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On the reduction theorems

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1. In this paper we shall give new proofs of the classical reduction theorems of affine and Riemannian geometry. This is an answer to a problem of J. Aczél and S. Gołąb (see [1], p. 154). The reduction theorems have a long history, for the bibliography see [12], p. 155 and 165. The first formulation of the reduction theorems is due to Christoffel in [4]. For further references see [2], [3], [6], [10], [11], [14], and [15].

Most authors consider the reduction theorems for the concomitants regarded as quantities, with additional assumptions concerning those objects.

In this work we shall prove the reduction theorems for the pure differential geometric objects of the first class. In our considerations we presume the minimal regularity assumptions on the affine (i.e. symmetric) connexion and metrical tensor. The proofs base on the theory of geometric objects and theory of functional equations.

We first recall the notion of the differential concomitant (see [6] and [7]). Let Ω^1 be any geometric object of the p-th class whose transformation law under a coordinate transformation $\xi^\lambda \to \xi^{\lambda'}$, $\lambda = 1, \ldots, n$, is

$$\Omega^{\Lambda'} = F^{\Lambda'}(\Omega^\Sigma, \xi^v, \xi^{v'}, L), \tag{1.1}$$

where $L \in \mathscr{L}_p^n$ ($\mathscr{L}_p^n$ denotes a differential group of rank p).

The components $(\Omega^\Sigma, \partial_{v_1}\Omega^\Sigma, \ldots, \partial_{v_p,\ldots,v_1}\Omega^\Sigma)$ determine a geometric object with the following transformation law,

$$\begin{aligned} \Omega^{\Lambda'} &= F^{\Lambda'}(\Omega^\Sigma, \xi^v, \xi^{v'}, L), \\ \partial_{v_1'}\Omega^{\Lambda'} &= \partial_{v_1'}F^{\Lambda'}(\Omega^\Sigma, \xi^v, \xi^{v'} L), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \partial_{v_p',\ldots,v_1'}\Omega^{\Lambda'} &= \partial_{v_p',\ldots,v_1'}F^{\Lambda'}(\Omega^\Sigma, \xi^v, \xi^{v'}, L), \end{aligned} \tag{1.2}$$

where the right-hand side of (1.2) is obtained by partial differentiation of F^Λ with respect to $\xi^{v'}$.

The geometric object $(\Omega^\Sigma, \partial_{v_1}\Omega^\Sigma, \ldots, \partial_{v_p,\ldots,v_1}\Omega^\Sigma)$ is called a *differential extension* of Ω^Σ of rank p, and will be denoted by $\partial^p\Omega$.

The geometric object Θ^{Σ} is called a *differential concomitant* of rank p of the geometric object Ω^{Λ} if Θ^{Σ} is a concomitant of $\partial^p \Omega$.

It follows ([7], p. 51) that Θ^{Σ} is given by the formula

$$\Theta^{\Sigma} = f^{\Sigma}(\partial^p \Omega^{\Lambda}), \tag{1.3}$$

where $f^{\Sigma}(\partial^p \Omega)$ is an invariant function, i.e.

$$\Theta^{\Sigma'} = f^{\Sigma'}(\partial^p \Omega^{\Lambda'}) = \delta^{\Sigma'}_{\Sigma} f^{\Sigma}(\partial^p \Omega^{\Lambda'}) \tag{1.4}$$

in every coordinate system ξ^{v} ($\delta^{\Sigma'}_{\Sigma}$ denotes the Kronecker symbol).

Our formulation of the reduction theorem for the affine connexion reads as follows:

THEOREM 1. *Let Ω be any pure differential geometric object of the first class. If Ω is a differential concomitant of rank p of the affine connexion $\Gamma^{\lambda}_{\mu v}$, then Ω is a concomitant of the tensors*

$$R^{\lambda}_{\varrho\mu v},\ \nabla_{v_1} R^{\lambda}_{\varrho\mu v},\ \ldots,\ \nabla_{v_{p-1},\ldots,v_1} R^{\lambda}_{\varrho\mu v},$$

where $R^{\lambda}_{\varrho\mu v}$, ∇ denote the curvature tensor and the covariant derivative with respect to $\Gamma^{\lambda}_{\mu v}$.

Let $\{\Phi_k\}_{k=1,\ldots,s}$ be a set of quantities (see [12], p. 68).

