ANNALES
POLONICI MATHEMATICI
LI (1990)

Metrization of $D_E[0, 1]$ by Hausdorff distance between graphs

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Abstract. The Skorokhod topology in the space of right-continuous functions $t \to x(t)$, $t \in [0, 1]$, without oscillatory discontinuities coincides with the topology determined by the metric $\delta(x, y) =$ Hausdorff distance between graphs of the maps $t \to (x(t-), x(t))$ and $t \to (y(t-), y(t))$.

1. The Skorokhod topology. Let E be a metric space. Following [2], we denote by $D_E[0, 1]$ the space of all E-valued functions on $[0, 1]$ which are right-continuous on $[0, 1)$, are left-continuous at 1, and have left-side limits everywhere on $(0, 1]$. For notational convenience, for any $x \in D_E[0, 1]$ we define $x(0-)$ as equal to $x(0)$.

The distance between elements x and y of $D_E[0, 1]$ can be defined as

$$d(x, y) = \inf_{\lambda \in \Lambda} \sup_{t \in [0,1]} |t - \lambda(t)| \vee r\big(x(t), y(\lambda(t))\big),$$

where r denotes the distance in E and Λ is the set of all continuous, strictly increasing real functions λ on $[0, 1]$, such that $\lambda(0) = 0$ and $\lambda(1) = 1$. Another, even more useful distance function in $D_E[0, 1]$ can be defined by

$$d_0(x, y) = \inf_{\lambda \in \Lambda_0} \operatorname*{ess\,sup}_{t \in [0,1]} \left|\log \frac{d\lambda(t)}{dt}\right| \vee r\big(x(t), y(\lambda(t))\big),$$

where Λ_0 is the subset of Λ consisting of Lipschitz functions with Lipschitz inverse. If (E, r) is complete, then $(D_E[0, 1], d_0)$ is complete, unlike $(D_E[0, 1], d)$, which is incomplete unless E is a singleton. Nevertheless, both these metrics, d and d_0, always determine in $D_E[0, 1]$ the same topology. Proofs of these statements and historical remarks can be found in the book of Billingsley [1], § 14.

The topology of $(D_E[0,1], d)$ is called *Skorokhod's topology*. The corresponding convergence of sequences in $D_E[0, 1]$ is called *Skorokhod's convergence*.

2. Hausdorff distance between graphs. For each $x \in D_E[0, 1]$, define on $[0, 1]$ the $E \times E$-valued function $\hat{x}$ so that

$$\hat{x}(t) = \big(x(t-),\, x(t)\big), \qquad t \in [0, 1].$$

For every x and y in $D_E[0, 1]$ define

$$h(x, y) = \sup_{t \in [0,1]} \inf_{s \in [0,1]} |t-s| \vee r\big(x(t-), y(s-)\big) \vee r\big(x(t), y(s)\big).$$

Then
$$\delta(x, y) = h(x, y) \vee h(y, x)$$

is the Hausdorff distance between the graphs of $\hat{x}$ and $\hat{y}$. Consequently we have $\delta(x, x) = 0$, $\delta(x, y) \geqslant 0$, $\delta(x, y) = \delta(y, x)$ and $\delta(x, z) \leqslant \delta(x, y) + \delta(y, z)$.

If $\delta(x, y) = 0$ then for each $t \in [0, 1]$ there is a sequence $s_1, s_2, \ldots$ in $[0, 1]$ such that $\lim s_n = t$ and $\lim y(s_n) = x(t)$. Since $\lim y(s_n) \in \{y(t-), y(t)\}$, we conclude that

$$x(t) \in \{y(t-), y(t)\}.$$

In the case of $t = 1$ this implies that $x(1) = y(1-) = y(1)$. If $t \in [0, 1)$ then for each $h \in (0, 1-t]$

$$x(t+h) \in \{y(t+h-), y(t+h)\}.$$

Since $\lim_{h \to +0} y(t+h-) = \lim_{h \to +0} y(t+h) = y(t)$ and $\lim_{h \to +0} x(t+h) = x(t)$, it follows that $x(t) = y(t)$. So, if $\delta(x, y) = 0$ then $x = y$. We conclude that δ is a metric in $D_E[0, 1]$.

3. The scope of the paper. Our purpose is

(1) to prove that the topology of $(D_E[0, 1], \delta)$ coincides with Skorokhod's topology, and

(2) to connect this fact with a direct characterization of Skorokhod's convergence, not involving any metric in $D_E[0, 1]$.

