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On the existence and uniqueness of solutions of some partial differential functional equations

by Zdzisław Denkowski and Andrzej Pelczar (Kraków)

Abstract. The existence and uniqueness of solutions of some initial-boundary problems for partial differential functional equations are considered. Some methods of successive approximations are used to prove the existence of solutions, under suitable assumptions on right-hand members of such equations. These assumptions reduce, in some classical special cases, to well-known conditions of Kamke type. The problems considered here generalize, among others, some classical initial problems for delay type ordinary differential equations and also initial-boundary problems for partial differential equations of the hyperbolic type.

INTRODUCTION

The purpose of the present paper is to give some results on partial differential functional equations of n-th order, with unknown functions of n variables, considered in certain sets in the n-dimensional Euclidean space, with initial-boundary conditions of the Cauchy–Darboux type. Special cases of such equations are partial and ordinary delay equations of various types (in particular, the classical). We discuss here the questions of the existence, uniqueness and the convergence of some successive approximations for such equations, and — more generally — for some integro-differential-functional equations, since Cauchy–Darboux problems under consideration can be transformed into problems of existence of solutions of certain integro-differential-functional equations. If the operators A_μ appearing in our equations (see Section 3) are identities in corresponding classes of functions and the sets V_μ reduce to the surface S (see the notation introduced in Section 1), then we obtain the classical differential equation, which can be written in the form:

$$u_{x_1\ldots x_n} = f(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, \ldots, u_{x_1\ldots x_{n-1}}, \ldots, u_{x_2\ldots x_n})$$

with the initial-boundary condition of the Cauchy–Darboux type; in particular, in the case $n = 2$ we have the Cauchy–Darboux problem (a direct generalization of the two classical problems: of Cauchy and Darboux) for the equation

$$u_{xy} = f(x, y, u, u_x, u_y)$$

investigated by many authors (see for instance: [1], [7]–[9], [23], [24], [26], [30]; further, a very rich bibliography can be found in references given in the above papers and in the book [30]). Some generalizations of the problem of Cauchy and Darboux and also of the Cauchy–Darboux problem generalizing the both, are problems considered (and introduced) by Szmydt [28] (see also Lasota [16] and Bielecki [1]). Our generalizations are of some other kind; we consider equations in which some operators are applied to unknown functions. Such equations for $n = 2$ have been considered by Kisielewicz [6], Mangeron [17], Mangeron and Krivosein [18], [19], Palczewski [22] and for n arbitrary by Kwapisz and Turo [14], Turo [29], Pelczar [26] (only in the case of such functions f which do not depend on the partial derivatives of unknown functions) and Kłapyta [10], [11]. Let us note here that before some papers on n-dimensional functional differential equations have been published, the natural generalizations of the classical Darboux problem for the n-dimensional case had been discussed; we quote for example: Castellano [2], Conlan and Diaz [3], Glick [5], Nappi [20], Kwapisz, Palczewski and Pawelski [13] and others (for further references see for instance [26]).

The questions of existence and uniqueness of solutions, as well as of the convergence of successive approximations, should be always considered with respect to some class of regularity; one can require various conditions of regularity which must be fulfilled by solutions, or — in other words — one can try to find solutions (unique or not) in various classes of regularity and try to prove the convergence of successive approximations in various functional spaces. For example, in [23] and [24] there are considered regular solutions (being of the so called class C^*) of equations in the classical sense; in [27] corresponding equations are essentially understood in the sense "almost everywhere". Generalized solutions are considered for example by Kisyński in [7] and by Kisyński and Pelczar in [9]. In the first part of the present paper we shall consider the classical solutions, in the second part there are investigated equations in the sense "almost everywhere".

We do not present here any complete classification of the references from all points of view; in particular, we do not state precisely which of the referred papers deal with equations in Banach spaces and which with of them finite dimensional spaces. In the present paper we limit ourselves to the finite dimensional case only; some natural generalizations for Banach spaces are possible.

We shall consider, as mentioned above, some method of successive approximations, which has been used in various versions and under various assumptions in many papers, with respect to classical and generalized problems concerning partial and ordinary differential equations, integro-differential equations, functional differential and functional integral equations and very general functional equations in some abstract spaces (see for instance: [9], [12], [14], [15], [21], [24], [26], [27], [29], [30]).

The general idea of the method used here is taken from the fundamental paper of Ważewski [31]; we use in Section 10 in the proof of Theorem 10.1 the method slightly modified and adopted to generalized integro-functional equations, presented by Kwapisz and Turo in [14] and [15] (see also Turo [29] and other papers of these authors referred in [29]), and in Section 12, in the proof of Theorem 12.1, we use the successive approximations method taken almost directly from papers [23], [24], [26], based on [31].

In the second part of the paper we use the same method as in Section 10; certain possibilities of modifications, analogous to those from Section 12, are just mentioned, without details. For some general remarks on the methods used here we refer to paper [25].

Since the equations considered here are similar to those occurring in [14], [15] and [29], we shall give some remarks on the correspondence between them. Formally, our results are neither special cases of the main results of [14], [15], [29], nor any generalizations of them. The main results of the present paper are, however, some generalizations of theorems which can be viewed as certain special cases of the results from [14], [15], [29]; we mean have the theorems obtained by an application of the general results for the n-dimensional space, to integro-functional-differential equations of type (5.1). Here we shall consider very general operators A_μ. In particular, we admit operators being essentially of the delay type (including some constant delay-deviation of the independent variables). This is ensured by conditions concerning the sets V_μ. One can extend our results to Banach spaces and also to more general equations.

We shall make a few remarks on this subject in the sequel.

Note, finally, that the problem of the existence of solutions of equations of the same type as that considered here, without uniqueness, has been discussed in [4].

PART I

1. Preliminaries

1.1. Notation. By R, R_*, R_+, N, N_0 we denote — respectively — the sets of: real numbers, real non-negative, real positive numbers, positive integers, non-negative integers.

For a set A we denote, as usual, by A^n the Cartesian product $A \times \ldots \times A$ (n-times). Hence $R^n = \{(x_1, \ldots, x_n): x_i \in R\}$ is identified with the n-dimensional Euclidean space. We write R^n_* R^n_+ and N^n_0 in place of $(R_*)^n, (R_+)^n, (N_0)^n$, respectively. For a set $A \subset R^n$ we denote by ∂A, $\operatorname{int} A$ and $\bar{A}$ — respectively — the boundary, the interior and the closure of A.

If $b = (b_1, \ldots, b_n) \in R^n$, we write

$$(1.1) \qquad (-\infty, b] \overset{\mathrm{df}}{=} (-\infty, b_1] \times \ldots \times (-\infty, b_n].$$

If $a = (a_1, \ldots, a_n) \in R^n$ and $b = (b_1, \ldots, b_n) \in R^n$, then

$$(1.2) \qquad \begin{aligned} &a \leqslant b \overset{\mathrm{df}}{\Leftrightarrow} a_i \leqslant b_i \quad \text{for every } i \in \{1, \ldots, n\}, \\ &a \overset{0}{<} b \overset{\mathrm{df}}{\Leftrightarrow} a_i < b_i \quad \text{for every } i \in \{1, \ldots, n\}, \\ &a < b \overset{\mathrm{df}}{\Leftrightarrow} a \leqslant b \quad \text{and} \quad a_i < b_i \quad \text{for some } i \in \{1, \ldots, n\}. \end{aligned}$$

For $a, b \in R^n$ such that $a \leqslant b$ we write

$$(1.3) \qquad [a, b] = [a_1, b_1] \times \ldots \times [a_n, b_n].$$

The same convention is adopted for n-dimensional intervals:

$$(a, b), \ (a, b], \ [a, b)$$

(under the assumption that $a \overset{0}{<} b$).

If $\mu \in R^n$ and $x \in R^n$, then we define

$$(1.4) \qquad \mu \cdot x = (\mu_1 x_1, \ldots, \mu_n x_n)$$

(we shall often write μx in place of $\mu \cdot x$) and

$$(1.5) \qquad \Lambda_\mu = \{\mu x:\ x \in R^n\}.$$

If $\mu \in N^n_0$, then we write

$$(1.6) \qquad 1 - \mu = (1-\mu_1, \ldots, 1-\mu_n).$$

In particular, we shall consider $\mu \in R^n$ such that $\mu_i \in \{0, 1\}$ ($i = 1, \ldots, n$); in this case $1-\mu$ is also of the same type, $(1-\mu)_i = 1-\mu_i \in \{0, 1\}$ ($i = 1, \ldots, n$). In this special case the sets

$$\Lambda_\mu = \{(\mu_1 x_1, \ldots, \mu_n x_n):\ x \in R^n\}$$
$$\text{and} \quad \Lambda_{1-\mu} = \{\big((1-\mu_1)x_1, \ldots, (1-\mu_n)x_n\big):\ x \in R^n\}$$

are subsets of the union of all hyperplanes H_i ($i = 1, \ldots, n$), where H_i is defined by the equality: $x_i = 0$.

If $M = \{M_{ij}\}$ is a $(k \times p)$-matrix and $z \in R^k$, then we denote by

$$M \times z$$

the usual product; hence, this is the element of R^p whose coordinates are given by the formulae

$$(M\times z)_i = \sum_{j=1}^{k} M_{ij} z_j \qquad (i = 1, \ldots, p).$$

When using multiindices, we shall apply the standard notation: for $\mu = (\mu_1, \ldots, \mu_n) \in N_0^n$, we write

$$|\mu| = \mu_1 + \ldots + \mu_n. \tag{1.7}$$

We shall extend this notation to arbitrary $x \in R^n$, putting

$$|x| = |x_1| + \ldots + |x_n|. \tag{1.8}$$

We shall also use another symbol: if $x \in R^n$, then

$$|x|_0 = (|x_1|, \ldots, |x_n|). \tag{1.9}$$

Hence $|x|_0$ belongs to R^n.

1.2. If a function w is defined in a set A and has its values in a set B, then for any subset D of the set A, we denote by $w|_D$ the restriction of w to the set D.

If two functions u and w are defined in some sets A and B, respectively, and have values in a set C and, moreover,

$$u|_{A\cap B} = w|_{A\cap B}$$

(it is possible that $A\cap B = \emptyset$, then the above condition is fulfiled trivially), then we define the union $u\cup w$ as the function given by the formula

$$(u\cup w)(x) = \begin{cases} u(x) & \text{for } x\in A, \\ w(x) & \text{for } x \in B. \end{cases}$$

If $w\colon A\to R^s$ is a given function, then $|w|$ or $|w|_0$ denote, respectively, the functions:

$$A \ni x \to |w(x)| \in R_*$$

and

$$A \ni x \to |w(x)|_0 \in R_*^s$$

(cf. (1.8) and (1.9)).

Moreover, we put for such a function w:

$$\max\{w(x)\colon x\in A\} = \big(\max\{w_1(x)\colon x\in A\}, \ldots, \max\{w_s(x)\colon x \in A\}\big).$$

The same convention is used for $\sup w$, $\inf w$ and $\min w$.

If the domain of a function w is known and fixed in a particular problem, then we shall write shrotly $\max w$ instead of $\max\{w(x)\colon x\in \text{domain of } w\}$.

Finally, we introduce the natural convention: if $u, w: A \to R^p$, then

$$u \leqslant w \overset{\mathrm{df}}{\Leftrightarrow} u(x) \leqslant w(x) \quad \text{for every } x \in A$$

(that means (see (1.2)) that $u_i(x) \leqslant w_i(x)$ for every $i \in \{1, \ldots, p\}$, $x \in A$).