THEOREM 2. *Let Ω be any pure differential geometric object of the first class. If Ω is a differential concomitant of rank p of $\{\Phi_k\}_{k=1,\ldots,s}$ and of the affine connexion $\Gamma^{\lambda}_{\mu v}$, then Ω is a concomitant of the quantities*

$$\{\nabla_{v_l,\ldots,v_1} \Phi_k\},\quad l = 0, 1, \ldots, p;\ k = 1, \ldots, s,$$
$$\{\nabla_{v_l,\ldots,v_1} R^{\lambda}_{\varrho\mu v}\},\quad l = 0, 1, \ldots, p-1,$$
$$\nabla_0 \Phi_k \overset{\mathrm{df}}{=} \Phi_k.$$

We generalize this theorem to the case of the linear connexion $\Gamma^{\lambda}_{\mu v}$. Write $\bar{\Gamma}^{\lambda}_{\mu v} = \Gamma^{\lambda}_{(\mu v)}$ and $S^{\lambda}_{\mu v} = \Gamma^{\lambda}_{[\mu v]}$.

THEOREM 3. *Let Ω be any pure differential geometric object of the first class. If Ω is a differential concomitant of rank p of $\{\Phi_k\}_{k=1,\ldots,s}$ and of the linear connexion $\Gamma^{\lambda}_{\mu v}$, then Ω is a concomitant of the quantities:*

$$\{\bar{\nabla}_{v_l,\ldots,v_1} \Phi_k\},\quad l = 0, 1, \ldots, p;\ k = 1, \ldots, s,$$
$$\{\bar{\nabla}_{v_l,\ldots,v_1} S^{\lambda}_{\mu v}\},\quad l = 0, 1, \ldots, p,$$
$$\{\bar{\nabla}_{v_l,\ldots,v_1} \bar{R}^{\lambda}_{\varrho\mu v}\},\quad l = 0, 1, \ldots, p-1,$$

where $\bar{R}^{\lambda}_{\varrho\mu v}$, $\bar{\nabla}$ denote the curvature tensor and the covariant derivative with respect to $\bar{\Gamma}^{\lambda}_{\mu v}$.

An important consequence of Theorem 2 is the reduction theorem for a tensor $a_{\lambda\mu}$ with the non-singular symmetric part $a_{(\lambda\mu)}$.

THEOREM 4. *Let a tensor $a_{\lambda\mu}$ have a non-singular symmetric part $g_{\lambda\mu} = a_{(\lambda\mu)}$. Denote by $s_{\lambda\mu}$ the skew-symmetric part of $a_{\lambda\mu}$. If the pure differential geometric object of the first class Ω is a differential concomitant of rank p of $a_{\lambda\mu}$, then Ω is a concomitant of tensors:*

$$R^{\lambda}_{\varrho\mu\nu}, \; \nabla_{v_1} R^{\lambda}_{\varrho\mu\nu}, \; \ldots, \; \nabla_{v_{p-2},\ldots,v_1} R^{\lambda}_{\varrho\mu\nu},$$
$$a_{\lambda\mu}, \; \nabla_{v_1} s_{\lambda\mu}, \; \ldots, \; \nabla_{v_p,\ldots,v_1} s_{\lambda\mu},$$

where $R^{\lambda}_{\varrho\mu\nu}$, ∇ denote a curvature tensor and the covariant derivative with respect to Christoffel's symbols determined by $g_{\lambda\mu}$.

In particular, for the non-singular symmetric tensor $g_{\lambda\mu}$ we obtain the reduction theorem of the Riemannian geometry.

THEOREM 5. *If the pure differential geometric object Ω of the first class Ω is a differential concomitant of rank p of $g_{\lambda\mu}$, then Ω is a concomitant of the tensors:*

$$g_{\lambda\mu}, \; R^{\lambda}_{\varrho\mu\nu}, \; \nabla_{v_1} R^{\lambda}_{\varrho\mu\nu}, \; \ldots, \; \nabla_{v_{p-2},\ldots,v_1} R^{\lambda}_{\varrho\mu\nu}.$$

In Section 2 we shall prove the fundamental Lemma 1. Sections 3—5 are devoted to the proofs of Theorems 1—5.

For the particular case $p = 1$, Theorems 1—5, where proved in [8] and [9].

In the sequel we restrict our consideration to pure differential geometric objects of the first class.

