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On a fixed-point theorem of Banach type

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THEOREM 1. *Let (E, ϱ) be a metric space, let X be a non-void subset of E and let $g\colon X \to X$ be a map with these properties:*

(i) $$\bigwedge_{x,y}\big(x, y \in X \Rightarrow \varrho\big(g(x), g(y)\big) \leqslant \varrho(x, y)\big)$$

(we call g contractive);

(ii) $$\bigwedge_{x,y}\Big(\big(x, y \in X \wedge g(x) \neq x \wedge g(y) = y\big) \Rightarrow \varrho\big(g(x), y\big) < \varrho(x, y)\Big);$$

(iii) $$\bigvee_{y_0}\big(y_0 \in X \wedge g(y_0) = y_0\big);$$

(iv) *There exist $x_0, x_1 \in X$ and a subsequence $\{k_n\}$ of N such that*
$$\lim_{n\to\infty}\{g^{k_n}(x_0)\} = x_1.$$

Then $g(x_1) = x_1$ and $\lim\limits_{n\to\infty}\{g^n(x_0)\} = x_1$.

That (ii) cannot be omitted is shown by the example $E\colon = \boldsymbol{R}$, $X\colon = \boldsymbol{R}$, $\varrho(x, y)\colon = |x-y|$, $g(x)\colon = -x$, $y_0\colon = 0$, $x_0\colon = 1$, $x_1\colon = 1$, $k_n\colon = 2n$. We have $g(1) \neq 1$, however, (ii) is not fulfilled as can be seen by choosing $y\colon = 0$, $x\colon = 1$.

Proof of Theorem 1. For any fixed-point $z \in X$ of g we have
$$0 \leqslant \varrho\big(g^n(x_0), z\big) \leqslant \varrho\big(g^n(x_0), g^n(z)\big) \underset{(i)}{\leqslant} \varrho\big(g^{n-1}(x_0), g^{n-1}(z)\big)$$
$$\leqslant \varrho\big(g^{n-1}(x_0), z\big) \underset{(i)}{\leqslant} \varrho(x_0, z)$$
(all $n \in \boldsymbol{N}$), i.e.

(*) $\big\{\varrho\big(g^n(x_0), z\big)\big\}$ is bounded and non-increasing and therefore convergent in $\boldsymbol{R}$.

Putting $z\colon = y_0$ we obtain for $n \in \boldsymbol{N}$:
$$\varrho\big(g^{k_n+1}(x_0), y_0\big) \leqslant \varrho\big(g^{k_n+1}(x_0), g(x_1)\big) + \varrho\big(g(x_1), y_0\big)$$
$$\underset{(i)}{\leqslant} \varrho\big(g^{k_n}(x_0), x_1\big) + \varrho\big(g(x_1), y_0\big)$$

and so by (*) and (iv)

$$\varrho(x_1, y_0) = \lim_{n\to\infty}\{\varrho(g^{k_n+1}(x_0), y_0)\}$$
$$\leqslant \lim_{n\to\infty}\{\varrho(g^{k_n}(x_0), x_1)\} + \varrho(g(x_1), y_0) \leqslant \varrho(g(x_1), y_0).$$

By (ii) this implies $g(x_1) = x_1$. For $z := x_1$ we have now

$$\lim_{n\to\infty}\{\varrho(g^n(x_0), x_1)\} = \lim_{n\to\infty}\{\varrho(g^{k_n}(x_0), x_1)\} = 0,$$

i.e.

$$\lim_{n\to\infty}\{g^n(x_0)\} = x_1,$$

q.e.d.

For a contractive mapping we give now as an application of Theorem 1 a more constructive formulation of Schauder's fixed-point theorem:

THEOREM 2. *Let $(E, \|\ \|)$ be a strictly-convex Banach-space, let X be a non-void bounded closed convex subset of E, let $f\colon X \to X$ be a contractive completely continuous mapping, let $x_0 \in X$ and $a \in (0, 1)$. Then the sequence $\{x_n\}_{n\in N}$ iteratively defined by*

$$x_n := af(x_{n-1}) + (1-a)x_{n-1} \qquad (n \geqslant 1)$$

converges (in norm) to a fixed-point of f.

Proof. We want to verify the hypothesis of Theorem 1. We set $\varrho(x, y) := \|x-y\|$ and define $g\colon X \to X$ by $g(x) := af(x) + (1-a)x$ (convexity of X). Since f is contractive, g is contractive, too, so (i) holds. Now let $x, y \in X$ $(x \neq y)$ and $\|g(x)-g(y)\| = \|x-y\|$. The inequalities

$$\|x-y\| = \|a(f(x)-f(y)) + (1-a)(x-y)\|$$
$$\leqslant \|a(f(x)-f(y))\| + \|(1-a)(x-y)\| \leqslant \|x-y\|$$

then imply

$$(*) \qquad \|a(f(x)-f(y))\| + \|(1-a)(x-y)\| = \|a(f(x)-f(y)) + (1-a)(x-y)\|$$

and

$$(**) \qquad \|a(f(x)-f(y)) + (1-a)(x-y)\| = \|x-y\|.$$

Since $(E, \|\ \|)$ is strictly-convex because of (*) a $\lambda \geqslant 0$ exists with $a(f(x)-f(y)) = \lambda(1-a)(x-y)$ and by (**) we obtain $\lambda = \dfrac{a}{1-a}$, i.e. $f(x)-f(y) = x-y$.

Because of g's definition this is equivalent to $g(x)-x = g(y)-y$. Now let $x, y \in X$ and $g(x) \neq x$, $g(y) = y$. Then we have $\|g(x)-g(y)\| < \|x-y\|$; otherwise we have $g(x)-x = g(y)-y = 0$, i.e. $g(x) = x$ which is contradictory: (ii) holds. By Schauder's fixed-point theorem a fixed-point $y_0 \in X$ of f exists, i.e. (iii) holds. To establish (iv) let us define

$M := \overline{C[\overline{f(X)} \cup \{x_0\}]}$ (convex hull). Because of the compactness of $\overline{f(X)}$ and the completeness of E, M is a (sequentially) compact convex subset of E and since we have $g^n(x_0) \in M$ there are $u_0, u_1 \in X$ and a subsequence $\{k_n\}$ of N such that $\lim_{n\to\infty} \{g^{k_n}(u_0)\} = u_1$, (iv) holds. Now from Theorem 1 we have $g(u_1) = u_1$, i.e. $f(u_1) = u_1$ and the convergence $\lim_{n\to\infty} \{x_n\} = \lim_{n\to\infty} \{g^n(u_0)\} = u_1$, q.e.d.

Remark. For $\alpha := 1/2$ we obtain a theorem of M. Edelstein, [1], and for arbitrary $\alpha \in (0, 1)$ and a uniformly convex Banach-space E we have a result proved by Schaefer, [3].

References

[1] M. Edelstein, *A remark on a theorem of M. A. Krasnoselski*, Amer. Math. Monthly 73 (1966), pp. 509-510.

[2] J. Reinermann, *Über Fixpunkte kontrahierender Abbildungen in uniformen Räumen und deren Darstellung durch konvergente Iterationsverfahren*, Gesellschaft für Mathematik und Datenverarbeitung, Bonn 1968.

[3] H. Schaefer, *Über die Methode der sukzessiven Approximationen*, Jahresber. der Deutsch. Mathematiker-Vereinigung 59 (1957), pp. 131-140.

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