1.3. Preliminary assumptions. We shall consider the dimension n of the space R^n in which we carry our investigations as fixed. Suppose that $b \in R^n_+$ is fixed. Let Δ be a subset of R^n, considered also as fixed throughout the paper. We assume that

$$\Delta \text{ is connected, closed} \tag{1.10}$$

and fulfils the following condition:

(W) if $x = (x_1, \ldots, x_n) \in \Delta \setminus \{b\}$, then there exists $t = (t_1, \ldots, t_n) \in R^n_+$ such that $[x, x+t] \subset \Delta$ and, moreover,

$$\begin{gathered} \big(\{b_1\} \times (-\infty, b_2] \times \ldots \times (-\infty, b_n]\big) \cap R^n_* \subset \Delta, \\ \big((-\infty, b_1] \times \{b_2\} \times (-\infty, b_3] \times \ldots \times (-\infty, b_n]\big) \cap R^n_* \subset \Delta, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \big((-\infty, b_1] \times \ldots \times (-\infty, b_{n-1}] \times \{b_n\}\big) \cap R^n_* \subset \Delta, \end{gathered} \tag{1.11}$$

$$\Delta \subset [0, b]. \tag{1.12}$$

It is clear that $0 \in \Delta$ if and only if $\Delta = [0, b]$.

An example of a set Δ which is not an n-dimensional cube, but fulfils conditions (1.10)–(1.12) is the triangle ABC on the plane R^2, where $A = (0, 1)$, $B = (1, 1)$, $C = (1, 0)$.

For a point $x \in \Delta$ we write

$$\Delta_x = [0, x] \cap \Delta. \tag{1.13}$$

Observe that $\Delta_b = \Delta$. By S we denote the set

$$\Delta \cap \overline{[0, b] \setminus (\operatorname{int} \Delta) \setminus \{x \in R^n \colon x_i = b_i \text{ for some } i \in \{1, \ldots, n\}\}}. \tag{1.14}$$

It is clear that $S \subset \partial\Delta$ and that S is compact.

We suppose that S, being obviously a hypersurface in R^n, has the form

$$S = \Sigma^0 \cup \bigcup_i \Sigma_i, \tag{1.15}$$

where

$$\Sigma_i = S \cap \{x \in R^n \colon x_i = 0\}, \qquad i \in \{1, \ldots, n\} \tag{1.16}$$

and Σ^0 is a hypersurface, which we shall describe more precisely below.

Let us consider the following projection mappings:

$$\operatorname{proj}_k \colon R^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in R^{n-1}. \tag{1.17}$$

In particular we have

$$(1.18)\qquad \operatorname{proj}_i(\Sigma_i) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R^{n-1}:\ (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in \Sigma_i\}.$$

The required properties of the surface Σ^0 are described by the following

ASSUMPTION (A_0). *It is assumed that there are n functions φ_i ($i \in \{1, \ldots, n\}$) with the following properties:*

$$(1.19)\qquad \varphi_i\colon S_i \to [0, b_i],$$

where

$$(1.20)\qquad S_i = \overline{\operatorname{proj}_i(S)\setminus\operatorname{proj}_i(\Sigma_i)} \quad (i \in \{1, \ldots, n\}),$$

$$(1.21)\qquad \varphi_i(x) = 0 \quad \textit{for } x \in S_i \cap \operatorname{proj}_i(\Sigma_i),$$

$$(1.22)\qquad \varphi_i \textit{ is continuous } (i \in \{1, \ldots, n\}),$$

$$(1.23)\qquad \varphi_i \textit{ is strictly decreasing with respect to each variable,}$$

that is: if

$$x_j < y_j,$$

then

$$\varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) > \varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n) \quad (i \in \{1, \ldots, n\}),$$

$$(1.24)\qquad x_j^0 = \varphi_j(x_1^0, \ldots, x_{j-1}^0, x_{j+1}^0, \ldots, x_n^0) \quad \textit{if and only if} \quad x_k^0 = \varphi_k(x_1^0, \ldots, x_{k-1}^0, x_{k+1}^0, \ldots, x_n^0) \quad \textit{for every } k \in \{1, \ldots, n\}.$$

Remark 1.1. In implication (1.23) it is obviously assumed that $j \neq i$, since φ_i is defined for all variables except x_i.

Remark 1.2. In virtue of (1.24) we have the set equality

$$(1.25)\qquad \Sigma^0 = \{(x_1, \ldots, x_n) \in S\colon \text{there is an } i \in \{1, \ldots, n\} \text{ such that } x_i = \varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\} = \{(x_1, \ldots, x_n) \in S\colon \text{for every } i \in \{1, \ldots, n\} \text{ we have } x_i = \varphi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\}.$$

1.4. Differential operators. For $\mu \in N_0^n \setminus \{0\}$ we define the differential operator D_μ by the formula

$$(1.26)\qquad D_\mu u = \frac{\partial^{|\mu|} u}{\partial x^\mu} = \frac{\partial^{\mu_1 + \ldots + \mu_n}}{\partial x_1^{\mu_1} \ldots \partial x_n^{\mu_n}} u$$

(for sufficiently regular u).

Furthermore, we put

$$D_0 = D_{(0,\ldots,0)} = \text{identity} \tag{1.26}$$

and

$$D = D_{(1,\ldots,1)}. \tag{1.27}$$

This means that $D_0u = u$ and $Du = \dfrac{\partial^n}{\partial x_1 \ldots \partial x_n} u$.

By I_n we denote the set of all elements μ belonging to R^n_* such that $\mu_i = 0$ or $\mu_i = 1$ for $i \in \{1, \ldots, n\}$ and $|\mu| = \mu_1 + \ldots + \mu_n \leqslant n-1$. In the sequel we shall consider D_μ only for $\mu \in I_n$ or for $\mu = (1, \ldots, 1)$.

We extend notation (1.25)–(1.27) for systems of m functions $(y_1, \ldots, y_m)$: if

$$y = (y_1, \ldots, y_m): R^n \to R^m$$

is sufficiently regular, then we put

$$D^i_\mu y = D_\mu y_i \quad (i \in \{1, \ldots, m\}) \tag{1.28}$$

and

$$D_\mu y = (D^1_\mu y, \ldots, D^m_\mu y). \tag{1.29}$$

We shall extend the above notation to one side partial derivatives in Section 2.

1.5. The family of sets $\{V_\mu\}$. There is given a family $\{V_\mu\}_{\mu \in I_n}$ of closed and connected subsets of the set

$$\overline{(R^n \setminus R^n_*)} \cup ([0, b] \setminus \operatorname{int} \Delta) \tag{1.30}$$

such that

$$V_\mu \cap \Delta = S \quad \text{for every } \mu \tag{1.31}$$

and, moreover, for every $\mu \in I_n$, $\mu \neq 0$, we have

$$\overline{(\operatorname{int} V_\mu)} \cap [\Lambda_{1-\mu} \cap (\partial V_\mu \setminus \Sigma^0)] = \emptyset. \tag{1.32}$$

Condition (1.32) means that in the set $\Lambda_{1-\mu} \cap (\partial V_\mu \setminus \Sigma^0)$ there are no accumulation points of the interior of V_μ. This family will be considered as given and fixed throughout the paper.

At the figure we give an interpretation of the conditions introduced above with respect to the set Δ and the sets V_μ, in the case $n = 2$. In the figure there is also given an example of an impossible situation: the position of $V_{(0,1)}$ marked by the segment-line, is excluded.

1.6. Some constants. Let us consider the mapping

$$\Delta \ni x \to \int_{\Delta_x} dt \ (= \text{measure of } \Delta_x) \in R_*. \tag{1.33}$$

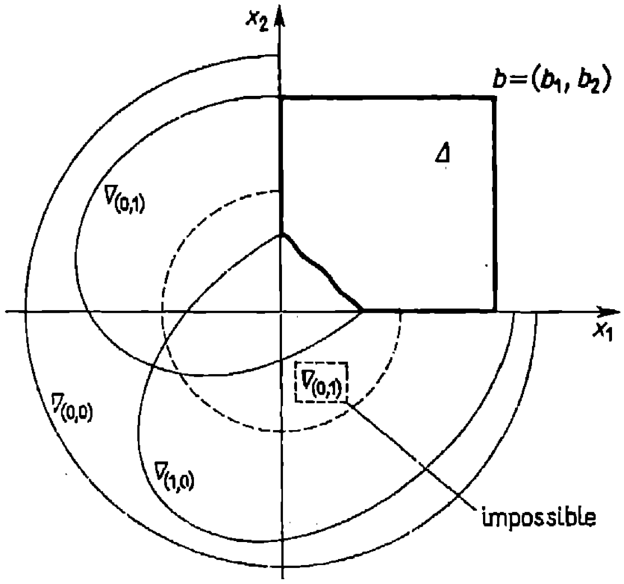

Fig. 1

It is well known that for every $\mu \in I_n$ and every $x \in \Delta$ there is a set $\Delta_{x,\mu} \subset R^{n-|\mu|}$, whose measure is uniquely determined by x and μ, such that

(1.34) $$D_\mu \int_{\Delta_x} dt = \int_{\Delta_{x,\mu}} ds$$

(more precisely:

$$\Big(D_\mu \int_{\Delta .} dt\Big)(x) = \int_{\Delta_{x,\mu}} (1-\mu)\, dt).$$

On the right-hand side of (1.34) we have the $(n-|\mu|)$-dimensional integral representing the $(n-|\mu|)$ measure of the set $\Delta_{x,\mu}$. Of course we have

$$\Delta_{x,\mu} = \mathrm{proj}_\mu(\Delta_x)$$

and this set considered as a subset of R^n is contained in $\Delta_{1-\mu}$ (see (1.5)); here by proj_μ we mean the composition of proj_i for such i for which $\mu_i = 1$ $(\mu = (\mu_1, \ldots, \mu_n))$.

For every $\mu \in I_n$ we put

(1.35) $$P^\mu = \int_{\Delta_{b,\mu}} ds \;\; (= \max\{\mathrm{meas}\, \Delta_{x,\mu} \colon x \in \Delta\}).$$

Let us now denote by J_μ (for any $\mu \in I_n$) the set of all $i \in \{1, \ldots, n\}$ for which $\mu_i = 0$; this means that $i \in J_\mu \Leftrightarrow$ [the i-th coordinate of the point μx is equal to zero for every $x \in \Delta$]. For $\mu \in I_n$ we denote by h_μ the mapping

(1.36) $$\Delta \ni (x_1, \ldots, x_n) \to \int_{\Delta_{x\,\mu}} ds \in R_*.$$

Now, for any $\mu \in I_n$ and $i \in J_\mu$ we define

(1.37) $T_i^\mu =$ [the least constant with which the mapping h_μ fulfils the Lipschitz condition with respect to the variable x_i].

2. Fundamental classes of functions

If $\mu \in N_0^n$, $\mu_i \in \{0, 1\}$, then by

$$C_\mu(\Delta \cup \Delta_\mu)$$

(or shortly C_μ) we denote the class of all real functions defined in $\Delta \cup \nabla_\mu$ which are of class C^μ, that is, for which all the partial derivatives

$$\frac{\partial^{|\nu|}}{\partial x^\nu} \quad \text{for } \nu = (\nu_1, \ldots, \nu_n),\ 0 \leqslant \nu \leqslant \mu,$$

exist and are continuous.

The meaning of the term: class C^μ, and thus also the precise sense of the definition of $C_\mu(\Delta \cup \nabla_\mu)$, will be explained in all details below.