2. In this section we shall prove some lemmas. The basic role played by the following

LEMMA 1. *If numbers $T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$ fulfil the equations*

$$(2.1) \quad \begin{array}{lrl} \text{(i)} & T^{\lambda}_{v_l,\ldots,v_1\varrho\mu} - T^{\lambda}_{v_l,\ldots,v_2\varrho v_1\mu} & = U^{\lambda}_{v_l,\ldots,v_1\varrho\mu}, \\ \text{(ii)} & T^{\lambda}_{(v_l,\ldots,v_1)\varrho\mu} & = T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}, \\ \text{(iii)} & T^{\lambda}_{v_l,\ldots,v_1(\varrho\mu)} & = T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}, \\ \text{(iv)} & T^{\lambda}_{(v_l,\ldots,v_1\varrho\mu)} & = 0, \end{array}$$

where $U^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$ are given numbers, then $T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$ are linear functions of $U^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$.

We denote this fact by

$$(2.2) \qquad T = E_l(U), \quad l = 1, 2, \ldots$$

If T and U are tensors, then the function E_l determines T as a concomitant of U.

Proof of Lemma 1. We establish an arbitrary set of to lower indices $\{v_l, \ldots, v_1\varrho\mu\}$ of $T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$. Thus, for the every subset of indices $1, \ldots, n$ there correspond many components of T whose indices belong to this subset. Now we have the essential

Remark. If the set of two last indices $\varrho\mu$ of the components of T coincide, then those components are equal.

Indeed; it follows from the symmetry properties (ii) and (iii). Denote by $\{\xi_k\}_{k=1,\ldots,N}$ the essential components of T. For ξ_k the equations (2.1) have the form

$$\xi_k - \xi_j = b_{kj}, \quad \sum_{k=1}^{N} n_k \xi_k = 0. \tag{2.1'}$$

The second equality of (2.1') express (iv) in terms of ξ_k (n_k denotes the number of components, of T which are equal ξ_k, $n_k \geqslant 1$). Numbers $b^{\,k}_{\,j}$ are obtained in the following way:

Case 1° Let $T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$ correspond to ξ_k and $T^{\lambda}_{v_l,\ldots,v_2\varrho v_1\mu}$ to ξ_j. Then, by (i), we put $a_{kj} = U^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$.

Case 2° Let $T^{\lambda}_{v_l,\ldots,v_1\varrho\mu}$ correspond to ξ_k and $T^{\lambda}_{v_l,\ldots,v_3\varrho\mu v_2 v_1}$ to ξ_j. Then, by the symmetry properties (ii) and (iii), we get the following equalities:

$$\begin{aligned}
\xi_k - \xi_j &= T^{\lambda}_{v_l,\ldots,v_1\varrho\mu} - T^{\lambda}_{v_l,\ldots,v_3\varrho\mu v_2 v_1} \\
&= T^{\lambda}_{v_l,\ldots,v_1\varrho\mu} - T^{\lambda}_{v_l,\ldots,v_2\varrho v_1\mu} + T^{\lambda}_{v_l,\ldots,v_2\varrho v_1\mu} - T^{\lambda}_{v_l,\ldots,v_3\varrho\mu v_2 v_1} \\
&= U^{\lambda}_{v_l,\ldots,v_1\varrho\mu} + T^{\lambda}_{v_l,\ldots,v_3\varrho v_2\mu v_1} - T^{\lambda}_{v_l,\ldots,v_3\varrho\mu v_2 v_1} \\
&= U^{\lambda}_{v_l,\ldots,v_1\varrho\mu} + U^{\lambda}_{v_l,\ldots,v_3\varrho v_2\mu v_1},
\end{aligned}$$

and we put $b_{kj} = U^{\lambda}_{v_l,\ldots,v_1\varrho\mu} + U^{\lambda}_{v_l,\ldots,v_3\varrho v_2\mu v_1}$.

In a similar way we proceed for every pair v_k, v_l.

In view of the preceding Remark we can reduce every case to that described above. By our assumptions there exists a solution of (2.1'). We observe that this solution is unique. Indeed, a difference Θ_k of two solutions ξ_k, η_k of (2.1') fulfils,

$$\Theta_k - \Theta_j = 0,$$
$$\sum_{k=1}^{N} n_k \Theta_k = 0.$$

By an elementary calculation we obtain a solution of (2.1'),

$$\xi_k = -\frac{1}{(l+2)!}\sum_{s=1}^{N}\sum_{l=s}^{N-1} n_s b_{l,l+1} + \sum_{l=k}^{N-1} b_{l,l+1}. \tag{2.3}$$

Consequently, the system (2.1) has the unique solution. We denote this fact by (2.2). Equations (2.1) have the invariant tensorial form. Therefore the function E_l determines the tensor T as a concomitant of U. This completes the proof.

