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Optimality conditions for a Bolza problem governed by a hyperbolic system of Darboux–Goursat type

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Abstract. In the paper we consider a linear optimization problem of Bolza type described by a system of partial differential equations in the space of functions absolutely continuous on the plane. We prove a necessary condition for optimality in the form of a maximum principle. The form of the corresponding theorem is analogous to the Pontryagin maximum principle for ordinary differential systems.

1. Introduction. Let us consider a dynamical system of Darboux–Goursat type

$$\text{(a)}\qquad \begin{aligned}\frac{\partial^2 z}{\partial t\partial z}(t, x) &= A_0(t, x)z(t, x)+A_1(t, x)\frac{\partial z}{\partial t}(t, x)\\ &\quad +A_2(t, x)\frac{\partial z}{\partial x}(t, x)+B(t, x)u(t, x),\end{aligned}$$

$$z(t, 0) = \varphi_1(t), \qquad z(0, x) = \varphi_2(x)$$

with a cost functional of the form

$$\text{(b)}\qquad \begin{aligned}F(z, u) &= \int_0^1\int_0^1\Big(c_0(t, x)z(t, x)+c_1(t, x)\frac{\partial z}{\partial t}(t, x)\\ &\quad +c_2(t, x)\frac{\partial z}{\partial x}(t, x)+d(t, x)u(t, x)\Big)dtdx\\ &\quad +\int_0^1\Big(e_1(t)z(t, 1)+e_2(t)\frac{\partial z}{\partial t}(t, 1)\Big)dt\\ &\quad +\int_0^1\Big(e_3(x)z(1, x)+e_4(x)\frac{\partial z}{\partial x}(1, x)\Big)dx.\end{aligned}$$

The system (a) will be considered in the space of functions absolutely continuous on the plane. The definition of this space was given in [8], [10], and its basic properties are proved in [9]. Among other things, it is shown there that, for any control $u \in L_\infty$, there exists a uniquely determined trajectory z which is an absolutely continuous function and depends continuously on the initial conditions φ_1 and φ_2. Systems of Darboux–Goursat type were the object of investigations of many mathematicians. Extensive references and a number of essential results concerning the existence of smooth solutions can be found in [1]. In [6], a sufficient condition for the existence of solutions in Sobolev spaces is proved.

In the present paper, the optimization system (a)–(b) is considered in the space of absolutely continuous functions. This permits the Dubovitskii–Milyutin method to be used to obtain necessary conditions for optimality, quite analogous to the Pontryagin maximum principle for ordinary systems. Bolza's problem in other function spaces was studied in [2], [7]. A comparison of our results with those obtained previously is given at the end of the paper (Remark 3).

2. Preliminaries. By P^2 we shall denote an interval in the space $\boldsymbol{R}^2$ of the form

$$P^2 = \{(t, x) \in \boldsymbol{R}^2;\ 0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant 1\}.$$

A function $z\colon P^2 \to \boldsymbol{R}$ is called *absolutely continuous on* P^2 if the function $F_z(Q)$ associated with z is an absolutely continuous function of an interval $Q \subset P^2$, and the functions $z(0, x)$ and $z(t, 0)$ are absolutely continuous as functions of one variable (cf. [8], [10]).

The space of all absolutely continuous functions on P^2 will be denoted by $W(t, x)$ or, shortly, by W.

The space W has the following properties:

(a) if $z \in W$, then there exist partial derivatives (in the classical sense) $\partial z/\partial t$, $\partial z/\partial x$, $\partial^2 z/\partial t \partial x$ and the total differential almost everywhere on P^2 (cf. [9]),

(b) if $z \in W$, then z is an absolutely continuous function in the sense of Tonelli,