Observe that, for every $x, y \in D_E[0, 1]$, $t \in [0, 1]$ and $\lambda \in \Lambda$, we have

$$\begin{aligned} \inf_{s \in [0,1]} &|t-s| \vee r\big(x(t-), y(s-)\big) \vee r\big(x(t), y(s)\big) \\ &\leqslant |t-\lambda(t)| \vee r\big(x(t-), y(\lambda(t)-)\big) \vee r\big(x(t), y(\lambda(t))\big) \\ &\leqslant \sup_{u \in [0,1]} |u-\lambda(u)| \vee r\big(x(u), y(\lambda(u))\big), \end{aligned}$$

whence $h(x, y) \leqslant d(x, y)$. Similarly, $h(y, x) \leqslant d(y, x) = d(x, y)$, and consequently

$$\delta(x, y) \leqslant d(x, y),$$

so that the topology of $(D_E[0, 1], \delta)$ cannot be stronger than the topology of Skorokhod. This reduces our point (1) to proving that the topology of $(D_E[0, 1], \delta)$ is not weaker than Skorokhod's topology.

Remark. Denote by $\varrho(x, y)$ the Hausdorff distance between graphs of x and y. The same argument as in the case of δ shows that also ϱ is a metric in $D_E[0, 1]$. Let e and e' be distinct elements of E. Define

$$x(t) = \begin{cases} e & \text{for } t \in [0, 1/2), \\ e' & \text{for } t \in [1/2, 1], \end{cases}$$

$$x_n(t) = \begin{cases} e & \text{for } t \in [0, 1/2) \cup [1/2 + 1/2^{n+1}, 1/2 + 1/2^n), \\ e' & \text{for } t \in [1/2, 1/2 + 1/2^{n+1}) \cup [1/2 + 1/2^n, 1]. \end{cases}$$

Then $\lim_{n\to\infty} \varrho(x_n, x) = 0$ and $\lim_{n\to\infty} h(x, x_n) = 0$, but $h(x_n, x) = r(e, e')$ for each n, and $\lim_{n\to\infty} \delta(x_n, x) = r(e, e')$. This shows that the topology of $(D_E[0, 1], \varrho)$ is essentially weaker than that of $(D_E[0, 1], \delta)$.

4. Connection with the Ethier–Kurtz characterization of Skorokhod's convergence. All sequences occurring below are labelled by natural numbers. Let $x \in D_E[0, 1]$ and let $\{x_n\}$ be a sequence in $D_E[0, 1]$. Take into account the statements:

(a) $\lim_{n\to\infty} r(x_n(t_n), x(t-)) \wedge r(x_n(t_n), x(t)) = 0$ for every sequence $\{t_n\}$ in $[0, 1]$ such that $\lim_{n\to\infty} t_n = t$,

(a$-$) $\lim_{n\to\infty} r(x_n(t_n-), x(t-)) \wedge r(x_n(t_n-), x(t)) = 0$ for every sequence $\{t_n\}$ in $[0, 1]$ such that $\lim_{n\to\infty} t_n = t$,

(b) for any sequences $\{u_n\}$ and $\{t_n\}$ in $[0, 1]$ such that $u_n \leqslant t_n$ for each n and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} t_n = t$, the equality $\lim_{n\to\infty} x_n(u_n) = x(t)$ implies the equality $\lim_{n\to\infty} x_n(t_n) = x(t)$,

(c) for any sequences $\{t_n\}$ and $\{v_n\}$ in $[0, 1]$ such that $t_n \leqslant v_n$ for each n and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} v_n = t$, the equality $\lim_{n\to\infty} x_n(v_n) = x(t-)$ implies the equality $\lim_{n\to\infty} x_n(t_n) = x(t-)$,

(d) $\lim_{n\to\infty} h(x_n, x) = 0$,

(e) $\lim_{n\to\infty} h(x, x_n) = 0$,

(f) for any pair of sequences $\{u_n\}$ and $\{v_n\}$ in $[0, 1]$ such that $u_n < v_n$ for each n, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$, $\lim_{n\to\infty} x_n(u_n) = \mathfrak{g}$ and $\lim_{n\to\infty} x_n(v_n) = \mathfrak{h}$, there is a sequence $\{s_n\}$ in $[0, 1]$ such that $u_n \leqslant s_n \leqslant v_n$ for each n, $\lim_{n\to\infty} x_n(s_n-) = \mathfrak{g}$ and $\lim_{n\to\infty} x_n(s_n) = \mathfrak{h}$,