Now we introduce the normal coordinate system of order p (see [12], p. 158).

DEFINITION. The coordinate system is called the *normal coordinate system of order p* for the affine connexion $\Gamma^\lambda_{\mu\nu}$ at a point ξ_0 if the equations

$$\Gamma^\lambda_{\mu\nu} = 0, \quad \vdots \quad \partial_{(v_p,\ldots,v_1}\Gamma^\lambda_{\mu v)} = 0 \tag{2.4}$$

are satisfied in these coordinates at the point ξ_0.

We recall the following (see [12], p. 158):

LEMMA 2. *The normal coordinate system of order p can be introduced by a C_p coordinate transformation $\xi^{\lambda'} = \varphi^{\lambda'}(\xi^\lambda)$ with partial derivatives at ξ_0 $(\delta^\lambda_{\lambda'}, A^\lambda_{v_1'\lambda'}, \ldots, A^\lambda_{v_{p-1}',\ldots,v_1'\lambda'})$, where $\partial\xi^\lambda/\partial\xi^{\lambda'} = \delta^\lambda_{\lambda'}$ is the Kronecker symbol.*

Proof of Lemma 2. We notice that, by a coordinate transformation with partial derivatives at ξ_0 $(\delta^\lambda_{\lambda'}, 0, \ldots, A^\lambda_{v_k',\ldots,v_1'\mu'v'}, 0, \ldots, 0)$, we have

$$\partial_{v_k',\ldots,v_1'}\Gamma^{\lambda'}_{\mu'v'} = \partial_{v_k,\ldots,v_1}\Gamma^\lambda_{\mu v} + A^\lambda_{v_k',\ldots,v_1'\mu'v'}, \tag{2.5}$$

where $v_j = v_j'$, $\lambda = \lambda'$, $\mu = \mu'$, $v = v'$, and the partial derivatives of $\Gamma^\lambda_{\mu v}$ of order $q < k$ do not change.

For $p = 0$ (see [5], p. 200) we obtain the result substituting $A^{\lambda'}_{\mu'v'} = -\Gamma^\lambda_{\mu v}$ to (2.5).

We suppose that there exists a coordinate transformation fulfilling the assumption of Lemma 2, for $p = 1, \ldots, k-1$, or $\Gamma^\lambda_{\mu v} = 0$, $\partial_{(v_1}\Gamma^\lambda_{\mu v)} = 0, \ldots, \partial_{(v_{k-1},\ldots,v_1}\Gamma^\lambda_{\mu v)} = 0$.

Now we apply (2.5) with $A^\lambda_{v_k',\ldots,v_1'\mu'v'} = -\partial_{(v_k,\ldots,v_1}\Gamma^\lambda_{\mu v)}$; where $v_j' = v_j$, $\mu' = \mu$, $v' = v$. This completes the induction and the proof of Lemma 2.

3. Proof of Theorem 1. Let Ω be an arbitrary pure differential geometric object of the first class with the followin transformation law

$$\Omega' = F(\Omega, L), \tag{3.1}$$

where $L \in Gl(n, R) = \mathscr{L}^n_1$. Let ξ_0 be a given point. The components of $\partial^k\Gamma$ in the normal coordinate system of order p at ξ_0 will be denoted by an asterisk overhead. By Lemma 2 we conclude

$$\Omega = f(\overset{*}{\Gamma}{}^\lambda_{\mu v}, \ldots, \partial_{v_p,\ldots,v_1}\overset{*}{\Gamma}{}^\lambda_{\mu v}), \tag{3.2}$$

where the components $\partial^k\overset{*}{\Gamma}$ fulfil equation (2.4).

For $p = 0$ there exist no a non-trivial concomitants of $\Gamma^\lambda_{\mu v}$, see [9]. Now we apply induction with respect to p. For $p = 1$ we have

$$\partial_\varrho\overset{*}{\Gamma}{}^\lambda_{\mu v} - \partial_\mu\overset{*}{\Gamma}{}^\lambda_{\mu v} = R^\lambda_{\varrho\mu v}.$$

We use Lemma 1 with $l = 1$. Hence we obtain $\partial_\varrho\overset{*}{\Gamma}{}^\lambda_{\mu v} = E_1(R^\lambda_{\varrho\mu v})$ and consequently $\Omega = f\big(0, E_1(R^\lambda_{\varrho\mu v})\big)$.