(c) a necessary and sufficient condition for z to be absolutely continuous on P^2 ($z \in W$) is that z possess the following integral representation:

$$z(t, x) = c + \int_0^t l^1(\tau)\,d\tau + \int_0^x l^2(s)\,ds + \int_0^t \int_0^x l^3(\tau, s)\,d\tau\,ds,$$

where $c \in \boldsymbol{R}$ and l^1, l^2, l^3 are integrable functions. Moreover,

$$\frac{\partial^2 z(t, x)}{\partial t \partial x} = l^3(t, x), \qquad \frac{\partial z(t, x)}{\partial t} = l^1(t) + \int_0^x l^3(t, s)\,ds,$$

$$\frac{\partial z(t, x)}{\partial x} = l^2(x) + \int_0^t l^3(\tau, x)\,d\tau \qquad \text{for } (t, x) \in P^2 \text{ a.e. (cf. [9]).}$$

In W we introduce the norm by the formula

$$(1)\qquad \|z\|_W = |c| + \left\|\frac{\partial z(\cdot, 0)}{\partial t}\right\|_{L^1(0,1)} + \left\|\frac{\partial z(0, \cdot)}{\partial x}\right\|_{L^1(0,1)} + \left\|\frac{\partial^2 z(\cdot, \cdot)}{\partial t \partial x}\right\|_{L^1(P)}.$$

It is easy to prove that W with this norm is a separable Banach space. By $W^n = W^n(t, x)$ we denote the nth Cartesian power of $W = W(t, x)$.

Let $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$ be $n \times n$ matrix functions essentially bounded on P^2, g a function integrable on P^2, and φ_1, φ_2 vector-valued functions absolutely continuous on [0, 1] and such that $\varphi_1(0) = \varphi_2(0)$. In $W^n(t, x)$ we shall consider a system of differential equations of the form

$$(2)\qquad \frac{\partial^2 z}{\partial t \partial x} = \bar{A}_0(t, x) z(t, x) + \bar{A}_1(t, x)\frac{\partial z}{\partial t}(t, x) + \bar{A}_2(t, x)\frac{\partial z}{\partial x}(t, x) + g(t, x)$$

with the boundary conditions

$$(3)\qquad z(t, 0) = \varphi_1(t), \quad z(0, x) = \varphi_2(x).$$

A function $z \in W^n$ is called a *solution of the Darboux–Goursat problem* (2)–(3) in the sense of Carathéodory if z satisfies equation (2) for a.e. $(t, x) \in P^2$ and the boundary conditions (3) for any t and x from the interval [0, 1].

The following theorem holds:

THEOREM 1. *The system* (2) *with boundary conditions* (3) *has a unique solution in the sense of Carathéodory in the space* $W^n(t, x)$ (cf. [8], [10]).

Remark 1. By a simple transformation of variables one can prove that (2) with the "end" boundary conditions

$$(4)\qquad z(t, 1) = \varphi_1(t), \quad z(1, x) = \varphi_2(x) \quad (\varphi_1(1) = \varphi_2(1))$$

has a unique solution $z \in W^n(t, x)$.

Remark 2. The definition of absolutely continuous functions of two variables is easily extended by induction to the space $\boldsymbol{R}^n$, $n > 2$ (cf. [8], [10]).

Theorem 1 gives a sufficient condition for the existence of a solution in the sense of Carathéodory for the Darboux–Goursat problem. The existence of a solution in the classical sense was considered in many papers and monographs. Much information on this subject can be found in [1]. In [6], a sufficient condition for the existence of a solution in the Sobolev space was given.

3. The maximum principle for a Bolza problem. In the space $W^n(t, x)$ let us consider a distributed control system of the form

$$(5)\qquad \frac{\partial^2 z}{\partial t \partial x}(t, x) = A_0(t, x) z(t, x) + A_1(t, x)\frac{\partial z}{\partial t}(t, x) + A_2(t, x)\frac{\partial z}{\partial x}(t, x) + B(t, x) u(t, x)$$

with the boundary conditions

(6) $$z(t, 0) = \varphi_1(t), \quad z(0, x) = \varphi_2(x)$$

and the performance index

(7) $$F(z, u) = \int_0^1\int_0^1 \left(c_0(t, x)z(t, x) + c_1(t, x)\frac{\partial z}{\partial t}(t, x) + c_2(t, x)\frac{\partial z}{\partial x}(t, x) + d(t, x)u(t, x) \right) dtdx + \int_0^1 \left(e_1(t)z(t, 1) + e_2(t)\frac{\partial z}{\partial t}(t, 1) \right) dt + \int_0^1 \left(e_3(x)z(1, x) + e_4(x)\frac{\partial z}{\partial x}(1, x) \right) dx.$$