(g) for every triple of sequences $\{u_n\}$, $\{v_n\}$ and $\{t_n\}$ in $[0, 1]$ the relations

$u_n \leqslant t_n \leqslant v_n$ for each n, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$ and $\lim_{n\to\infty} x_n(u_n) = \lim_{n\to\infty} x_n(v_n) = \mathfrak{g}$ imply $\lim_{n\to\infty} x_n(t_n) = \mathfrak{g}$.

In context of the space $D_E[0, \infty)$, statements (a), (b) and (c) occur in the book of Ethier and Kurtz [2], p. 125, where it is proved that they are jointly equivalent to convergence of $\{x_n\}$ to x in sense of the metric defined by formula (5.2) of [2], p. 117. Equality (e) corresponds to the equality $\lim_{n\to\infty} \varepsilon_n = 0$ occurring in [2], ε_n being defined by formulas (6.16) and (6.17) of [2], p. 125–126.

Statement (f) is convenient for organizing the proofs. It is instructive to check which ones of conditions (a)–(g) are satisfied and which are not in the situation from the Remark at the end of Section 3.

LEMMA 1. *Statement* (d) *implies* (a), (a$-$), (b), (c), (e), (f) *and* (g).

Proof. (d)$\Rightarrow$(a). If (d) holds and $\{t_n\}$ is a sequence in $[0, 1]$ converging to t, then there is a sequence $\{s_n\}$ in $[0, 1]$ converging to t such that $\lim_{n\to\infty} r(x_n(t_n), x(s_n)) = 0$. But

$$\lim_{n\to\infty} r(x(s_n), x(t-)) \wedge r(x(s_n), x(t)) = 0$$

and therefore

$$\lim_{n\to\infty} r(x_n(t_n), x(t-)) \wedge r(x_n(t_n), x(t)) = 0.$$

Remark. Note that we have in fact proved the stronger implication $(\mathrm{d}_0)\Rightarrow$(a), where (d_0) stands for $\lim_{n\to\infty} h_0(x_n, x) = 0$, with $h_0(x, y)$ being defined by

$$h_0(x, y) = \sup_{t\in[0,1]} \inf_{s\in[0,1]} |t-s| \vee r(x(t), y(s)),$$

so that $h_0(x, y) \vee h_0(y, x) = \varrho(x, y)$. See the Remark at the end of Section 3.

(a)$\Rightarrow$(a$-$). It is sufficient to observe that for every sequence $\{t_n\}$ in $[0, 1]$ there is a sequence $\{s_n\}$ in $[0, 1]$ such that $t_n - 1/n \leqslant s_n \leqslant t_n$ and $r(x_n(s_n), x_n(t_n-)) \leqslant 1/n$.

(a)$\Rightarrow$(f). If $\mathfrak{g} = \mathfrak{h}$ then it is sufficient to choose each s_n so that $u_n < s_n \leqslant v_n$ and $r(x_n(s_n-), x_n(u_n)) \vee r(x_n(s_n), x_n(u_n)) \leqslant 1/n$. This is possible thanks to right continuity of x_n. Now suppose that $\mathfrak{g} \neq \mathfrak{h}$. Then, by (a), either $(\mathfrak{g}, \mathfrak{h}) = (x(t-), x(t))$ or $(\mathfrak{g}, \mathfrak{h}) = (x(t), x(t-))$, where $t = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$. Put

$$A_n = \{s\in[u_n, v_n]:\ r(\mathfrak{g}, x_n(s)) \geqslant \tfrac{1}{2} r(\mathfrak{g}, \mathfrak{h})\}.$$