Functions f and E_1 are invariant with respect to the action of $\mathscr{L}^n_1$. This completes the proof for $p = 1$.

Write $\nabla_k R = \nabla_{v_k,\ldots,v_1} R^{\lambda}_{\varrho\mu\nu}$.

We suppose that Theorem 1 is true for $p = 1, \ldots, k-1$ or, in the normal coordinates of order k, we have

$$\partial_{v_l,\ldots,v_1} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = H_l(R, \ldots, \nabla_l R) \quad \text{for } l = 1, \ldots, k-1, \tag{3.3}$$

where H_l are invariant functions with respect to the action of $\mathscr{L}^n_1$. By a differentiation of the formula defining the curvature tensor

$$R^{\lambda}_{\varrho\mu\nu} = 2\partial_{[\varrho}\Gamma^{\lambda}_{\mu]\nu} + 2\Gamma^{\lambda}_{[\varrho|\omega|}\Gamma^{\omega}_{\mu]\nu}$$

we obtain the equality

$$\nabla_{v_{k-1},\ldots,v_1} R^{\lambda}_{\varrho\mu\nu} = \partial_{v_{k-1},\ldots,v_1\varrho}\Gamma^{\lambda}_{\mu\nu} - \partial_{v_{k-1},\ldots,v_1\mu}\Gamma^{\lambda}_{\varrho\nu} + W_{k-1}(\partial^{k-1}\Gamma), \tag{3.4}$$

where W_{k-1} is a polynomial.

Of course, this formula is invariant with respect to the action of $\mathscr{L}^n_k$. Hence we have

$$\partial_{v_{k-1},\ldots,v_1\varrho} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} - \partial_{v_{k-1},\ldots,v_1\mu} \overset{*}{\Gamma}{}^{\lambda}_{\varrho\nu} = \nabla_{v_{k-1},\ldots,v_1} R^{\lambda}_{\varrho\mu\nu} - W_{k-1}(\partial^{k-1}\overset{*}{\Gamma}). \tag{3.5}$$

From the symmetry of the partial derivatives and of the affine connexion (on the lower indices) it follows that components $\partial_{v_{k-1},\ldots,v_1\varrho} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu}$ fulfil the equations of Lemma 1 with

$$U^{\lambda}_{v_{k-1},\ldots,v_1\varrho\mu\nu} = \nabla_{v_{k-1},\ldots,v_1} R^{\lambda}_{\varrho\mu\nu} - W_{k-1}(\partial^{k-1}\overset{*}{\Gamma}).$$

Hence we obtain

$$\partial_{v_{k-1},\ldots,v_1\varrho} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = E_k\big(\nabla_{k-1} R + W_{k-1}(\partial^{k-1}\overset{*}{\Gamma})\big). \tag{3.6}$$

From our assumptions we conclude that

$$\partial_{v_{k-1},\ldots,v_1\varrho} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = E_k\Big(R + W_{k-1}\big(0, E_1(R), H_1(R, \nabla_1 R), \ldots, H_{k-1}(R, \ldots, \nabla_{k-1} R)\big)\Big). \tag{3.7}$$

Every function on the right-hand side of (3.7) is invariant with respect to the action of $\mathscr{L}^n_1$. Substituting (3.7) into (3.2) we complete the proof of Theorem 1.

Remark. We can give more express formulas for the components of $\partial^p \overset{*}{\Gamma}$ in the normal coordinates of order p:

$$\overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = 0,$$

$$\partial_\varrho \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = E_1(R),$$

$$\partial_{v_1\varrho} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = E_2\big(\nabla_1 R - W_1(0, E_1(R))\big),$$

. . . .

$$\partial_{v_p,\ldots,v_1\varrho} \overset{*}{\Gamma}{}^{\lambda}_{\mu\nu} = E_{p+1}\Big(\nabla_p R - W_p\Big(0, E_1(R), E_2\big(\nabla_1 R - W_1(0, E_1(R))\big), \ldots, E_p\big(\nabla_{p-1} R - W_{p-1}(0, \ldots)\big)\Big)\Big).$$

4. Proof of Theorem 2. We apply induction with respect to p. Let Ω be any pure differential geometric object of the first class with transformation law (3.1). We shall give the proof for a pair of objects Φ, Γ. The reasoning for a system of objects $\{\Phi_k\}_{k=1,\dots,s}, \Gamma$ is analogous.