We shall assume that:

(a1) A_0, A_1, A_2 are $n \times n$ matrices with entries absolutely continuous on P^2 and with essentially bounded derivatives $\frac{\partial A_1}{\partial t}, \frac{\partial A_2}{\partial x}$ ($\frac{\partial A_1}{\partial t}(t, x)$ and $\frac{\partial A_2}{\partial x}(t, x)$ exist almost everywhere on P^2 by Lemma 3 of [9]),

(a2) B is an $n \times m$ matrix function integrable on P^2,

(a3) φ_1, φ_2 are absolutely continuous functions on [0, 1], and $\varphi_1(0) = \varphi_2(0)$,

(a4) u is an m-dimensional control vector such that $u(t, x) \in M$ for a.e. $(t, x) \in P^2$ and u is an essentially bounded function on P^2 ($u \in L^m_\infty(P^2)$); here M is a convex and closed subset of $\mathbf{R}^m$,

(a5) z is a trajectory of system (5) which belongs to the space $W^n = W^n(t, x)$ (z is the solution of (5) in the sense of Carathéodory; cf. Theorem 1),

(a6) c_0 is an integrable function on P^2, c_1 and c_2 are absolutely continuous on P^2 and $\partial c_1/\partial t$, $\partial c_2/\partial x$ are essentially bounded,

(a7) $d = d(t, x)$ is an essentially bounded vector-valued function,

(a8) $e_1 = e_1(t)$, $e_3 = e_3(x)$ are integrable functions on [0, 1], $e_2 = e_2(t)$, $e_4 = e_4(x)$ are absolutely continuous on [0, 1], and $e_4(1) = 0$.

Besides system (5)–(6) we shall consider an equation of the form

(8) $$\frac{\partial^2 \psi}{\partial t \partial x}(t, x) = -(c_0(t, x) - A_0^*(t, x)\psi(t, x)) + \frac{\partial}{\partial t}(c_1(t, x) - A_1^*(t, x)\psi(t, x)) + \frac{\partial}{\partial x}(c_2(t, x) - A_2^*(t, x)\psi(t, x))$$

with the end boundary conditions

$$\psi(t, 1) = \alpha_1(t), \qquad \psi(1, x) = \alpha_2(x), \tag{9}$$

where α_1 and α_2 satisfy the Volterra equations

$$\begin{gathered} -\alpha_1(t) + \int_t^1 \big(A_1^*(t, 1)\alpha_1(t) - e_1(t) - c_2(t, 1)\big)dt - e_2(t) = 0, \\ \alpha_1(1) - \alpha_2(x) + \int_x^1 \big(A_1^*(1, x)\alpha_2(x) - e_3(x) - c_1(1, x)\big)dx - e_4(x) = 0. \end{gathered} \tag{10}$$

System (8) will be referred to as the *conjugate system* to the Bolza optimal control problem (5)–(7).

We shall prove

LEMMA 1. *The conjugate system* (8) *with boundary conditions* (9) *has a unique solution* ψ *in the space of absolutely continuous functions on* P^2 ($\psi \in W^n(t, x)$).

Proof. The functions e_i satisfy assumption (a8). So, system (10) has a unique solution $\alpha_1 = \alpha_1(t)$, $\alpha_2 = \alpha_2(x)$, and the functions α_1, α_2 are absolutely continuous on [0, 1]. Moreover, $\alpha_1(1) = \alpha_2(1)$. Applying Theorem 1 and Remark 1, we obtain the assertion.