(I) We say that a function $u\colon \Delta \to R$ is *of class* $C^\mu(\Delta)$ (u belongs to $C^\mu(\Delta)$) if and only if the following conditions hold true:

(a) u is of class C^μ in the ordinary sense, in the interior of Δ;

(b) if $\xi \in \Delta$ is such that $\xi_k = b_k$ for k belonging to $\{i_1, \ldots, i_r\}$ (where of course $r \leqslant n$ and $i_1, \ldots, i_r \in \{1, \ldots, n\}$), $\xi_k < b_k$ for $k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_r\}$ and $\xi_k > 0$ for $k \in \{1, \ldots, n\}$, then u has at the point ξ the continuous partial derivatives of the first order $\partial u/\partial x_j$ for these indices j for which $\xi_j < b_j$ and $\mu_j > 0$, and u has at the point ξ the continuous partial left-hand derivatives (left) $-\partial u/\partial x_j$ for these indices j for which $\xi_j = b_j$ and $\mu_j > 0$;

each derivative (left-hand derivative) $\partial \mu/\partial x_j$ has at the point ξ the continuous partial derivatives $\dfrac{\partial}{\partial x_s}\left(\dfrac{\partial u}{\partial x_j}\right)$ for s such that $\mu_s > 0$, $s \notin \{i_1, \ldots, i_r, j\}$, and continuous left-hand derivatives $\dfrac{\partial}{\partial x_t}\left(\dfrac{\partial u}{\partial x_j}\right)$ for t such that $\mu_t > 0$, $t \neq j$, $t \in \{i_1, \ldots, i_r\}$, etc. (cf. Pelczar [26], Definition 0.4.1);

(c) if $\xi \in \Delta$ is such that $\xi_k = 0$ for $k \in \{i_1, \ldots, i_r\}$ and $\xi_k \neq b_k$ for $k \in \{1, \ldots, n\}$, then u has at the point ξ the continuous partial derivatives of the first order $\partial u/\partial x_j$ for j such that $\xi_j > 0$, $\mu_j > 0$, and u has at the point ξ the right-hand derivatives (right) $-\partial u/\partial x_j$ for j such that $\xi_j = 0$, $\mu_j > 0$, etc. (see Definition 0.4.1 in [26]);

(d) If $\xi \in \Sigma^0$ and $\xi_k \neq b_k$ for $k \in \{1, \ldots, n\}$, then u has at the point ξ all continuous partial right-hand derivatives (right) $-\partial^{|\nu|}/\partial x^\nu$ for $\nu \leqslant \mu$, $\nu \in N_0^n$;

(e) if $\xi \in S$ and $\xi_k = b_k$ for $k \in \{i_1, \ldots, i_r\}$, then we assume that all partial derivatives $\left(\frac{\partial^{|\nu|}}{\partial x^\nu} u\right)(\cdot)$ regarded as functions defined in int Δ, have limits at ξ as $x \to \xi$, $x \in \Delta$;

these limits are denoted also by $\left(\frac{\partial^{|\nu|}}{\partial x^\nu} u\right)(\xi)$;

(f) it is supposed that the functions $\frac{\partial^{|\nu|}}{\partial x^\nu} u$, $\nu \leqslant \mu$, $\nu \in N_0^n$ (including the case $\nu = (0, \ldots, 0)$), where the symbols of partial derivatives denote in some cases left-hand derivatives, right-hand derivatives or limits of derivatives (see (b), (d), (e)) (and, in the case $\nu = 0$, denote the function itself) are continuous in Δ.

(II) We say that a function $w\colon V_\mu \to R$ is *of class* $C^\mu(V_\mu)$ (w belongs to $C^\mu(V_\mu)$) if and only if w is the restriction of a function $v\colon \tilde{V}_\mu \to R$, where $\tilde{V}_\mu$ is a set containing the set V_μ, open in $\overline{(R^n \setminus R_*^n) \cup ([0, b] \setminus \Delta)}$, such that $\tilde{V}_\mu \cap R_*^n = V_\mu \cap R_*^n$, and v is of class C^μ in the ordinary sense in $(\tilde{V}_\mu \setminus R_*^n) \cup$ $\cup ([0, b] \setminus \Delta)$ and has the continuous partial derivatives of suitable orders at points belonging to $\tilde{V}_\mu \cap R_*^n$ in the sense that — for instance — if ξ is such that $\xi_{i_1} = \xi_{i_2} = \ldots = \xi_{i_r} = 0$, then the partial derivatives with respect to $x_{i_1}, \ldots, x_{i_r}$ are taken as left-hand derivatives (cf. (b) in (I) above), etc.

(III) Now we define precisely the class $C_\mu(\Delta \cup V_\mu)$ as the set of all functions

$$u\colon \Delta \cup V_\mu \to R \tag{2.1}$$

such that, putting

$$v = u|_\Delta, \quad w_\mu = u|_{\Gamma_\mu}, \tag{2.2}$$

we obtain

$$v \in C^\mu(\Delta) \tag{2.3}$$

and

$$w_\mu \in C^\mu(V_\mu) \tag{2.4}$$

in the sense of (I) and (II), respectively, and, moreover,

$$\frac{\partial^{|\nu|} w_\mu}{\partial x^\nu}(\xi) = \frac{\partial^{|\nu|} v}{\partial x^\nu}(\xi) \quad (\nu \leqslant \mu) \tag{2.5}$$

for every $\xi \in S$ $(= V_\mu \cap \Delta)$, where the partial derivatives of w and v are defined as in (I) and (II), respectively.

Finally, we can define the class $\hat{C}_\mu$ with which we shall be concerned in the sequel.

DEFINITION 2.1. Let $\mu \in I_n$. We say that a function u is *of the class* $\hat{C}_\mu$ (shortly $u \in \hat{C}_\mu$) if and only if

$$u\colon \Delta \cup \nabla \to R, \tag{2.6}$$

where

$$\nabla = \bigcup \{\nabla_\nu\colon 0 \leqslant \nu \leqslant \mu, \nu \in N_0^n\} \tag{2.7}$$

and every restriction

$$u|_{\nabla_\nu \cup \Delta} \quad (0 \leqslant \nu \leqslant \mu,\ \nu \in N_0^n) \tag{2.8}$$

belongs to the class $C_\nu(\Delta \cup \nabla_\nu)$; this means that one can write shortly (with some inessential "non-formality") that

$$\hat{C}_\mu = \bigcap_{0 \leqslant \nu \leqslant \mu} C_\nu(\Delta \cup \nabla_\nu). \tag{2.9}$$

For any $p \in N$ we denote by $\hat{C}_\mu^p$ the Cartesian product $(\hat{C}_\mu)^p$, where

$$(\hat{C}_\mu)^p = \hat{C}_\mu \times \ldots \times \hat{C}_\mu \quad (p\text{-times}).$$

DEFINITION 2.2. We say that u *belongs to* $\hat{C}_{(1,\ldots,1)}$ if and only if $u \in \hat{C}_\mu$ for every $\mu \in I_n$ and, moreover, the restriction of u to the set Δ is of class $C^{(1,\ldots,1)}(\Delta)$ in the sense of (I).

DEFINITION 2.3. For $\mu \in I_n \cup \{1, \ldots, 1\}$ we denote by $\check{C}_\mu$ the class of all functions

$$v\colon \nabla \to R$$

such that

$$v|_{\nabla_\nu} \in C^\nu(\nabla_\nu)$$

(see (II)) for every $\nu \in N_0^n$, $\nu \leqslant \mu$.

By $\check{C}_\mu^k$ we denote the Cartesian product $(\check{C}_\mu)^k$.

Now we shall use the convention that the symbol D_μ is extended to derivatives in the sense of (I)–(III); this means that $D_\mu u$ can be equal to right-hand or left-hand derivative or can denote a suitable limit of derivatives.

In $\hat{C}_\mu$ we introduce the natural topology of uniform convergence of all partial derivatives D_ν for $\nu \leqslant \mu$, $\nu \in N_0^n$; of course, the convergence of D_ν is considered only in the set $\Delta \cup \nabla_\nu$.

This means that if $\{u_k\} \subset \hat{C}_\mu$, $k = 1, 2, \ldots$, $u \in \hat{C}_\mu$, then

$$u = \lim_{k\to\infty} u_k$$

in the above sense if and only if $D_\nu u_k$ tends uniformly in $\varDelta \cup V_\nu$ to $D_\nu u$ for each $\nu \in N_0^n$, $\nu \leqslant \mu$.

The same topology is considered in $\check{C}_\mu$.

For any $p \in N$ we introduce in $\hat{C}_\mu^p$ and $\check{C}_\mu^p$ the usual topology of the Cartesian product.

If V_μ is unbounded, then uniform convergence has to be replaced by almost uniform convergence.

We shall use the following abreviations:

$$(2.10) \qquad C_{0,\mu} = C_0(\varDelta \cup V_\mu), \qquad C_{0,\mu}^s = (C_{0,\mu})^s \qquad \text{for } \mu \in I_n,\ s \in N$$

(here $o = (0, \ldots, 0) \in R^n$).

3. Formulation of problem

Let m be a positive integer, fixed in the sequel, let k_μ $(\mu \in I_n)$ be positive integers, also fixed throughout the paper. We shall consider continuous operators

$$(3.1) \qquad A_\mu\colon\ C_{0,\mu}^m \to C_{0,\mu}^{mk_\mu}, \qquad \mu \in I_n.$$

Let

$$(3.2) \qquad f\colon\ \varDelta \times R^{m\Sigma k_\mu} \to R^m,$$

where

$$(3.3) \qquad \sum k_\mu = \sum \{k_\mu\colon \mu \in I_n\},$$

be a function having suitable regularity proprieties. We shall assume that f is continuous (in the first part of the paper) or that f fulfils the Carathéodory conditions (in the second part); the details will be formulated below.

Let

$$\varphi\colon\ V \to R^m$$

be a given function belonging to the class $\check{C}_{(1,\ldots,1)}^m$.

We shall consider the (generalized) Cauchy–Darboux Problem:

Problem (P). *Given a function* $\varphi \in \check{C}_{(1,\ldots,1)}^m$, *find a function* $y \in \hat{C}_{(1,\ldots,1)}^m$ *fulfilling the following equations:*

$$(3.4) \qquad (Dy)(x) = f\big(x, (A_0 y)(x), (A_{(1,0,\ldots,0)} D_{(1,0,\ldots,0)} y)(x), (A_{(0,1,0,\ldots,0)} D_{(0,1,0,\ldots,0)} y)(x), \ldots, (A_{(0,\ldots,0,1)} D_{(0,\ldots,0,1)} y)(x), \ldots, (A_\nu D_\nu y)(x), \ldots, (A_{(0,1,\ldots,1)} D_{(0,1,\ldots,1)} y)(x)\big) \quad \text{for } x \in \varDelta,$$

$$(3.5) \qquad y(x) = \varphi(x) \quad \text{for } x \in V,$$

$$(3.6) \qquad D_\nu (y|_\varDelta)(x) = (D_\nu \varphi)(x) \quad \text{for } x \in \sum\nolimits^0, \text{ and } \nu \in I_n.$$

4. Some special case of the Cauchy–Darboux problem

Let a function $\varphi \in \check{C}^m_{(1,\ldots,1)}$ be given. Consider the following Cauchy–Darboux problem:

$$(4.1) \qquad (Du)(x) = 0 \qquad \text{for } x \in \Delta,$$

$$(4.2) \qquad u(x) = \varphi(x) \qquad \text{for } x \in S,$$

$$(4.3) \qquad (D_\nu u)(x) = D_\nu \varphi(x) \qquad \text{for } x \in \sum\nolimits^0, \ \nu \in I_n.$$

PROPOSITION 4.1. *For every $\varphi \in \check{C}^m_{(1,\ldots,1)}$ problem* (4.1)–(4.3) *has exactly one solution of class $C^m_{(1,\ldots,1)}(\Delta)$.*

The proof of this proposition is quite elementary and will be omitted.

DEFINITION 4.1. *For a given $\varphi \in \check{C}^m_{(1,\ldots,1)}$ we shall denote by λ_φ the unique (in the class $C^m_{(1,\ldots,1)}(\Delta)$) solution of* (4.1)–(4.3).

5. Integral equations equivalent to the Cauchy–Darboux problem

5.1. We shall replace Problem (P) by some integral equation. It is a simple equation being a natural generalization of the classical integral equations considered in the theory of ordinary differential equations.