From (1.3) it follows that,

$$\Omega = f(\partial^p \Phi, \partial^p \Gamma). \tag{4.1}$$

For $p = 0$ we have

$$\Omega = f(\Phi, \Gamma).$$

In the normal coordinates of order p is

$$\Omega^* = F\big(\Omega, (\delta^{\lambda}_{\lambda'})\big) = \Omega = f(\Phi^*, 0).$$

But for the quantities we have $\Phi^* = \Phi$ and consequently $\Omega = f(\Phi, 0)$. This proves Theorem 2 for $p = 0$.

We next suppose that Theorem is true for $k \leqslant p-1$ and that in the normal coordinates of order p the following holds true

$$\partial_{v_k,\dots,v_1}\Phi = S_k(\Phi, \nabla_1\Phi, \dots, \nabla_k\Phi, R, \nabla_1 R, \dots, \nabla_{k-1}R), \tag{4.3}$$

where S_k is an invariant function with respect to the action of $\mathscr{L}^n_1$.

Further consideration will be preceded by

Remark. Covariant derivative of quantity Φ can be written invariantly with respect to the action of $\mathscr{L}^n_p$ in the form

$$\nabla_{v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} = \partial_{v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} + P_1(\partial^{l-1}\Phi, \partial^{l-1}\Gamma), \tag{4.4}$$

where P_l is a polynomial.

Indeed, we have the formula ([5], p. 190)

$$\begin{aligned}\nabla_{v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} &= \partial_{v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} - a\Gamma_{v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} + \Gamma^{\lambda_1,\dots,\lambda_q}_{v_1\varrho_1,\dots,\varrho_q}\Phi^{\varrho_1,\dots,\varrho_q}_{\mu_1,\dots,\mu_r} + \\ &\quad + \Gamma^{\sigma_1,\dots,\sigma_r}_{v_1\mu_1,\dots,\mu_r}\Phi^{\lambda_1,\dots,\lambda_q}_{\sigma_1,\dots,\sigma_r},\end{aligned} \tag{4.5}$$

where

$$\Gamma^{\lambda_1,\dots,\lambda_q}_{v_1\beta_1,\dots,\beta_q} = \sum_{i=1}^{q}\Gamma^{a_i}_{v_1\beta_i}\delta^{a_1}_{\beta_1}\dots\delta^{a_{i-1}}_{\beta_{i-1}}\delta^{a_{i+1}}_{\beta_{i+1}}\dots\delta^{a_q}_{\beta_q}, \qquad \Gamma_{v_1} = \Gamma^{\lambda}_{v_1\lambda}$$

and a is the weight of Φ.

This proves the Remark for $l = 1$. By a differentiation of formula (4.4) with respect to v_{l+1} we obtain,

$$\begin{aligned}\nabla_{v_{l+1},\dots,v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} &= \partial_{v_{l+1},\dots,v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r} + \partial_{v_{l+1}}P_l + \\ &\quad + \Gamma^{\lambda_1,\dots,\lambda_q}_{v_{l+1}\varrho_1,\dots,\varrho_q}\nabla_{v_l,\dots,v_1}\Phi^{\varrho_1,\dots,\varrho_q}_{\mu_1,\dots,\mu_r} - \Gamma^{\sigma_1,\dots,\sigma_l,\tau_1,\dots,\tau_r}_{v_{l+1},v_l,\mu_1,\dots,\mu_r}\nabla_{\sigma_1,\dots,\sigma_l}\Phi^{\lambda_1,\dots,\lambda_q}_{\tau_1,\dots,\tau_r} + \\ &\quad + a\Gamma_{v_{l+1}}\nabla_{v_l,\dots,v_1}\Phi^{\lambda_1,\dots,\lambda_q}_{\mu_1,\dots,\mu_r}.\end{aligned}$$