THEOREM 2 (The maximum principle for a Bolza problem). *If*

(i) *a control* u^* *and the corresponding trajectory* z^* *are optimal for the control problem* (5)–(7),

(ii) *system* (5)–(7) *satisfies assumptions* (a1)–(a8),

then there exists a unique absolutely continuous function $\psi = \psi(t, x)$ *which satisfies the conjugate system* (8), the *transversality conditions* (9) *and the maximum condition*

$$\begin{aligned} &[B^*(t, x)\psi(t, x) - d(t, x)]u^*(t, x) \\ &\qquad = \max_{u \in M} [B^*(t, x)\psi(t, x) - d(t, x)]u \quad \textit{for a.e. } (t, x) \in P^2. \end{aligned} \tag{11}$$

Proof. Without loss of generality one can put $\varphi_1 = \varphi_2 = 0$ in (5)–(7). In the proof of the maximum principle we shall apply the Dubovitskii–Milyutin method (cf. [3]). Denote by $W_0^n(t, x)$ the subspace of $W^n(t, x)$ consisting of functions $z = z(t, x)$ such that $z(t, 0) \equiv 0$ and $z(0, x) \equiv 0$. Put $E = W_0^n \times L_\infty^m$,

$$Q_1 = \{(z, u) \in E;\ u(t, x) \in M \text{ for a.e. } (t, x) \in P^2\},$$

$$Q_2 = \{(z, u) \in E;\ z_{tx} - A_0 z - A_1 z_t - A_2 z_x - Bu = 0\}.$$

Our problem is thus to minimize the integral functional $F = F(z, u)$ (cf. (7)) on $Q = Q_1 \cap Q_2$. Just as in [3], Ch. 12, we can calculate the cone of decrease K_0 and the dual cone K_0^* at (z^*, u^*).

We have

$$(12)\quad K_0 = \{(\bar z, \bar u)\in E;\ \int_0^1\int_0^1 (c_0\bar z + c_1\bar z_t + c_2\bar z_x + d\bar u)dtdx + \int_0^1 (e_1\bar z(t,1) + e_2\bar z_t(t,1))dt + \int_0^1 (e_3\bar z(1,x) + e_4\bar z_x(1,x))dx < 0\}$$

$$= \{(\bar z, \bar u)\in E;\ F(\bar z, \bar u) < 0\},$$

$$(13)\quad K_0^* = \{f_0\in E^*;\ f_0(\bar z, \bar u) = -\lambda_0\cdot F(\bar z, \bar u),\ \lambda_0 \geqslant 0\}.$$

Let K_1 be the cone of feasible directions for Q_1 at (z^*, u^*). Then if $f_1\in K_1^*$, it follows that $f_1 = (0, f_1')$, where $f_1'\in(L_\infty^m)^*$ is a support to $Q_1^* = \{u\in L^m;\ u(t, x)\in M\}$ at u^* (cf. [3], Th. 10.5). From the Lyusternik theorem it follows that the tangent subspace K_2 to the set Q_2 is of the form

$$(14)\quad K_2 = \{(\bar z, \bar u)\in E;\ \bar z_{tx} = A_0\bar z + A_1\bar z_t + A_2\bar z_x + B\bar u\}.$$

The dual cone K_2^* consists of all functionals $f_2\in E^*$ such that $f_2(\bar z, \bar u) = 0$ for $(\bar z, \bar u)\in K_2$. The application of Theorem 6.1 of [3] to our problem implies that there exist $f_0, f_1, f_2\in E^*$, not all zero, such that, for all $(\bar z, \bar u)\in E$,

$$(15)\quad f_0(\bar z, \bar u) + f_1(\bar z, \bar u) + f_2(\bar z, \bar u) = 0,$$

where f_0 is given by (13), $f_1(\bar z, \bar u) = f_1'(\bar u)$ is a support to Q_1' at u^* and f_2 vanishes on K_2 (cf. (14)). It is easy to notice that $K_1\cap K_2 \neq 0$. Thus, without loss of generality, we can put $\lambda_0 = 1$ (cf. [3], Ch. 6, Remark 3) and the Euler equation (15) takes the form

$$(16)\quad -\int_0^1\int_0^1 (c_0\bar z + c_1\bar z_t + c_2\bar z_x + d\bar u)dtdx - \int_0^1 (e_1\bar z(t,1) + e_2\bar z_t(t,1))dt - \int_0^1 (e_3\bar z(1,x) + e_4\bar z_x(1,x))dx + f_1'(\bar u) + f_2(\bar z, \bar u) = 0$$

for $(\bar z, \bar u)\in E$.

