ANNALES
POLONICI MATHEMATICI
XLIX (1988)

On certain classes of analytic functions

by R. Parvatham and S. Radha (Madras)

Abstract. We prove that if q is a regular function of the form $q(z) = 1+b_1 z = \dots$ $|z| = r \in (0, 1)$ and for $\alpha, c, \theta_1, \theta_2$ with $\alpha \geqslant 1$, $\operatorname{Re} c \geqslant 0$, $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ satisfies

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{q(z)+\frac{\alpha z q'(z)}{c\alpha+q(z)}\right\} d\theta > -\pi,$$

then $\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z)\, d\theta > -\pi$, and we show some application of this result.

Let f be analytic in the open unit disc E with $f(0) = 0 = f'(0)-1$ and $f(z)f'(z)/z \neq 0$ for z in E. Denote by S the class of these functions.

Let $P(\alpha)$ denote the class of these functions $f \in S$, satisfying the condition

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\} d\theta > -\pi$$

whenever $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, $z = re^{i\theta}$, $r < 1$, α being a non-negative real number. Functions in $P(\alpha)$ are called α-close-to-convex and investigated by Bharati [1]. These functions in $P(\alpha)$ unify the well-known classes of CS^*-close-to-starlike ($\alpha = 0$) defined in [5] and C-close-to-convex ($\alpha = 1$) of Kaplan [3].

In this paper we prove the following theorem and consider certain applications of the same.

Theorem 1. *Let q be a regular function of the form $q(z) = 1+b_1 z+ \dots$ If for every $r \in (0, 1)$, α, c, θ_1, θ_2 with $\alpha \geqslant 1$, $\operatorname{Re} c \geqslant 0$, $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{q(z)+\frac{\alpha z q'(z)}{c\alpha+q(z)}\right\} d\theta > -\pi, \tag{1}$$

then $\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z)\, d\theta > -\pi$, $z = re^{i\theta}$.

Proof. The function $h(z) = z\exp\int_0^z \frac{q(t)-1}{t}\,dt = z+ \dots$ is regular in E and we have $zh'(z)/h(z) = q(z)$; $h(z)/z \neq 0$ for $z \in E$. The differential equation $z^{\alpha} F^{1-\alpha}(z) F'^{\alpha}(z) = h(z)$ has a holomorphic solution $f(z)$ in E and so we have

$$\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)+(1-\alpha)\frac{zF'(z)}{F(z)} = \frac{zh'(z)}{h(z)} = q(z).$$

Consider the function $f(z)$ defined by the differential equation

$$z^{-1/\alpha} zF'(z) F^{1/\alpha-1}(z) = \alpha f^{1/\alpha}(z) - c\alpha F^{1/\alpha}(z) z^{-1/\alpha}, \tag{2}$$

where c is any complex number with $\operatorname{Re} c \geqslant 0$ and $\alpha \geqslant 1$. On differentiating and simplifying (2) we obtain

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) = q(z)+\frac{\alpha z q'(z)}{q(z)+c\alpha}. \tag{3}$$

From (1) and (3) it follows that $f(z) \in P(\alpha)$. Padmanabhan and Bharati [4] proved that $F(z)$ defined by (2) belongs to $P(\alpha)$, whenever $f(z)$ is in $P(\alpha)$ for $\alpha \geqslant 1$ and $\operatorname{Re} c \geqslant 0$. Thus, when $\alpha \geqslant 1$ we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{(1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)\right\} d\theta > -\pi, \quad |z| < 1$$

for every $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ and $z = re^{i\theta}$, which implies $\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z)\, d\theta > -\pi$.

Remark. When $\alpha = 1$ this reduces to a result of Blezu and Pascu [2].

THEOREM 2. *For $\alpha \geqslant 1$, we have $P(\alpha) \subset P(0) = CS^*$.*

Proof. Let $Q(z) \in P(\alpha)$, $\alpha \geqslant 1$ and $q(z) = zQ'(z)/Q(z)$. On taking logarithmic differentiations, we obtain

$$q(z)+\frac{\alpha z q'(z)}{q(z)} = \alpha\left(1+\frac{zQ''(z)}{Q'(z)}\right)+(1-\alpha)\frac{zQ'(z)}{Q(z)}.$$

Since $Q(z) \in P(\alpha)$, it follows that

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{q(z)+\frac{\alpha z q'(z)}{q(z)}\right\} d\theta = \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{(1-\alpha)\frac{zQ'(z)}{Q(z)}+\alpha\left(1+\frac{zQ''(z)}{Q'(z)}\right)\right\} d\theta > -\pi,$$

$0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, $|z| = r < 1$. Applying Theorem 1 with $c = 0$, we have

$\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z)\, d\theta > -\pi$ which is equivalent to $\int_{\theta_1}^{\theta_2} \operatorname{Re}[zQ'(z)/Q(z)]\, d\theta > -\pi$.

This completes the proof of the theorem.