Then $v_n \in A_n$ and $\inf A_n > u_n$ for every n greater than some n_0. For $n = 1, 2, \ldots, n_0$ define s_n as any point of $[u_n, v_n]$. For $n > n_0$, following Ethier and Kurtz [2], p. 126, formula (6.20), define

$$s_n = \inf A_n.$$

By (a) and (a−), each of the sequences $\{x_n(s_n-)\}$ and $\{x_n(s_n)\}$ either converges to $\mathfrak{g}$ or to $\mathfrak{h}$, or has exactly two cluster points $\mathfrak{g}$ and $\mathfrak{h}$. The definition of s_n immediately implies

$$r(\mathfrak{g}, x_n(s_n-)) \leqslant \tfrac{1}{2}r(\mathfrak{g}, \mathfrak{h}) \quad \text{and} \quad r(\mathfrak{g}, x_n(s_n)) \geqslant \tfrac{1}{2}r(\mathfrak{g}, \mathfrak{h})$$

for every $n > n_0$. Consequently $\mathfrak{h}$ cannot be a cluster point of $\{x_n(s_n-)\}$, $\mathfrak{g}$ cannot be a cluster point of $\{x_n(s_n)\}$, and therefore $\lim_{n\to\infty} x_n(s_n-) = \mathfrak{g}$ and $\lim_{n\to\infty} x_n(s_n) = \mathfrak{h}$.

(a) ⇒ (e). In spite of different organization, our proof of this implication follows the reasonings of Ethier and Kurtz [2], p. 126, and is included only for completeness. We proceed *ad absurdum.* Suppose that for some $x \in D_E[0, 1]$ and for some sequence $\{x_n\}$ in $D_E[0, 1]$ statement (a) is true and (e) is not. The former means that there exist: an $\varepsilon > 0$, a sequence $\{t_n\}$ in [0, 1], and a subsequence $\{y_n\}$ of $\{x_n\}$, such that

$$|t_n - s| \vee r(x(t_n-), y_n(s-)) \vee r(x(t_n), y_n(s)) \geqslant \varepsilon \quad \text{for every } s \in [0, 1] \text{ and every natural } n.$$

By choosing a subsequence, we may assume that $\lim_{n\to\infty} t_n = t$, $\lim_{n\to\infty} x(t_n-) = \mathfrak{g}$, $\lim_{n\to\infty} x(t_n) = \mathfrak{h}$, and that

$$(*) \qquad |t - s| \vee r(y_n(s-), \mathfrak{g}) \vee r(y_n(s), \mathfrak{h}) \geqslant \tfrac{1}{2}\varepsilon$$

for every $s \in [0, 1]$ and every natural n. Now, for every n take u_n and v_n in [0, 1] such that

$$t_n - 1/n \leqslant u_n \leqslant t_n \leqslant v_n \leqslant t_n + 1/n, \quad u_n < v_n,$$

$$r(x(u_n-), x(t_n-)) \vee r(x(u_n), x(t_n-)) \leqslant 1/n,$$

$$r(x(v_n-), x(t_n)) \vee r(x(v_n), x(t_n)) \leqslant 1/n.$$

By (a) we can construct a strictly increasing sequence $\{k_n\}$ of naturals such that

$$r(y_{k_n}(u_n), x(u_n-)) \wedge r(y_{k_n}(u_n), x(u_n)) \leqslant 1/n,$$

$$r(y_{k_n}(v_n), x(v_n-)) \wedge r(y_{k_n}(v_n), x(v_n)) \leqslant 1/n$$

for each n. We then have $\lim_{n\to\infty} y_{k_n}(u_n) = \mathfrak{g}$ and $\lim_{n\to\infty} y_{k_n}(v_n) = \mathfrak{h}$. Since we already know that (a) implies (f), we conclude that there is a sequence $\{s_n\}$ in [0, 1] such that $\lim_{n\to\infty} s_n = t$, $\lim_{n\to\infty} y_{k_n}(s_n-) = \mathfrak{g}$ and $\lim_{n\to\infty} y_{k_n}(s_n) = \mathfrak{h}$. But this is absurd; it is evident from (∗) that such a sequence $\{s_n\}$ cannot exist.

(d) ⇒ (b). We again proceed *ad absurdum.* Suppose that (d) holds and (b) is not true. Then there are sequences $\{u_n\}$ and $\{t_n\}$ in [0, 1] such that $u_n < t_n$ for each n, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} t_n = t$, $\lim_{n\to\infty} x_n(u_n) = x(t)$ and $\{x_n(t_n)\}$ does not converge to $x(t)$. Since (d) implies (a), we must have $x(t-) \neq x(t)$ and, by choosing a subsequence, we may assume that

$$\lim_{n\to\infty} x_n(t_n) = x(t-).$$

Then the implications (d) $\Rightarrow$ (a) $\Rightarrow$ (f) already proved permit to conclude that there is a sequence $\{\tau_n\}$ in $[0, 1]$ such that

$$\lim_{n\to\infty} \tau_n = t, \quad \lim_{n\to\infty} x_n(\tau_n-) = x(t) \quad \text{and} \quad \lim_{n\to\infty} x_n(\tau_n) = x(t-).$$