We first formulate the following easy

THEOREM 5.1. *Suppose that all asumptions introduced in the previous sections are satisfied. Assume that the function f* (see (3.2)) *is continuous and $\varphi \in \check{C}^m_{(1,\ldots,1)}$.*

Then Problem (P) *is equivalent to the following*

PROBLEM (P*). *Find a function*

$$y\colon \Delta \cup V \to R^m$$

belonging to $\hat{C}^m_\mu$ for every $\mu \in I_n$ such that (3.5) *and* (3.6) *are satisfied, and, moreover, satisfying, the following equality*:

$$(5.1) \qquad y(x) = \lambda_\varphi + \int\limits_{\Delta_x} f\big(t, (A_{(0,\ldots,0)}y)(t), \ldots, (A_{(0,1,\ldots,1)}D_{(0,1,\ldots,1)}y)(t)\big)\,dt \qquad \textit{for } x \in \Delta,$$

where λ_φ is defined as in Section 4.

We shall omit a trivial proof of this theorem.

5.2. Now we shall slightly modify equation (5.1).

It is easy to see that the number of elements of I_n is equal to

$$\sum_{k=0}^{n-1} \binom{n}{k} = 2^n - 1.$$

Let us put, m being fixed previously (cf. Section 3),

$$q = m(2^n - 1). \tag{5.2}$$

Let

$$Z = (Z_{(0,\dots,0)}, Z_{(1,0,\dots,0)}, Z_{(0,1,\dots,0)}, \dots, Z_{(0,1,\dots,1)}): \Delta \to R^q$$

be such that for every $\mu \in I_n$ the function

$$Z_\mu: \Delta \to R^m$$

belongs to the space $(C_0(\Delta))^m$ (shortly, $Z_\mu \in C_0^m$),

We shall speak shortly that such a mapping

$$Z: \Delta \to R^q$$

is continuous.

Suppose that φ is a function belonging to the class $\check{C}^m_{(1,\dots,1)}$. We assume that

$$Z_\mu|_S = D_\mu \varphi|_S \quad \text{for } \mu \in I_n \tag{5.3}$$

and we denote

$$\tilde{Z}_\mu = Z_\mu \cup D_\mu \varphi \quad \text{for } \mu \in I_n. \tag{5.4}$$

We shall write shortly

$$\int_{\Delta_x} f\big(t, (A\tilde{Z})(t)\big)\,dt \tag{5.5}$$

instead of

$$\int_{\Delta_x} f\big(t, (A_{(0,\dots,0)}\tilde{Z}_{(0,\dots,0)})(t), \dots, (A_\nu \tilde{Z}_\nu)(t), \dots, (A_{(0,1,\dots,1)}\tilde{Z}_{(0,1,\dots,1)})(t)\big)\,dt. \tag{5.6}$$

Now we define

$$F_0(x, Z) = \int_{\Delta_x} f\big(t, (A\tilde{Z})\big)(t)\,dt \quad \text{for } x \in \Delta, \tag{5.7}$$

and

$$F_\mu(x, Z) = \big(D_\mu F_0(\cdot, Z)\big)(x) \quad \text{for } x \in \Delta,\ \mu \in I_n. \tag{5.8}$$

Example. Let $n = 2$ and let

$$\Delta = \Delta \cup V = \{(x_1, x_2): g(x_1) \leqslant x_2 \leqslant b_2\} = \{(x_1, x_2): h(x_2) \leqslant x_1 \leqslant b_1\},$$

where g (resp. h) is a strictly decreasing continuous function from $[0, b_1]$ onto $[0, b_2]$ (resp. from $[0, b_2]$ onto $[0, b_1]$) such that $g = h^{-1}$, A_μ = identity (for $\mu = (0, 0), (1, 0), (0, 1)$), and, finally, let $f: R^2 \times R^3 \to R$ be a continuous function. Then, for

$$Z = (Z_{(0,0)}, Z_{(1,0)}, Z_{(0,1)}) = (u, v, w): \Delta \to R^3$$

and $x = (x_1, x_2)$, we have:

$$F_{(0,0)}(x, Z) = \int\limits_{\Delta\cap([0,x_1]\times[0,x_2])} f\big(s, t, u(s,t), v(s,t), w(s,t)\big)\,ds\,dt,$$

$$F_{(1,0)}(x, Z) = \int\limits_{g(x_1)}^{x_2} f\big(x_1, t, u(x_1,t), v(x_1,t), w(x_1,t)\big)\,dt,$$

$$F_{(0,1)}(x, Z) = \int\limits_{h(x_2)}^{x_1} f\big(s, x_2, u(s,x_2), v(s,x_2), w(s,x_2)\big)\,ds.$$

Let us turn our attention again to the general situation and consider the following

PROBLEM $(\tilde{\mathrm{P}})$. *Given a function $\varphi \in \check{C}^m_\mu$ or every $\mu \in I_n$, find a continuous function*

$$Z = (Z_{(0,\dots,0)}, Z_{(1,0,\dots,0)}, \dots, Z_{(0,1,\dots,1)}):\ \Delta \to R^q$$

such that

$$Z_\mu(x) = (D_\mu \lambda_\varphi)(x) + F_\mu(x, Z) \quad \text{for } x \in \Delta,\ \mu \in I_n, \tag{5.9}$$

where λ_φ is, as previously, the unique solution of (4.1)–(4.3).

THEOREM 5.2. *Under the assumption of Theorem* 5.1 *problems* (P) *and* $(\tilde{\mathrm{P}})$ *are equivalent.*

The proof is trivial; it is a simple consequence of the definitions of F_μ, Z_μ and $\tilde{Z}_\mu$.

6. Some generalized integral equations

The system of equations (5.9) can be considered as a special case of some more general system of equations. Since those generalized integral equations can be solved by the same methods and under — essentially — the same, or even more general, assumptions as equations (5.9), we shall consider in the sequel only this generalizations, obtaining also, implicitly, corresponding theorems for (5.9) as a special case.

Let us assume that the functions φ_i, $i \in \{1, \dots, n\}$, defining the set Σ^0 (see Section 1.3) fulfil the conditions of (A_0).

Suppose that there are given continuous functions

$$\lambda^\mu:\ \Delta \to R^m \quad (\mu \in I_n) \tag{6.1}$$

and continuous functions

$$\psi^\mu:\ \nabla_\mu \to R^m \quad (\mu \in I_n) \tag{6.2}$$

such that

$$\psi^\mu|_S = \lambda^\mu|_S \quad (\mu \in I_n). \tag{6.3}$$

We shall denote shortly by λ and ψ the corresponding systems

$$(\lambda^{(0,\ldots,0)}, \ldots, \lambda^{(0,1,\ldots,1)}) \quad \text{and} \quad (\psi^{(0,\ldots,0)}, \ldots, \psi^{(0,1,\ldots,1)})$$

of functions (6.1) and (6.2).

PROBLEM (Q). *Given λ and ψ as above, find a continuous function*

$$Z = (Z_{(0,\ldots,0)}, \ldots, Z_{(0,1,\ldots,1)}): \Delta \to R^q$$

such that

$$(6.4) \qquad Z_\mu(x) = \lambda^\mu(x) + \Phi_\mu^\psi(x, Z) \quad \textit{for} \quad x \in \Delta,\ \mu \in I_n,$$

where

$$(6.5) \qquad \Phi_\mu^\psi(x, Z) = D_\mu \int_{\Delta x} f\big(t, (A\hat{Z})(t)\big)\,dt \quad \textit{for} \quad x \in \Delta,\ \mu \in I_n,$$

with

$$(6.6) \qquad \hat{Z} = (\hat{Z}_{(0,\ldots,0)}, \ldots, \hat{Z}_{(0,1,\ldots,1)}), \quad \hat{Z}_\mu = Z_\mu \cup \psi^\mu,\ \mu \in I_n.$$

In order to exclude any confusion, let us explain precisely that (6.5) means

$$\Phi_0^\psi(x, Z) = \int_{\Delta x} f\big(t, (A\hat{Z})(t)\big)\,dt \quad \text{and} \quad \Phi_\mu^\psi(x, Z) = \big(D_\mu \Phi_0^\psi(\cdot, Z)\big)(x)$$

$$\text{for } x \in \Delta,\ \mu \in I_n,\ \mu > 0.$$

Here, similarly to the case of the definition of F_μ (see Section 5), we use the natural notation

$$\int_{\Delta_x} f\big(t, (A\hat{Z})(t)\big)\,dt = \int_{\Delta_x} f\big(t, (A_{(0,\ldots,0)}\hat{Z}_{(0,\ldots,0)})(t), \ldots, (A_\nu \hat{Z}_\nu)(t), \ldots$$
$$\ldots, (A_{(0,1,\ldots,1)}\hat{Z}_{(0,1,\ldots,1)})(t)\big)\,dt.$$

We shall use a further abbreviation: instead of

$$\big(\Phi^\psi_{(0,\ldots,0)}(\cdot, Z), \ldots, \Phi^\psi_{(0,1,\ldots,1)}(\cdot, Z)\big)$$

we shall write shortly $\Phi^\psi(Z)$; using this notation we can write the system (6.4) in the natural form

$$(6.7) \qquad Z = \lambda + \Phi^\psi(Z),$$

and formulate Problem (Q) as follows:

Given λ and ψ, find Z which is continuous and fulfils (6.7). Recording ψ as given, we shall sometimes write shortly Φ in place of Φ^ψ and $\Phi_\mu(x, Z)$ in place of $\Phi_\mu^\psi(x, Z)$.

Remark 6.1. Putting $\psi^\mu = D_\mu \varphi$ for a function φ belonging to the class $\check{C}^m_{(1,\ldots,1)}$, we obtain $\hat{Z}$ in the form of a $\tilde{Z}$ given by (5.4) and then we can write: if $\psi^\mu = D_\mu \varphi$, then $\Phi_\mu^\psi(x, Z) = F_\mu(x, Z)$ with F_μ given by (5.7) and (5.8). Hence Problem (Q) is a direct generalization of Problem ($\tilde{\mathrm{P}}$). In order to underline this fact, we do not use here the same notation as in Section 5; F_μ are replaced by Φ_μ.

7. Lipschitz conditions, boundedness and semicontinuity

Let a family of non-negative constants

$$\{M_i^{\mu,k}\}_{\mu\in I_n, k\in\{1,\ldots,m\}, i\in\{1,\ldots,n\}}$$

be given. For any fixed $\mu \in I_n$ and $k \in \{1, \ldots, m\}$ we shall denote by

$$M^{\mu,k}$$

the system $(M_1^{\mu,k}, \ldots, M_n^{\mu,k})$; hence

$$\{M^{\mu,k}\}_{\mu\in I_n, k\in\{1,\ldots,m\}} = \{(M_1^{\mu,k}, \ldots, M_n^{\mu,k})\}_{\mu\in I_n, k\in\{1,\ldots,m\}} \subset R_*^n.$$

Furthermore, we denote by M^μ the matrix

$$\{M_i^{\mu,k}\}_{i,k};$$

hence

$$\{M^\mu\}_{\mu\in I_n} = \{(M^{\mu,1}, \ldots, M^{\mu,m})\}_{\mu\in I_n}; \tag{7.1}$$

and finally the set (more precisely, the finite sequence) of all matrices M^μ will be denoted by M; this means that

$$M = (M^{(0,\ldots,0)}, \ldots, M^{(0,1,\ldots,1)}). \tag{7.2}$$

Suppose, moreover, that there is given a matrix

$$K = (K^{(0,\ldots,0)}, \ldots, K^{(0,1,\ldots,1)}) \tag{7.3}$$

with

$$K^\mu = (K_1^\mu, \ldots, K_m^\mu) \in R_*^m, \quad \mu \in I_n. \tag{7.4}$$

Let us consider a mapping

$$Z = (Z_{(0,\ldots,0)}, \ldots, Z_{(0,1,\ldots,1)}) \colon \Delta \to R^q.$$

DEFINITION 7.1. We say that the mapping Z *belongs to the class*

$$[\mathscr{B}(K), \mathscr{L}(M)]$$

(saying also that Z is bounded by K and fulfils the Lipschitz condition with the constant M) if and only if

$$|Z_\lambda(x)|_0 \leqslant K^\lambda \quad \text{for } x \in \Delta,\ \lambda \in I_n \tag{7.5}$$

and

$$(7.6)\qquad \big|Z_\mu\big(\mu x+(1-\mu)x\big)-Z_\mu\big(\mu x+(1-\mu)\tilde{x}\big)\big|_0 \leqslant \big|M^\mu\times\big((1-\mu)(x-\tilde{x})\big)\big|_0$$

for $\mu\in I_n$, $x,\tilde{x}\in\Delta$.