Let $\bar u$ be arbitrary. Then $\bar u$ determines a solution $\bar z\in W_0^n$ of the equation

$$(17)\quad \bar z_{tx} = A_0\bar z + A_1\bar z_t + A_2\bar z_x + B\bar u.$$

With this choice of $\bar z$ and $\bar u$, we have $f_2(\bar z, \bar u) = 0$ (cf. (14)), and so, equation (16) becomes

$$-\int_0^1\int_0^1 (c_0\bar z + c_1\bar z_t + c_2\bar z_x + d\bar u)dtdx - \int_0^1 (e_1\bar z(t,1) + e_2\bar z_t(t,1))dt - \int_0^1 (e_3\bar z(1,x) + e_4\bar z_x(1,x))dx + f_1'(\bar u) = 0.$$

Integrating by parts and using the conjugate system (8), we obtain

$$\int_0^1\int_0^1 \big([\psi_{tx}+(\psi A_1)_t+(\psi A_2)_x-\psi A_0]\bar{z}-d\bar{u}\big)dtdx$$
$$-\int_0^1 c_1(1,x)\bar{z}(1,x)dx-\int_0^1 c_2(t,1)\bar{z}(t,1)dt$$
$$-\int_0^1 e_1\bar{z}(t,1)dt-\int_0^1 e_2\bar{z}_t(t,1)dt$$
$$-\int_0^1 e_3\bar{z}(1,x)dx-\int_0^1 e_4\bar{z}_x(1,x)dx+f_1'(\bar{u})=0.$$

By a simple transformation and by formula (17) the Euler equation (15) takes the form

$$\int_0^1\int_0^1 (B^*\psi-d)\bar{u}\,dt\,dx+\int_0^1 [\psi(1,1)-\psi(1,x)$$
$$+\int_x^1 (A_1^*(1,x)\psi(1,x)-e_3(x)-c_1(1,x))dx-e_4(x)]\,\bar{z}_x(1,x)\,dx$$
$$+\int_0^1 [\int_t^1 (A_2^*(t,1)\psi(t,1)-e_1(t)-c_2(t,1))dt$$
$$-\psi(t,1)-e_2(t)]\,\bar{z}_t(t,1)dt+f_1'(\bar{u})=0. \tag{18}$$

The functions $\psi(t, 1)$ and $\psi(1, x)$ satisfy equations (9)–(10); thus the Euler equation (18) reduces to

$$\int_0^1\int_0^1 (B^*\psi-d)\bar{u}\,dt\,dx+f_1'(\bar{u})=0.$$

Since the functional f_1' is a support to $Q_1'=\{u\in L_\infty^m;\ u(t,x)\in M \text{ a.e.}\}$ at $u^*\in Q_1'$, we obtain the assertion.

Remark 3. Necessary conditions for optimality for systems of Darboux–Goursat type were investigated in [2], [5], [7]. In [5] and [2], the maximum principle for piecewise continuous controls is proved under the assumption that the system of partial differential equations considered possesses a unique solution in the class of continuous functions. Moreover, the existence of an optimal control is assumed. In [7], the optimization problem is considered in the space of measurable controls. The solutions of the system of partial differential equations (trajectories) are understood in the generalized sense and belong to a special Sobolev space. Making use of a variational method analogous to that used in [2], the author of [7] proves the existence of (in general, infinitely many) Lagrange multipliers in L_∞ and the maximum condition.

In our paper, the optimization problem is considered in the space of essentially bounded controls, and in the space of absolutely continuous trajectories. Similarly to ordinary linear systems, the existence of Lagrange multipliers in the space of absolutely continuous functions and the maximum condition are proved. Owing to the fact that the optimization problem is considered in the space of absolutely continuous functions on the plane, it was possible to apply the Dubovitskii–Milyutin method. There is no difficulty either in proving the existence of an optimal process (cf. [4]). The proof is analogous to the case of ordinary linear systems considered in the space of absolutely continuous functions of one variable.

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