Now, for each n we choose $s_n \in [0, 1]$ so that

$$|\tau_n - s_n| \vee r\big(x_n(\tau_n-), x(s_n-)\big) \vee r\big(x_n(\tau_n), x(s_n)\big) < 1/n + h(x_n, x)$$

and we obtain

$$\lim_{n\to\infty} s_n = t, \quad \lim_{n\to\infty} x(s_n-) = x(t) \quad \text{and} \quad \lim_{n\to\infty} x(s_n) = x(t-),$$

which is absurd, since $x(t-) \neq x(t)$.

(d) $\Rightarrow$ (c). The reasoning is similar to the case of (d) $\Rightarrow$ (b).

(a) & (b) & (c) $\Rightarrow$ (g). Let $t = \lim_{n\to\infty} t_n$. By (a), either $g = x(t)$, or $g = x(t-)$. If $g = x(t)$, then $\lim_{n\to\infty} x_n(t_n) = g$, by (b). If $g = x(t-)$, then $\lim_{n\to\infty} x_n(t_n) = g$, by (c).

LEMMA 2 (Implicitly contained in [2], p. 126, lines 9–14). *Suppose that for some* $x \in D_E[0, 1]$ *and some sequence* $\{x_n\}$ *in* $D_E[0, 1]$ *condition* (a) *is satisfied. Then, for every* $t \in [0, 1]$ *and for* $g \in \{x(t-), x(t)\}$, *there is a sequence* $\{t_n\}$ *in* $[0, 1]$ *such that* $\lim t_n = t$ *and* $\lim x_n(t_n) = g$.

Proof. Take any sequence $\{s_k\}$ in $[0, 1]$ such that $\lim s_k = t$ and $\lim x(s_k-) = \lim x(s_k) = g$. Then, by (a), we can construct a strictly increasing sequence $n_1 < n_2 < \ldots$ of naturals such that

$$r\big(x_n(s_k), x(s_k-)\big) \wedge r\big(x_n(s_k), x(s_k)\big) \leqslant 1/k$$

for every $k = 1, 2, \ldots$ and every $n \geqslant n_k$. Now, let $t_n \in [0, 1]$ be arbitrary if $n < n_1$, and put $t_n = s_k$ if $n_k \leqslant n < n_{k+1}$. Then $\lim t_n = \lim s_k = t$ and $r\big(x_n(t_n), g\big) \leqslant 1/k + r\big(x(s_k-), g\big) \vee r\big(x(s_k), g\big)$ whenever $n_k \leqslant n < n_{k+1}$, so that $\lim x_n(t_n) = g$.

LEMMA 3. *Suppose that for some* $x \in D_E[0, 1]$ *and some sequence* $\{x_n\}$ *in* $D_E[0, 1]$ *conditions* (a) *and* (g) *are satisfied. Then every subsequence of* $\{x_n\}$ *also satisfies* (g).

Proof. Let $\{m_k\}$ be a strictly increasing sequence of naturals. Let $\{u_{m_k}\}$, $\{t_{m_k}\}$ and $\{v_{m_k}\}$ be sequences in [0, 1] labelled by elements of $\{m_k\}$, such that $u_{m_k} \leqslant t_{m_k} \leqslant v_{m_k}$ for every $k = 1, 2, \ldots$, $\lim u_{m_k} = \lim t_{m_k} = \lim v_{m_k} = t$ and $\lim x_{m_k}(u_{m_k}) = \lim x_{m_k}(v_{m_k}) = g$. We have to prove that then $\lim x_{m_k}(t_{m_k}) = g$.