From our assumptions it follows that the polynomial

$$P_{l+1} = \partial_{v_{l+1}} P_l + \Gamma^{\lambda_1,\ldots,\lambda_q}_{v_{l+1},\varrho_1,\ldots,\varrho_q} \nabla_{v_l,\ldots,v_1} \Phi^{\varrho_1,\ldots,\varrho_q}_{\mu_1,\ldots,\mu_r} - \\ - \Gamma^{\sigma_1,\ldots,\sigma_l,\tau_1,\ldots,\tau_r}_{v_{l+1},v_l,\ldots,v_1,\mu_1,\ldots,\mu_r} \nabla_{\sigma_l,\ldots,\sigma_1} \Phi^{\lambda_1,\ldots,\lambda_q}_{\tau_1,\ldots,\tau_r} + a\Gamma_{v_{l+1}} \nabla_{v_l,\ldots,v_1} \Phi^{\lambda_1,\ldots,\lambda_q}_{\mu_1,\ldots,\mu_r}$$

fulfils (4.4) for $l+1$. This completes the induction.

Now we finish the proof of Theorem 2. From (4.4) we obtain,

$$\partial_{v_p,\ldots,v_1}\Phi = \nabla_{v_p,\ldots,v_1}\Phi - P_p(\partial^{p-1}\Phi, \partial^{p-1}\Gamma). \tag{4.6}$$

Thus, in the normal coordinates of order p, we have, by (4.3),

$$\partial_{v_p,\ldots,v_1}\Phi = \nabla_{v_p,\ldots,v_1}\Phi - P_p\big(\Phi, S_1(\Phi, \nabla_1\Phi), S_2(\Phi, \nabla_1\Phi, \nabla_2\Phi', R), \\ \ldots, S_{p-1}(\Phi, \nabla_1\Phi, \ldots, \nabla_{p-1}\Phi, R, \ldots, \nabla_{p-2}R)\big). \tag{4.7}$$

Therefore the partial derivatives $\partial_{v_p,\ldots,v_1}\Phi$ are expressed by the quantity Φ, the curvature tensor R and their covariant derivatives $\nabla_k\Phi$, $k = 1, \ldots, p$, $\nabla_l R$, $l = 1, \ldots, p-1$. Every function on the right-hand of (4.7) is an invariant function with respect to the action of $\mathscr{L}^n_1$. Substituting formula (4.7) to (4.1) we conclude our supposition for p. This completes the induction and the proof of Theorem 2.

5. Proof of Theorem 3. We notice that the object: linear connexion $\Gamma^\lambda_{\mu v}$ is equivalent to the system of objects $\overline{\Gamma}^\lambda_{\mu v}$ and $S^\lambda_{\mu v}$, where $\overline{\Gamma}^\lambda_{\mu v} = \Gamma^\lambda_{(\mu v)}$ and $S^\lambda_{\mu v} = \Gamma^\lambda_{[\mu v]}$. We search for the differential concomitant of rank p of the objects $\overline{\Gamma}^\lambda_{\mu v}$, $S^\lambda_{\mu v}$, $\{\Phi_k\}_{k=1,\ldots,s}$. Now we apply Theorem 2.

Proof of Theorem 4. Let $\Gamma^\lambda_{\mu v}$ denote Christoffel's symbols determined by $g_{\lambda\mu}$. We consider the system of objects; $\partial^p a$, $\partial^{p-1}\Gamma$. Object $\partial^{p-1}\Gamma$ is the differential concomitant of rank p of the tensor $a_{\lambda\mu}$. Hence we obtain the following simple;

Remark. Every differential concomitant of rank p of $a_{\lambda\mu}$ is a concomitant of the system $\partial^p a, \partial^{p-1}\Gamma$.

Now we apply Theorem 2 to the system of objects $a_{\lambda\mu}$, $\Gamma^\lambda_{\mu v}$. We conclude that the pure differential geometric object Ω of the first class is a differential concomitant of rank p of the tensor a, if Ω is a concomitant of the tensors:

$$a_{\lambda\mu}, \nabla_{v_1}a_{\lambda\mu}, \ldots, \nabla_{v_p,\ldots,v_1}a_{\lambda\mu}, R^\lambda_{\varrho\mu v}, \ldots, \nabla_{v_{p-2},\ldots,v_1}R^\lambda_{\varrho\mu v}.$$

But for Christoffel's symbols we have $\nabla_{v_l,\ldots,v_1}g_{\lambda\mu} = 0$, $l = 1, \ldots,$ and consequently $\nabla_{v_l,\ldots,v_1}a_{\lambda\mu} = \nabla_{v_l,\ldots,v_1}s_{\lambda\mu}$. This completes the proof of Theorem 4.

References

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