

[1] C. E. Aull, *The first symmetric derivatives*, Amer. Math. Monthly 74 (1967), p. 708–711.

[2] A. M. Bruckner, J. G. Ceder and M. Weiss, *Uniform limits of Darboux functions*, Colloq. Math. 15 (1966), p. 65–77.

[3] T. M. Flett, *A mean value theorem*, Math. Gaz. 42 (1958), p. 38–39.

[4] I. S. Gal, *On the fundamental theorem of the calculus*, Trans. Amer. Math. Soc. 86 (1957), p. 309–320.

[5] A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fund. Math. 9 (1927), p. 212–279.

[6] N. K. Kundu, *On some conditions of symmetric derivatives implying monotonicity of functions*, Rev. Roum. Math. Pures Appl. 15 (1970), p. 561–568.

[7] S. N. Mukhopadhyay, *On Schwarz differentiability, IV*, Acta Math. Acad. Sci. Hung. 17 (1966), p. 129–136.

[8] S. Reich, *On mean value theorems*, Amer. Math. Monthly 76 (1969), p. 70–73.

[9] S. Saks, *Theory of the integral*, Warszawa 1937.

DEPARTMENT OF MATHEMATICS
NETAJI COLLEGE, ARAMBAGH
HOOGHLY, WEST BENGAL
INDIA

Reçu par la Rédaction le 23. 3. 1971