Remark 7.1. Condition (7.5) is equivalent to the following m systems of inequalities:

$$(7.5.j)\qquad |Z_{\lambda,j}(x)|\leqslant K_j^\lambda,\qquad x\in\Delta,\ \lambda\in I_n\ (j=1,\ldots,m).$$

Here, of course,

$$(*)\qquad Z_\lambda(x)=\big(Z_{\lambda,1}(x),\ldots,Z_{\lambda,m}(x)\big)\quad \text{for } x\in\Delta,\ \lambda\in I_n.$$

Conditions (7.6) are equivalent to the system of m systems of scalar inequalities

$$(7.6.k)\quad \big|Z_{\mu,k}\big(\mu x+(1-\mu)x\big)-Z_{\mu,k}\big(\mu x+(1-\mu)\tilde{x}\big)\big|\leqslant |M^{\mu,k}(1-\mu)(x-\tilde{x})|$$

for $x,\tilde{x}\in\Delta$, $\mu\in I_n$ (see $(*)$) $(k=1,\ldots,m)$.

On the left-hand side of (7.6.k) there is the usual absolute value, on the right-hand side we have

$$\sum_{j=1}^{n} M_j^{\mu k}(1-\mu_j)|x_j-\tilde{x}_j|$$

(see Section 1.1).

EXAMPLES. We give two examples for (7.6).

1° Consider the case $m=3$, $n=2$, $\mu=(1,0)$. Then condition (7.6) means that

$$|Z_{(1,0),i}(x_1,x_2)-Z_{(1,0),i}(x_1,\tilde{x}_2)|\leqslant M_2^{(1,0),i}|x_2-\tilde{x}_2|$$

for (x_1,x_2), $(x_1,\tilde{x}_2)\in\Delta$, $i=1,2,3$.

2° Let $m=3$, $n=3$, $\mu=(0,0,1)$. Then (7.6) means that

$$|Z_{(0,0,1),i}(x_1,x_2,x_3)-Z_{(0,0,1),i}(\tilde{x}_1,\tilde{x}_2,x_3)|\leqslant M_1^{(0,0,1),i}|x_1-\tilde{x}_1|+M_2^{(0,0,1),i}|x_2-\tilde{x}_2|$$

for (x_1,x_2,x_3), $(\tilde{x}_1,\tilde{x}_2,x_3)\in\Delta$, $i=1,2,3$.

Indeed, we have for $\mu_1=\mu_2=0$, $\mu_3=1$

$$M_1^{(0,0,1),i}(1-\mu_1)|x_1-\tilde{x}_1|+M_2^{(0,0,1),i}(1-\mu_2)|x_2-\tilde{x}_2|+M_3^{(0,0,1),i}(1-\mu_3)|x_3-\tilde{x}_3|$$
$$=M_1^{(0,0,1),i}|x_1-\tilde{x}_1|+M_2^{(0,0,1),i}|x_2-\tilde{x}_2|.$$

Remark 7.2. The Lipschitz condition with the "constant" M means in fact (cf. Examples) that our function fulfils the Lipschitz condition with respect to some variables; namely, with respect to the variables having indices i such that $\mu_i=0$.

DEFINITION 7.2. Let $\mu \in I_n$ be fixed and let a function

$$W\colon \Delta \to R^m$$

be given. We say that W *belongs to the class*

$$USC_\mu$$

if and only if for every $k \in \{1, \ldots, m\}$ and every $t \in \Delta$ the mapping

$$\Delta \ni x \to W_k\big((1-\mu)t+\mu x\big) \in R \tag{7.7}$$

is upper semicontinuous.

We say that a mapping

$$Z = (Z_{(0,\ldots,0)}, \ldots, Z_{(0,1,\ldots,1)})\colon \Delta \to R^q$$

belongs to the class

$$USC$$

if and only if for every $\mu \in I_n$

$$Z_\mu \in USC_\mu.$$

Remark 7.3. The definition of the class USC can be expressed shortly in the form:

$$USC = USC_{(0,\ldots,0)} \times USC_{(1,0,\ldots,0)} \times \ldots \times USC_{(0,1,\ldots,1)}.$$

DEFINITION 7.3. We denote

$$\mathscr{D}(K, M) = [\mathscr{B}(K), \mathscr{L}(M)] \cap USC. \tag{7.8}$$

DEFINITION 7.4. Let $\mu \in I_n$ be given. We say that a function

$$Z_\mu\colon \Delta \to R^m$$

belongs to the class

$$\mathscr{C}_\mu$$

if and only if for every $k \in \{1, \ldots, m\}$ and every $t \in \Delta$, the mapping

$$\Delta \ni x \to Z_{\mu,k}\big(\mu t+(1-\mu)x\big) \in R \tag{7.9}$$

is continuous.

We say that a mapping

$$Z = (Z_{(0,\ldots,0)}, \ldots, Z_{(0,1,\ldots,1)})\colon \Delta \to R^q$$

belongs to the class

$$\mathscr{C}$$

if and only if for every $\mu \in I_n$

$$Z_\mu \in \mathscr{C}_\mu.$$

DEFINITION 7.5. We denote

$$\mathscr{D}_* = \mathscr{C} \cap USC, \qquad \mathscr{D}_{*\mu} = \mathscr{C}_\mu \cap USC_\mu. \tag{7.10}$$

Definition 7.6. For every $\mu \in I_n$ we define the class $\boldsymbol{D}_\mu$ as follows:

(7.11) $$\boldsymbol{D}_\mu = \{U\colon \Delta \cup V_\mu \to R^m\colon\ U|_\Delta \in \mathscr{D}_{*,\mu} \text{ and } U|_{V_\mu} \text{ is continuous}\}.$$

8. Comparative functions

We shall consider a function

(8.1) $$\omega\colon\ \Delta \times R_*^{m\Sigma k_\mu} \to R_*^m$$

(k_μ are the same as in Section 3), on which we shall assume some conditions. The letter ω is reserved in the whole paper to denote this function. On account of condition (δ) (see Assumption (A_1) below) the function ω is very often called a *comparative function* for the function f.

We now introduce certain conditions, which we collect under the titles: Assumptions.

Assumption (A_1).

(α) *ω is continuous,*

(β) *if $x \in \Delta$, $u, v \in R^{m\Sigma k_\mu}$ and $u \leqslant v$, then $\omega(x, u) \leqslant \omega(x, v)$,*

(γ) *$\omega(x, 0) = 0$ for every $x \in \Delta$,*

(δ) *for every $x \in \Delta$, $u, v \in R^{m\Sigma k_\mu}$ the inequality*

(8.2) $$|f(x, u) - f(x, v)|_0 \leqslant \omega(x, |u-v|_0)$$

holds true.

Remark 8.1. The symbol $|\cdot|_0$ in (8.2) is applied to points in the spaces of dimension m (on the left-h nd side) and $m\sum k_\mu$ (on the right-hand side); this cannot lead to misunderstanding.

Suppose that there is a family of operators $\{B_\mu\}_{\mu\in I_n}$, which we shall consider as fixed throughout the paper; we shall regard this family as a finite sequence, ordered as previously:

$$(B_{(0,\ldots,0)}, B_{(1,0,\ldots,0)}, \ldots, B_{(0,1,\ldots,1)}).$$

Assumption (A_2).

(a) $B_\mu\colon \boldsymbol{D}_\mu \to \mathscr{D}_{*\mu}^{k_\mu}$, $\mu \in I_n$ (*here* $\mathscr{D}_\mu^s = (\mathscr{D}_\mu)^s$),

(b) *if Z, W belong to $C_{0\mu}^m$ (see Section 3) and*

$$Z|_{V_\mu} = W|_{V_\mu},$$

then the inequality

(8.3) $$|A_\mu Z - A_\mu W|_0 \leqslant B_\mu(|Z-W|_0)$$

holds true ($\mu \in I_n$),

(c) *for each $\mu \in I_n$ the operator B_μ is increasing, in the sense that if $U \leqslant V$, then $B_\mu U \leqslant B_\mu V$.*

In order to formulate further assumptions concerning the function ω, we introduce notation similar to that used in Sections 5 and 6.

Let $V = (V_{(0,\dots,0)}, \dots, V_{(0,1,\dots,1)})$ be a system of functions such that $V_\mu \in \boldsymbol{D}_\mu$. We denote:

$$(8.4)\quad \Omega_0(x, V) = \int_{\Delta_x} \omega\big(t, (BV)(t)\big)\,dt \qquad \text{for } x \in \Delta \ (o = (0, \dots, 0) \in R^n)$$

$$\Big(= \int_{\Delta_x} \omega\big(t, (B_{(0,\dots,0)} V_{(0,\dots,0)})(t), \dots, (B_\nu V_\nu)(t), \dots\big)\,dt\Big).$$

We adopt this notation for functions defined only in Δ: if

$$U = (U_{(0,\dots,0)}, \dots, U_{(0,1,\dots,1)}): \Delta \to R^q_*$$

belongs to USC and

$$U_\mu|_S = 0 \qquad \text{for } \mu \in I_n,$$

then we put

$$(8.5)\quad \Omega_0(x, U) = \int_{\Delta_x} \omega\big(t, (B\tilde{U})(t)\big)\,dt \qquad \text{for } x \in \Delta \ (o = (0, \dots, 0) \in R^n),$$

where

$$(8.6)\qquad \tilde{U}_\mu = U_\mu \cup \{\text{the zero function: } V_\mu \to \{0\}\}.$$

The notation (8.5) and (8.6) corresponds directly to the notation used in Section 6 for the functions f and Φ. Since the domain of mapping under consideration is always explicitly indicated, there is no possibility of any confusion concerning (8.5) and (8.4).

Similarly to the notion in Sections 5 and 6 we put

$$\Omega_\mu(x, V) = D_\mu \int_{\Delta_x} \omega\big(t, (BV)(t)\big)\,dt$$

and

$$\Omega_\mu(x, U) = D_\mu \int_{\Delta_x} \omega\big(t, (B\tilde{U})(t)\big)\,dt$$

for V and U as above.

Furthermore, instead of

$$\big(\Omega_{(0,\dots,0)}(\cdot, V), \dots, \Omega_{(0,1,\dots,1)}(\cdot, V)\big)$$

we shall write shortly $\Omega(V)$; the same convention is used also for $U: \Delta \to R^q$.