THEOREM 3. *For $\alpha \geqslant 1$ and $\alpha \geqslant \beta > 0$ we have the inclusion relation:* $P(\alpha) \subset P(\beta)$.

Proof. Let $f(z) \in P(\alpha)$ with $\alpha \geqslant 1$. Since the case $\beta = 0$ was considered in Theorem 2, here we can assume that $\beta \neq 0$. Let us consider the identity

$$\beta\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\beta)\frac{zf'(z)}{f(z)} = \frac{\beta}{\alpha}\left\{\left(\frac{\alpha}{\beta}-1\right)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{zf'(z)}{f(z)}\right\}.$$

Since $f \in P(\alpha)$ and also $P(\alpha) \subset P(0)$ for $\alpha \geqslant 1$, we have

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\beta\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\beta)\frac{zf'(z)}{f(z)}\right\}d\theta \geqslant \frac{\beta}{\alpha}\left\{\left(\frac{\alpha}{\beta}-1\right)(-\pi)-\pi\right\} = -\pi,$$

thereby proving the theorem.

Next, let us see another application of Theorem 1 which establishes the fact that the class of close-to-star functions is closed under St. Ruscheweyh's integral operator [6].

THEOREM 4. *If $g(z) \in CS^*$, then for every $\alpha \geqslant 1$ and $\mathrm{Re}\, c \geqslant 0$ the function $G(z)$ defined by*

$$G(z) = \left\{\frac{c+1/\alpha}{z^c}\int_0^z t^{c-1} g^{1/\alpha}(t)\,dt\right\}^{\alpha}, \qquad z \in E \tag{4}$$

is also in the class CS^*

Proof. G is holomorphic in a neighbourhood of $z = 0$ and satisfies $G(0) = 0 = G'(0)-1$. Thus there exists an $R > 0$ such that $G(z) \neq 0$ in $0 < |z| < R$. We begin by showing that G is close-to-star in $|z| < R$. Hence $G(z) = p(z)\Phi(z)$, where $\Phi(z)$ is starlike and $\mathrm{Re}\, p(z) > 0$ in $|z| < R$,

$$|G(z)| > \frac{(1-|z|)|z|}{(1+|z|)^3} \tag{5}$$

in $|z| < R$. If possible, let $G(z_0) = 0$ with $|z_0| = R < 1$. Then for any $\varepsilon > 0$ there exists a neighbourhood of z_0 in which $|G(z)| < \varepsilon$. This contradicts (5). Hence $G(z) \neq 0$ in E.

On differentiating (4) and putting $q(z) = zG'(z)/G(z)$, we have:

$$\left(\frac{G(z)}{z}\right)^{1/\alpha}\left(\frac{q(z)}{\alpha}+c\right) = \left(c+\frac{1}{\alpha}\right)\left(\frac{g(z)}{z}\right)^{1/\alpha}.$$

Logarithmic differentiation of this yields

$$q(z)+\frac{zq'(z)}{(q(z)/\alpha)+c}=\frac{zg'(z)}{g(z)},\qquad \text{where } g(z)\in CS^*.$$

Hence

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left\{q(z)+\frac{zq'(z)}{(q(z)/\alpha)+c}\right\}d\theta=\int_{\theta_1}^{\theta_2}\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\}d\theta>-\pi$$

for any θ_1, θ_2 such that $0\leqslant\theta_1<\theta_2\leqslant 2\pi$, $z=re^{i\theta}$ and for $r<1$ and $\alpha\geqslant 1$. By an application of Theorem 1, we conclude that

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\, q(z)\,d\theta>-\pi\qquad \text{for } \alpha\geqslant 1,\ \mathrm{Re}\, c\geqslant 0;$$

that is,

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left\{\frac{zG'(z)}{G(z)}\right\}d\theta>-\pi$$

which shows that $G(z)\in CS^*$ for $\alpha\geqslant 1$ and $\mathrm{Re}\, c\geqslant 0$.

References

[1] R. Bharati, *On α-close-to-convex functions*, Proc. Indian Acad. Sci. 88 A (1979), 93-103.

[2] D. Blezu and N. Pascu, *Integral of univalent functions*, Mathematica (Cluj) 23 (1981), 5–8.

[3] W. Kaplan, *Close-to-convex Schlicht functions*, Michigan Math. J. 1 (1952), 169-185.

[4] K. S. Padmanabhan and R. Bharati, *On α-close-to-convex functions II*, Glasinik Mathematicki 16 (1981), 235-244.

[5] M. O. Reade, *On close-to-convex univalent functions*, Michigan Math. J. 3 (1955), 59-62.

[6] St. Ruscheweyh, *Eine Invarianzeigenschaft der Basilevic-Funktionen*, Math. Z. 134 (1973), 215-219.

THE RAMANUJAN INSTITUTE
UNIVERSITY OF MADRAS
MADRAS, INDIA

Reçu par la Rédaction le 20.05.1986