Put $M = \{m_1, m_2, \ldots\}$ and define $\tilde{u}_n = u_n$ if $n \in M$, $\tilde{u}_n = t$ if $n \in N \backslash M$. Then $\lim \tilde{u}_n = t$ and, since $\lim x_{m_k}(u_{m_k}) = g$, g is a cluster point of the sequence $x_n(\tilde{u}_n)$, so that, by (a), $g \in \{x(t-), x(t)\}$. Consequently, by Lemma 2, there is a sequence $\{\tau_n\}$ in [0, 1] such that $\lim \tau_n = t$ and $\lim x_n(\tau_n) = g$. The numbers u_n, t_n and v_n being a priori given for every $n \in M$, we define $u_n = t_n = v_n = \tau_n$ whenever $n \in N \backslash M$. Then $u_n \leqslant t_n \leqslant v_n$ for every $n \in N$, $\lim u_n = \lim v_n$ and $\lim x_n(u_n) = \lim x_n(v_n) = g$, whence $\lim x_n(t_n) = g$, by (g), and consequently $\lim x_{m_k}(t_{m_k}) = g$.

5. The main result.

THEOREM. *The topology of $(D_E[0, 1], \delta)$ coincides with Skorokhod's topology.*

Proof. We have to show that for every sequence $\{x_n\}$ in $D_E[0, 1]$ and for every $x \in D_E[0, 1]$ the two implications hold:

$$\lim_{n \to \infty} d(x_n, x) = 0 \Rightarrow \lim_{n \to \infty} \delta(x_n, x) = 0 \Rightarrow \lim_{n \to \infty} d_0(x, x_n) = 0.$$

The first of them is a consequence of inequality $\delta \leqslant d$ proved in Section 3. For the proof of the second implication, in the main it is sufficient to follow the reasoning of the final part (pp. 126–127) of proof of Proposition 6.5 of Ethier and Kurtz [2], p. 125. We do this, for completeness.

Suppose that $\lim_{n \to \infty} \delta(x_n, x) = 0$. Then, by Lemma 1 of our Section 4, conditions (a) and (g) are satisfied. We shall show that these two conditions imply $\lim_{n \to \infty} d_0(x_n, x) = 0$. We know from Section 4 that (a) implies (e). (Obviously (e) is also a trivial consequence of the equality $\lim_{n \to \infty} \delta(x_n, x) = 0$, but we wish to rely only on (a) and (g).) Hence, we can choose a sequence $\{\delta_n\}$ of positive numbers such that $\lim_{n \to \infty} \delta_n = 0$ and $\delta_n^2 > h(x, x_n)$ for each n. Rejecting, if necessary, a finite number of terms, we may assume that $\delta_n < \frac{1}{2}$ for each n. Since $\lim_{n \to \infty} \delta_n = 0$, by Lemma 1 of Billingsley [1], §14, there is a sequence of partitions

$$\pi_n\colon\ 0 = t_{n,0} < t_{n,1} < \ \ldots\ < t_{n,l(n)} = 1$$

of the interval [0, 1] such that

$$\delta_n \leqslant t_{n,k} - t_{n,k-1} \leqslant 2\delta_n$$

for every n and $k = 1, \ldots, l(n)$, and

$$\lim_{n \to \infty} \max_{k = 1, \ldots, l(n)} \sup_{s,t \in [t_{n,k-1}, t_{n,k})} r(x(s), x(t)) = 0.$$

Since $h(x, x_n) < \delta_n^2$, we can choose for every n and $k = 1, \ldots, l(n)-1$ a number

$s_{n,k} \in [0, 1]$ so that

(2) $$|t_{n,k}-s_{n,k}| \vee r\big(x(t_{n,k}-), x_n(s_{n,k}-)\big) \vee r\big(x(t_{n,k}), x_n(s_{n,k})\big) < \delta_n^2.$$

Moreover, we set $s_{n,0} = 0$, $s_{n,l(n)} = 1$. Then

$$s_{n,k}-s_{n,k-1} \geqslant t_{n,k}-t_{n,k-1}-2\delta_n^2 \geqslant \delta_n-2\delta_n^2 > 0,$$

so that for each n we have

$$0 = s_{n,0} < s_{n,1} < \ldots < s_{n,l(n)} = 1.$$

Now, for each n we define a function $\lambda_n \in \Lambda_0$ by the two conditions:

(1) $\lambda_n(t_{n,k}) = s_{n,k}$ for $k = 0, 1, \ldots, l(n)$;
(2) λ_n is linear on each of the intervals $[t_{n,k-1}, t_{n,k}]$, $k = 1, \ldots, l(n)$.