Remark 8.2. If ω fulfils (A_1) and B_μ fulfil (A_2), then for every continuous mappings

$$Z, W: \Delta \cup V \to R^q$$

such that

$$Z_\mu|_{V_\mu} = W_\mu|_{V_\mu} \qquad (\mu \in I_n)$$

we have the following inequalities:

$$(8.7) \qquad |\Phi_\mu(x, Z) - \Phi_\mu(x, W)|_0 \leqslant \Omega_\mu(x, |Z-W|_0), \qquad \mu \in I_n.$$

Instead of the systems of inequalities (8.7) we can write, equivalently, the following inequality:

$$(8.8) \qquad |\Phi(Z) - \Phi(W)|_0 \leqslant \Omega(|Z-W|_0).$$

In the proof of (8.7) we use the following simple observation

$$D_\mu \int_{\Delta_x} \chi(t)\,dt = \int_{\Delta_{x,\mu}} \chi\big((1-\mu)t+\mu x\big)(1-\mu)\,dt$$

(see (1.6)):

DEFINITION 8.1. Let λ and ψ be as in Section 6 (see (6.1)–(6.3)). We say that a continuous mapping

$$Z = (Z_{(0,\ldots,0)}, \ldots, Z_{(0,1,\ldots,1)})\colon \Delta \to R^q$$

belongs to the set

$$E(\psi, \lambda)$$

if and only if there exists a mapping

$$U = (U_{(0,\ldots,0)}, \ldots, U_{(0,1,\ldots,1)})\colon \Delta \to R^q_*$$

belonging to $\mathscr{D}_*$, such that

$$(8.9) \qquad |\Phi^\psi(Z+\lambda) - Z|_0 \leqslant U - \Omega(U).$$

Remark 8.3. Clearly, (8.9) means that for every $x \in \Delta$ and every $\mu \in I_n$

$$|\Phi^\psi_\mu(x, Z+\lambda) - Z_\mu(x)|_0 \leqslant U_\mu(x) - \Omega_\mu(x, U).$$

Accordingly to the convention from Section 6, we write (from now on) Φ and Φ_μ instead of Φ^ψ and Φ^ψ_μ.

In theorems on the existence and uniqueness of solutions of the equation

$$Z = \lambda + \Phi(Z)$$

it will be assumed that the sets of the type $E(\psi, \lambda)$ are non-empty, and also that the only solution (in some class) of the equation

$$U = \Omega(U)$$

(called the *comparative equation*) is zero. The details will be stated precise by below.

Let M and K be given by (7.2) and (7.3). We introduce the following

ASSUMPTION $\big(\mathrm{A}_3(M, K)\big)$. *If*

$$U = (U_{(0,\ldots,0)}, \ldots, U_{(0,1,\ldots,1)})\colon \Delta \to R^q_*$$

belongs to $\mathscr{D}(K, M)$ *and*

$$U = \Omega(U), \tag{8.10}$$

then $U = 0$ (*that is*, $U_\mu(x) = 0 \in R^m$ *for every* $x \in \Delta$ *and* $\mu \in I_n$).

9. Technical lemmas

LEMMA 9.1. *If* λ *and* ψ *are as in Section* 6,

$$Z = (Z_{(0,\dots,0)}, \dots, Z_{(0,1,\dots,1)}): \Delta \to R^q \quad \textit{is continuous} \tag{9.1}$$

and

$$Z_\mu|_S = \lambda_\mu|_S = \psi_\mu|_S, \tag{9.2}$$

then putting

$$W_\mu(x) = \lambda^\mu(x) + \Phi_\mu(x, Z) \quad \textit{for } x \in \Delta,\ \mu \in I_n, \tag{9.3}$$

we obtain a continuous mapping

$$W = (W_{(0,\dots,0)}, \dots, W_{(0,1,\dots,1)}): \Delta \to R^q$$

such that

$$W - \lambda \in [\mathscr{B}(K), \mathscr{L}(M)], \tag{9.4}$$

where M *and* K *are given by* (7.2) *and* (7.3) *with*

$$M_s^{\mu,j} = T_s^\mu \cdot \max\{|f_j(x, Y)|:\ x \in \Delta,\ Y \in R^q,\ |Y|_0 \leqslant \overset{0}{K}\}, \tag{9.5}$$

$$K_j^\mu = P^\mu \cdot \max\{|f_j(x, Y)|:\ x \in \Delta,\ Y \in R^q,\ |Y|_0 \leqslant \overset{0}{K}\}; \tag{9.6}$$

here

$$\begin{aligned} \overset{0}{K} &= (\overset{0}{K}{}^{(0,\dots,0)}, \dots, \overset{0}{K}{}^{(0,1,\dots,1)}) \\ &= \big((\overset{0}{K}{}^{(0,\dots,0),1}, \dots, \overset{0}{K}{}^{(0,\dots,0),k_{(0,\dots,0)}}), \dots, (\overset{0}{K}{}^{\nu,1}, \dots, \overset{0}{K}{}^{\nu,k_\nu}), \dots \\ &\qquad \dots, (\overset{0}{K}{}^{(0,1,\dots,1),1}, \dots, \overset{0}{K}{}^{(0,1,\dots,1),k_{(0,1,\dots,1)}})\big) \end{aligned} \tag{9.7}$$

is given by

$$\overset{0}{K}{}^\lambda = \max |A_\lambda \hat{Z}_\lambda|_0, \quad \hat{Z}_\lambda = \hat{Z}_\lambda \cup \psi^\lambda \tag{9.8}$$

(cf. the notation form Section 1.2).

Remark 9.1. Of course,

$$\overset{0}{K}{}_1^{\lambda,p} = (\overset{0}{K}{}_1^{\lambda,p}, \dots, \overset{0}{K}{}_m^{\lambda,p}).$$

LEMMA 9.2. *If* $U: \Delta \to R^q_*$ *belongs to* $\mathscr{D}_*$ *and is such that*

$$U_\mu|_S = 0 \quad (\mu \in I_n),$$

then putting

$$V = \Omega(U) \tag{9.9}$$

we obtain a function belonging to the class $\mathscr{D}(\tilde{K}, \tilde{M})$, *where* $\tilde{M}$ *and* $\tilde{K}$ *are given by* (7.2) *and* (7.3) (*with the substitution* $M = \tilde{M}$, $K = \tilde{K}$), *respectively, with*

$$M_k^{\mu,i} = T_k^{\mu} \max\{\omega_i(x, Y)\colon x \in \Delta,\ Y \in R_*^q,\ Y \leqslant L^0\}, \tag{9.10}$$

$$K_i^{\mu} = P^{\mu} \max\{\omega_i(x, Y)\colon x \in \Delta,\ Y \in R_*^q,\ Y \leqslant L^0\}; \tag{9.11}$$

here

$$L^0 = (L^{0(0,\ldots,0)}, \ldots, L^{0(0,1,\ldots,1)}) \tag{9.12}$$

is given by

$$L^{0\lambda} = \max|B_\lambda \tilde{U}_\lambda|; \quad \tilde{U}_\lambda = U_\lambda \cup \{\textit{zero function}\colon \Delta_\lambda \to \{0\}\} \tag{9.13}$$

(cf. (9.7), (9.8) and Remark 9.1).

The proofs of the above lemmas are trivial and will be omitted.

Remark 9.2. We shall say in such a situation that $\tilde{M}$ and $\tilde{K}$ are given by the function U.

Lemma 9.3. *If functions* $U^k\colon \Delta \to R_*^q$ *are such that*

$$\{U^k\}_{k=1,2,\ldots} \subset \mathscr{D}(L, N)$$

for some L, N *and if*

$$U^{k+1} \leqslant U^k \quad \textit{for } k = 1, 2, \ldots, \tag{9.14}$$

which means that

$$U^{k+1}(x) \leqslant U^k(x) \quad \textit{for } x \in \Delta,\ \mu \in I_n,\ k \in N,$$

then for every $x \in \Delta$ *there exists*

$$\lim_{k\to\infty} U^k(x) = (\lim_{k\to\infty} U^k_{(0,\ldots,0)}, \ldots, \lim_{k\to\infty} U^k_{(0,1,\ldots,1)}) \tag{9.15}$$

and

$$U = \lim_{k\to\infty} U^k \in \mathscr{D}(L, N). \tag{9.16}$$

Moreover, the convergence of $\{U^k\}$ *is uniform with respect to those* x, *for which* U^k *fulfils the Lipschitz condition.*

The proof is trivial.

Lemma 9.4. *If* $\{U^k\}$ *is as in Lemma* 9.3, *then for every* $x \in \Delta$ *and every* $\mu \in I_n$ *there exists*

$$\lim_{k\to\infty} \Omega_\mu(x, U^k(x))$$

and this limit is equal to

$$\Omega(x, U(x)),$$

where U *is given by* (9.16).

The proof is trivial in virtue of the classical Lebesgue theorem.

10. The main results

Consider a function f (see (3.2)), ω (see (8.1)), operators A_μ and B_μ (see (3.1) and Section 8).

THEOREM 10.1. *Suppose that:*

(a) *f is continuous, ω fulfils Assumption* (A_1), *A_μ and B_μ fulfil Assumption* (A_2), *λ, ψ are given and satisfy conditions* (6.1)–(6.3);

(b) *the set $E(\psi, \lambda)$ is non-empty. Assume, moreover, that $\overset{0}{Z} \in E(\psi, \lambda)$ and $U^0\colon \Delta \to R^q_*$ are such that $U^0 \in \mathscr{D}_*$, and for $Z = \overset{0}{Z}$ and $U = U^0$ condition* (8.9) *is fulfilled;*

(c) *Assumption* $(A_3(M, K))$ *is satisfied for $\tilde{M}$ and $\tilde{K}$ given by the function U^0, accordingly to the terminology from Section* 9 (see (9.10)–(9.13) witch $U = U^0$, and Remark 9.2).

Under the above assumptions there exists a solution

$$Z^* = (Z^*_{(0,\dots,0)}, \dots, Z^*_{(0,1,\dots,1)})$$

of Problem (Q), *the uniform limit*

$$Z^* = \lim_{k\to\infty} Z^k \tag{10.1}$$

of the following sequence of successive approximations:

$$Z^{k+1} = \lambda + \Phi(Z^k), \qquad k = 0, 1, 2, \dots, \tag{10.2}$$

where $Z^0 = \overset{0}{Z} + \lambda$.

Moreover,

$$Z^* - \lambda \in [\mathscr{B}(K), \mathscr{L}(M)] \tag{10.3}$$

with M given by (9.5) (*with Z replaced by $\overset{0}{Z}$*), *and K such that*

$$K^\lambda = \max U^0_\lambda + \max \overset{0}{Z}_\lambda,$$

and finally,

$$|Z^k - Z^0|_0 \leqslant U^0 \quad \textit{for } k = 0, 1, 2, \dots \tag{10.4}$$

Proof. Let us consider the following sequence of continuous functions from Δ into R^q_*:

$$U^k = \Omega(U^{k-1}), \qquad k = 1, 2, \dots; \tag{10.5}$$

we have (by induction):

$$0 \leqslant U^k \leqslant U^{k-1} \leqslant U^0 \quad \text{for } k = 1, 2, \dots \tag{10.6}$$

(see the notation from Section 1.2).

Hence

$$U^k \in \mathscr{D}(\tilde{K}, \tilde{M}) \quad \text{for } k = 1, 2, \ldots$$

with $\tilde{K}$, $\tilde{M}$ as in the formulation of the theorem. (See Lemma 9.2.) Moreover (also by induction),

$$\Omega(U^k) \leqslant U^k, \quad k = 0, 1, 2, \ldots \tag{10.7}$$

In virtue of Lemmas 9.3 and 9.4, we have

$$U^k \to U^* \in \mathscr{D}(\tilde{K}, \tilde{M}) \quad \text{as } k \to \infty \tag{10.8}$$

and then, passing in (10.5) to the limit as $k \to \infty$, we obtain

$$U^* = \Omega(U^*). \tag{10.9}$$

Because of $(A_3(M, K))$, the function U^* must be equal to zero. Hence the convergence (10.7) is uniform, since every decreasing sequence of upper semicontinuous functions convergent to a continuous function, is uniformly convergent.

Now, we shall show by induction that (10.4) holds true. Indeed, for $k = 0, 1$ inequality (10.4) is true, since for $k = 0$ it is trivial and for $k = 1$ it is assumed (in the stronger form (8.9)). Assume (10.4) for $k = p$ and consider $|Z^{p+1} - Z^0|_0$.

We have

$$\begin{aligned} |Z^{p+1} - Z^0|_0 &\leqslant |\Phi(Z^p) - \Phi(Z^0)|_0 + |\Phi(Z^0) - \overset{0}{Z}|_0 \\ &\leqslant \Omega(|Z^p - Z^0|_0) + U^0 - \Omega(U^0) \\ &\leqslant \Omega(U^0) + U^0 - \Omega(U^0) = U^0 \end{aligned}$$

and the proof of (10.4) is complete.

We shall now show, again by induction, that

$$|Z^{p+k} - Z^k|_0 \leqslant U^k \quad \text{for } k = 0, 1, 2, \ldots, \; p = 1, 2, \ldots \tag{10.10}$$

For any $p \in N$ and $k = 0$, inequality (10.10) is a consequence of (10.4) and (10.7). Suppose (10.10) for $k = l$ and $p \in N$. Consider $|Z^{p+l+1} - Z^{l+1}|_0$.

We have

$$|Z^{p+l+1} - Z^{l+1}|_0 = |\Phi(Z^{p+l}) - \Phi(Z^l)|_0 \leqslant \Omega(|Z^{p+l} - Z^l|_0) \leqslant \Omega(U^l) = U^{l+1}$$

and so the proof of (10.10) is finished.

From the uniform convergence of the sequence $\{U^k\}$ to zero it follows, in virtue of (10.10), that $\{Z^k\}$ fulfils the Cauchy condition and then it is uniformly convergent to a function Z^*. Passing to the limit in (10.2) as $k \to \infty$, we obtain

$$Z^* = \lambda + \Phi(Z^*),$$

which means that Z^* is a solution of Problem (Q).

Relation (10.3) results immediately from the form of $Z^* = \lim Z^k$. Thus the proof of the theorem is complete.

THEOREM 10.2. *Suppose that assumptions* (a) *of Theorem* 10.1 *are satisfied and assume that* $U^0 \in \mathscr{D}_*$ *and Assumption* $(\mathrm{A}_3(\tilde{M}, \tilde{K}))$ *for* $\tilde{M}$ *and* $\tilde{K}$ *given by the function* U^0 (see Remark 9.2) *is satisfied. Suppose, moreover, that*

$$\Omega(U^0) \leqslant U^0. \tag{10.11}$$

Under these assumptions, if Z *and* W *are two solutions of Problem* (Q) *such that*

$$|Z-W|_0 \leqslant U^0, \tag{10.12}$$

then $Z = W$.

Proof. Since $Z = \lambda + \Phi(Z)$ and $W = \lambda + \Phi(W)$, we have

$$|Z-W|_0 = |\Phi(Z)-\Phi(W)|_0 \leqslant \Omega(|Z-W|_0) \leqslant \Omega(U^0) \leqslant U^0.$$

Considering the sequence (10.5), we obtain

$$|Z-W|_0 \leqslant U^k \quad \text{for every } k \tag{10.13}$$

(the proof proceeds by induction). Then, passing to the limit, we obtain

$$|Z-W|_0 = 0$$

and so the proof is finished.

11. Some corollaries

In order to formulate further results, we introduce the following

ASSUMPTION (A_4). *Assuming that the function* ω (see (8.1)) *is given, we suppose that for every continuous function*

$$Z\colon \Delta \to R^q$$

there exists a function $U^0 \in \mathscr{D}_*$ *such that*

$$|Z|_0 \leqslant U^0$$

and

$$\Omega(U^0) \leqslant U^0.$$

THEOREM 11.1. *Suppose that all assumptions of Theorem* 10.1 *are satisfied, and so are Assumption* (A_4) *and Assumption* $(\mathrm{A}_3(M, K))$ *for every* M *and* K. *Then there exists exactly one solution of Problem* (Q).

The proof is trivial in virtue of Theorem 10.1 and Theorem 10.2.

Remark 11.1. Assumption (A_4) is satisfied for a large class of functions ω. The trivial case is that in which ω is bounded. But also for a lot of unbounded functions ω this condition is satisfied. This is the case, for instance, for functions ω which are, in a sense, "similar to linear". One can find such examples in [8], [9], [26], [30].

Remark 11.2. If the surface S is such that Σ^0 reduces to one point, that is, if we consider a special case of Problem (Q), corresponding to a generalization of the Darboux problem, then under the assumption that the operators A_μ are of the delay type:

$$(A_\mu U_\mu)(t) = U_\mu(t-a_\mu),$$

where a_μ are non-negative, if we wish to find only local solutions (solutions in a sufficiently small subset Δ_{x^0} of Δ), we can assume, without loss of generality, that ω is bounded. Indeed, since we can regard (for sufficiently small x^0) $f|_{(\Delta_{x^0}\times F)}$ as bounded by suitable constants, we can consider the function $\omega_G = \max(\omega, G)$ (the notation is not precise; we omit the details) in place of ω, where G denotes a corresponding system of constants, such that

$$\sup\{|f(x, Z)-f(x, \tilde{Z})|_0\colon x \in \Delta_{x^0},\ Z, \tilde{Z} \in F\} \leqslant G,$$
$$F = \{Z \in R^q\colon |Z|_0 \leqslant U^0\}.$$

12. Some modification of the main results

Let ω be a function of the type (8.1).

Assumption (A_5). *For every function $U \in \mathscr{D}(K, M)$ with an arbitrary pair (K, M), and every $Z\colon \Delta \to R^q_*$ continuous such that*

$$Z \leqslant \Omega(Z) + U, \tag{12.1}$$

there exists $\tilde{U} \in \mathscr{D}_*$ such that

$$Z \leqslant \tilde{U} \tag{12.2}$$

and

$$U + \Omega(\tilde{U}) \leqslant \tilde{U}. \tag{12.3}$$

Assumption (A_6). *For every $U \in \mathscr{D}(M, K)$, with an arbitrary pair (M, K), there exists a maximal solution W_U of the equation*

$$Z = \Omega(Z) + U$$

in the class $\mathscr{D}_*$; this means that $W_U \in \mathscr{D}_*$ and W_U fulfils the above equation, and for every $Z \in \mathscr{D}_*$ fulfilling this equation, we have

$$Z \leqslant W_U.$$

Now we can formulate the following

Theorem 12.1. *Suppose that assumption* (a) *of Theorem* 10.1 *are satisfied. Assume that there exist $U^0 \in \mathscr{D}_*$ and a continuous function Z^0: $\Delta \to R^q$, such that*

$$|Z^0 - \lambda - \Phi(Z^0)|_0 \leqslant U^0 \tag{12.4}$$

and

$$\Omega(U^0) \leqslant U^0, \tag{12.5}$$

and, moreover, Assumption $(\mathrm{A}_3(M, K))$ *for every* M *and* K *and Assumptions* (A_5) *and* (A_6) *are satisfied. Then there exists exactly one solution* Z *of Problem* (Q); *this solution is given as the uniform limit*

$$Z = \lim_{k\to\infty} Z^k,$$

where $\{Z^k\}$ *are defined by* (10.2), *but now* Z^0 *is the function given above — not by* $(\overset{0}{Z}+\lambda)$.

Proof. We use the method of Ważewski [31] used with respect to the partial differential equations in [24].

Define $\{U^k\}$ as in Section 10.:

$$U^k = \Omega(U^{k-1}), \quad k = 1, 2, \ldots \tag{12.6}$$

It is easy to show by the induction that

$$U^k \leqslant U^{k-1}, \quad k = 1, 2, \ldots \tag{12.7}$$

and

$$|Z^{k+1} - Z^k|_0 \leqslant U^k, \quad k = 1, 2, \ldots \tag{12.8}$$

Furthermore, we easily prove that for $r \leqslant s$

$$|Z^r - Z^s|_0 \leqslant 2U^r + \Omega(|Z^r - Z^s|_0) \tag{12.9}$$

and then putting

$$S_{rs} = |Z^r - Z^s|_0 \tag{12.10}$$

and

$$V_r = 2U^r \tag{12.11}$$

we obtain

$$S_{rs} \leqslant V_r + \Omega(S_{rs}). \tag{12.12}$$

Now we apply Assumption (A_5). We find $W_{rs} \in \mathscr{D}_*$ such that

$$S_{rs} \leqslant W_{rs} \tag{12.13}$$

and

$$\Omega(W_{rs}) + V_r \leqslant W_{rs}. \tag{12.14}$$

Putting

$$W_{rs}^0 = W_{rs}, \quad W_{rs}^k = \Omega(W_{rs}^{k-1}), \quad k = 1, 2, \ldots \tag{12.15}$$

we obtain a decreasing sequence, which is convergent to a solution $\hat{W}_{rs}$ of the equation

$$W = \Omega(W) + V_r. \tag{12.16}$$

Obviously $\hat{W}_{rs} \in \mathscr{D}_*$.

It is easy to see that the following is true:

LEMMA 12.1. *If $X\colon \Delta \to R^q_*$ is a summable function such that*

$$X \leqslant V_r + \Omega(X) \quad \textit{and} \quad X \leqslant W_{rs}, \tag{12.17}$$

then

$$X \leqslant \hat{W}_{rs}. \tag{12.18}$$

Indeed, by induction we obtain

$$X \leqslant W^k_{rs} \quad \text{for } k = 0, 1, 2, \ldots \tag{12.19}$$

and then, passing to the limit in (12.19) as $k \to \infty$, we obtain the required inequality (12.18).

Hence, in view of (12.14), we obtain

$$S_{rs} \leqslant \hat{W}_{rs}. \tag{12.20}$$

Now, we apply Assumption (A_6) and we denote by $\hat{W}_r$ the maximal solution in $\mathscr{D}_*$ of (12.16).

Let us observe that $\hat{W}_r$ tends to zero as $r \to \infty$. Indeed, $\hat{W}_r \leqslant \hat{W}_{r-1}$, since $U^r \leqslant U^{r-1}$ (the same Lemma 12.1). Hence, $\{\hat{W}_r\}$ is decreasing and — in virtue of the obvious inequality — $\hat{W}_r \geqslant 0$, bounded. So $\{\hat{W}_r\}$ is convergent.

Passing to the limit in the relation

$$\hat{W}_r = V_r + \Omega(\hat{W}_r)$$

we obtain for $\hat{W} = \lim\limits_{r\to\infty} W_r$

$$\hat{W} = \Omega(\hat{W}).$$

Of course, $\hat{W} \in \mathscr{D}(M, K)$ for some M, K.

Hence, in virtue of Assumption $(A_3(M, K))$ (assumed for every M, K) we obtain $W = 0$ and therefore the convergence is uniform. Hence and from (12.20) it follows that S_{rs} tends uniformly to zero as $r, s \to \infty$.

This means that the sequence $\{Z^k\}$ fulfils the Cauchy condition and then it is uniformly convergent to a solution Z of Problem (Q).

But this solution is unique, because of Assumption (A_5).

The proof of the uniqueness is just the same as in the case of Theorem 10.2.

Remark 12.1. One can assume conditions some what weaker than the assumptions of Theorem 12.1, namely one can assume (A_5) with respect to some classes of functions defined by suitable constants M and K, not for every M, K.

Remark 12.2. Assumption (A_6) is fulfilled for a large class of functions ω; for instance for bounded functions ω (compare Remark 11.1 and Remark 11.2).

One can prove that Assumption (A_0) follows from the following two assumptions (A_7) and (A_5') (the second is stronger than (A_5)).

Assumption (A_7). *For every $U \in \mathscr{D}(M, K)$, in the partially ordered space of all summable functions $Z\colon \Delta \to R_*^q$ there exists*

$$\sup\{Z\colon Z \leqslant \Omega(Z) + U\}.$$

Assumption (A_5'). *For every $U \in \mathscr{D}(M, K)$, with an arbitrary pair (M, K), and for every summable function $Z\colon \Delta \to R_*^q$ such that* (12.1) *is satisfied, there exists a summable function $\tilde{U}\colon \Delta \to R_*^q$ such that $\Omega(\tilde{U}) \in \mathscr{D}_*$ and* (12.2) *and* (12.3) *are satisfied.*

13. Generalizations

It is easy to see that we can generalize Theorems 10.1 and 10.2 assuming some conditions for Ω and Φ, not for ω and f. More precisely, we can forget the definitions of Ω and Φ, and we can formulate the corresponding assumptions directly for Ω and Φ.