The relations $t_{n,k}-t_{n,k-1} \geqslant \delta_n > 0$, $|t_{n,k}-s_{n,k}| < \delta_n^2$ and $\lim_{n\to\infty} \delta_n = 0$ immediately imply

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in[0,1]} \left|\log \frac{d\lambda_n(t)}{dt}\right| = 0.$$

Moreover, it follows from (a) and from (2) that

(3) $$\lim_{n\to\infty} \sup_{t\in\pi_n} r\big(x(t), x_n(\lambda_n(t))\big) \vee r\big(x(t-), x_n(\lambda_n(t)-)\big) = 0.$$

To complete the proof it remains to show that the former limit relation remains true also when the supremum is extended over the whole interval [0, 1], i.e.,

(4) $$\lim_{n\to\infty} \sup_{t\in[0,1]} r\big(x(t), x_n(\lambda_n(t))\big) = 0.$$

We again proceed *ad absurdum*. Suppose that (4) is not true. Then, by (a) and by the uniform convergence of $\{\lambda_n\}$ to the identity function, there exist a sequence $\{t_n\}$ in [0, 1] and a strictly increasing sequence of naturals $\{m_n\}$ such that the four limits exist:

(5) $$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \lambda_{m_n}(t_n) = t, \quad \lim_{n\to\infty} x(t_n) = \mathfrak{g} \quad \text{and} \quad \lim_{n\to\infty} x_{m_n}\big(\lambda_{m_n}(t_n)\big) = \mathfrak{h},$$

t being a point of jump of x, and $(\mathfrak{g}, \mathfrak{h})$ being equal either to $\big(x(t), x(t-)\big)$ or to $\big(x(t-), x(t)\big)$.

Put

$$u_n = \max\{t \in \pi_{m_n}\colon t \leqslant t_n\}, \quad v_n = \min\{t \in \pi_{m_n}\colon t_n < t\}.$$

Then

(6) $$u_n \leqslant t_n < v_n$$

and $v_n - u_n \leqslant 2\delta_{m_n}$, so that

(7) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \lambda_{m_n}(u_n) = \lim_{n\to\infty} \lambda_{m_n}(t_n) = \lim_{n\to\infty} \lambda_{m_n}(v_n) = t.$$

Moreover, by (1) and (3) we have

$$\text{(8)} \qquad \lim r\big(x(u_n), x(v_n-)\big) = \lim r\big(x(u_n), x_{m_n}(\lambda_{m_n}(u_n))\big) = \lim r\big(x(v_n-), x_{m_n}(\lambda_{m_n}(v_n)-)\big) = 0.$$

By (6) and (7), either $\lim x(u_n) = g$ or $\lim x(v_n-) = g$, whence, by (8), $\lim x_{m_n}(\lambda_{m_n}(u_n)) = \lim x_{m_n}(\lambda_{m_n}(v_n)-) = g$. For every n choose v_n' so that $t_n < v_n' < v_n$ and $r\big(x_{m_n}(\lambda_{m_n}(v_n')), x_{m_n}(\lambda_{m_n}(v_n)-)\big) \leqslant 1/n$. Then

$$\text{(9)} \qquad \lambda_{m_n}(u_n) \leqslant \lambda_{m_n}(t_n) \leqslant \lambda_{m_n}(v_n') \quad \text{for each } n,$$

$$\lim \lambda_{m_n}(u_n) = \lim \lambda_{m_n}(v_n') \quad \text{and} \quad \lim x_{m_n}(\lambda_{m_n}(u_n)) = \lim x_{m_n}(\lambda_{m_n}(v_n')) = g.$$

Since, by assumption, x and $\{x_n\}$ satisfy (a) and (g), the sequence $\{x_{m_n}\}$ also satisfies (g), by Lemma 3. Consequently, (9) implies that $\lim x_{m_n}(\lambda_{m_n}(t_n)) = g$, which is incompatible with (5). This contradiction finishes the proof.

COROLLARY. *For every $x \in D_E[0, 1]$ and every sequence $\{x_n\}$ in $D_E[0, 1]$ the three statements are equivalent*:

(i) $\{x_n\}$ *converges to x in the sense of Skorokhod,*

(ii) *conditions* (a), (b) *and* (c) *are satisfied,*

(iii) *conditions* (a) *and* (g) *are satisfied.*

Indeed, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are all contained in the proofs of our Theorem and of Lemma 1.

References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, Inc., 1968.

[2] S. N. Ethier and T. G. Kurtz, *Markov Processes, Characterization and Convergence*, John Wiley & Sons, Inc., 1986.

Reçu par la Rédaction le 10.05.1988