In other words, instead of assumptions concerning the functions ω and f, we can introduce conditions which are conclusions from lemmas of Section 9, giving corresponding properties of Ω and Φ. The assertions in that case would be the existence and uniqueness of solutions of the equation

$$Z = \lambda + \Phi(Z)$$

(without correspondence with integral equations).

Such a formulation of our results is similar to those of Kwapisz [12].

PART II

1. Preliminaries

We adopt the notation and definitions of Part I. In addition, for $m \in N_1$ and $\mu \in I_n$ we denote

$$E_\mu^m(\Delta) = \Big\{W\colon \Delta \to R^m\colon \text{there exists } \tilde{W} \in L_1^m(\Delta) \text{ such that}$$
$$W(x) = \int_{\Delta_{x,\mu}} \tilde{W}\big((1-\mu)t + \mu x\big)(1-\mu)\,dt\Big\}.$$

Remark 1.1. It is easy to see that the set $E_\mu^m(\Delta)$ can be written in the form

$$E_\mu^m(\Delta) = \{W\colon \Delta \to R^m\colon D_{1-\mu}W \in L_1^m(\Delta)\}$$

or

$$E_\mu^m(\Delta) = \big\{W \in L_1^m(\Delta)\colon \operatorname{proj}_{1-\mu}^0(\Delta) \ni (1-\mu)x \to W\big(\mu x + (1-\mu)x\big) \in R^m$$
$$\text{is absolutely continuous for almost every } \mu x \in \operatorname{proj}_\mu^0(\Delta)\big\}.$$

Here, $L_1^m(\Delta)$ denotes the set of all summable functions defined on Δ with range in R^m, and by $\operatorname{proj}_\nu^0$ $(\nu = (\nu_1, \ldots, \nu_n) \in I_n)$ we denote the mapping

$$R^n \ni x \to \nu x = (\nu_1 x_1, \ldots, \nu_n x_n) \in R^n.$$

Further, we write

$$\hat{E}^m_{(0,\dots,0)} = \{V\colon\ V_{(0,\dots,0)}\cup\Delta\to R^m\colon\ V|_\Delta\in E^m_{(0,\dots,0)}(\Delta),\ V|_{V_{(0,\dots,0)}}\in C^m_{(0,\dots,0)}(V_{(0,\dots,0)})\}$$

and for $\mu\in I_n\setminus\{0,\dots,0\}$

$$\hat{E}^m_\mu = \{V\colon\ V_\mu\cup\Delta\to R^m\colon\ V|_\Delta\in E^m_\mu(\Delta),\ V|_{V_\mu}\in L^m_1(V_\mu)\}.$$

As usual, we denote by

$$\mathop{\mathsf{X}}_{\mu\in I_n} E^m_\mu(\Delta)$$

the Cartesian product of the sets $E^m_\mu(\Delta)$ and we put

$$\mathscr{E} = \Big\{U = (U_{(0,\dots,0)},\dots,U_{(0,1,\dots,1)})\in\mathop{\mathsf{X}}_{\mu\in I_n} E^m(\Delta)\colon \text{ there exists } U_{(1,\dots,1)}\in L^m_1(\Delta) \text{ such that for } \mu\in I_n \text{ we have } U_\mu(x) = \int\limits_{\Delta_{x\mu}} U_{(1,\dots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt \text{ a.e. in } \Delta\Big\}$$

(here "a.e." stands for "almost everywhere").

Thus, an element of the set $\mathscr{E}$ is a system $U = (U_\mu)_{\mu\in I_n}$ of functions U_μ which may be obtained by the suitable integration of the same function $U_{(1,\dots,1)}$ summable in Δ; then $U_{(1,\dots,1)}$ is equal to $D_{(1,\dots,1)}U_{(0,\dots,0)}$.

In other words, we have

$$\mathscr{E} = \{U = (U_\mu)_{\mu\in I_n}\in\mathop{\mathsf{X}}_{\mu\in I_n} E^m_\mu(\Delta)\colon \text{ there exists } U_{(1,\dots,1)}\in L^m_1(\Delta), \text{ such that for } \mu\in I_n \text{ we have } D_{1-\mu}U = U_{(1,\dots,1)} \text{ a.e.}\}$$

Now, we define

$$\mathscr{E}_* = \{U\colon\ \Delta\to R^{m(2^n-1)}_*\colon\ U\in\mathscr{E}\}$$

and, for $h\in L^m_1(\Delta)$, $h\geqslant 0$,

$$\mathscr{E}(h) = \{U\in\mathscr{E}\colon\ U_{(1,\dots,1)}(x)\leqslant h(x) \text{ a.e. in } \Delta\}$$

and

$$\mathscr{E}_*(h) = \mathscr{E}(h)\cap\mathscr{E}_*.$$

Similarly, we set

$$\hat{\mathscr{E}} = \{U\in\mathop{\mathsf{X}}_{\mu\in I_n}\hat{E}^m_\mu\colon\ U|_\Delta\in\mathscr{E}\},$$

and by $\hat{\mathscr{E}}_*$, $\hat{\mathscr{E}}(h)$ we denote the subsets of $\hat{\mathscr{E}}$, containing only those elements of $\hat{\mathscr{E}}$ whose restrictions to the set Δ belong to $\mathscr{E}_*$ and $\mathscr{E}(h)$, respectively.

Let a system of positive integers $\{k_\mu\}_{\mu\in I_n}$ be given. Let us put

$$r = m\sum_\mu{}' k_\mu.$$

Suppose that a function

$$f\colon \Delta \times R^r \to R^m$$

satisfies the following (Carathéodory type)

CONDITION (W).

(i) *for every fixed* $u \in R^r$ *the mapping* $\Delta \ni x \mapsto f(x, u) \in R^m$ *is measurable,*

(ii) *for every* $x \in \Delta$ *the mapping* $R^r \ni u \mapsto f(x, u) \in R^m$ *is continuous,*

(iii) *the mapping* $\Delta \ni x \mapsto f(x, 0) \in R^m$ *is summable.*

We admit that a comparative (for f) function

$$\omega\colon \Delta \times R^r \mapsto R^m_*$$

satisfies the following

ASSUMPTION $(\tilde{\mathrm{A}}_1)$.

(α) (i′) $\Delta \ni x \mapsto \omega(x, u) \in R^m_*$ *is measurable for every fixed* $u \in R^r$,

(ii′) $R^r \ni u \mapsto \omega(x, u) \in R^m_*$ *is continuous for every* $x \in \Delta$,

(iii′) *there exists a function* $g_\omega \in L^m_1(\Delta)$ *such that for every* $u \in R^r$ *the inequality* $\omega(x, u) \leqslant g_\omega(x)$ *is satisfied almost everywhere in the set* Δ.

(β) *If* $x \in \Delta$, $u, v \in R^r$ *and* $u \leqslant v$, *then* $\omega(x, u) \leqslant \omega(x, v)$,

(γ) $\omega(x, 0) = 0$ *for every* $x \in \Delta$,

(δ) $|f(x, u) - f(x, v)|_0 \leqslant \omega(x, |u-v|_0)$ *a.e. in* Δ, $u, v \in R^r$.

Remark 1.2. From (W)–(iii) and $(\tilde{\mathrm{A}}_1)$–(δ) it follows easily that the inequality

$$|f(x, u)| \leqslant |f(x, 0)| + \omega(x, |u|_0)$$

holds a.e. in Δ, which immediately implies the following estimation:

$$|f(x, u)| \leqslant g(x) \quad \text{a.e. in } \Delta \text{ for every } u \in R^r, \tag{1.1}$$

where g is the function summable on Δ and given by the formula:

$$g(\cdot) = |f(\cdot, 0)| + g_\omega(\cdot). \tag{1.2}$$

For mappings

$$A_\mu, B_\mu\colon \hat{E}^m_\mu \to \mathop{\times}_{\mu \in I_n} \mathscr{M}^{m\cdot k_\mu}(\Delta), \qquad \mu \in I_n$$

(where $\mathscr{M}^{m\cdot k_\mu}(\Delta)$ denotes the space of all measurable functions defined on Δ and with values in $R^{m\cdot k_\mu}$) we admit the following

ASSUMPTION $(\tilde{\mathrm{A}}_2)$.

(a) *If* $Z, W \in \hat{E}^m_\mu$ *and* $Z|_{V_\mu} = W|_{V_\mu}$ a.e. *in* V_μ, *then*

$$|A_\mu Z - A_\mu W|_0 \leqslant B_\mu(|Z - W|_0) \quad \text{a.e. in } \Delta,$$

(b) *if* $Z, W \in \hat{E}_\mu^m$ *and* $Z \leqslant W$ *a.e. in* $\Delta \cup V_\mu$, *then*

$$B_\mu Z \leqslant B_\mu W \quad \text{a.e. in } \Delta,$$

(c) *if* $\{Z_\mu^k\}_{k=1,2,\ldots}$, $\{Z_\mu\} \subset \hat{E}_\mu^m$ *and* $Z_\mu^k \to Z_\mu$ *a.e. in* $\Delta \cup V_\mu$ *as* $k \to \infty$ *then* $A_\mu Z_\mu^k \to A_\mu Z_\mu$ *a.e. in* Δ, *as* $k \to \infty$,

(d) *if* $\{Z_\mu^k\}_{k=1,2,\ldots}$, $\{Z_\mu\} \subset \hat{E}_\mu^m$ *and* $Z_\mu^k \searrow Z_\mu$ *a.e. in* $\Delta \cup V_\mu$ *as* $k \to \infty$, *then* $B_\mu Z_\mu^k \to B_\mu Z_\mu$ *a.e. in* Δ, *as* $k \to \infty$.

In condition (d) the sign $\searrow$ denotes the monotone convergence (decreasing) and, of course, we have $B_\mu Z_\mu^k \searrow B_\mu Z_\mu$. Now, we define a mapping

$$\Omega = (\Omega_{(0,\ldots,0)}, \ldots, \Omega_{(0,1,\ldots,1)}): \hat{\mathscr{E}} \to \mathscr{E}$$

setting for $V \in \hat{\mathscr{E}}$ and $\mu \in I_n$, $x \in \Delta$

$$\Omega_\mu(x, V) = \int_{\Delta_{x,\mu}} \omega\big((1-\mu)t+\mu x, (BV)\big((1-\mu)t+\mu x\big)\big)(1-\mu)\,dt, \tag{1.3}$$

where

$$BV = (B_\nu V_\nu)_{\nu \in I_n}.$$

We have, of course,

$$\Omega_\mu: \hat{\mathscr{E}} \to E_\mu^m(\Delta).$$

Remark 1.3. We denote also by Ω the mapping from the set

$$\mathscr{E}_0 = \{U \in \mathscr{E}: U_{(0,\ldots,0)}(x) = 0 \text{ for } x \in V_{(0,\ldots,0)} \cap \Delta\}$$

into the set $\mathscr{E}$ setting for $U \in \mathscr{E}_0$

$$\Omega(U) = \Omega(\tilde{U}), \tag{1.4}$$

where

$$\tilde{U} = (\tilde{U}_\mu)_{\mu \in I_n}, \qquad \tilde{U}_\mu = U_\mu \cup \{0: V_\mu \to \{0\}\} \quad (\mu \in I_n).$$

We assume that for a function $h \in L_1^m(\Delta)$, $h \geqslant 0$, the mapping Ω satisfies the following

ASSUMPTION $(\bar{A}_3(h))$. *If* $U \in \mathscr{E}(h)$ *and* $U = \Omega(U)$, *then* $U = 0$: *i.e.*, $U_{(0,\ldots,0)}(x) = 0$ *for every* $x \in \Delta$, *and for* $\mu \in I_n \setminus \{0, \ldots, 0\}$, $U_\mu(x) = 0$ *for every* $(1-\mu)x \in \operatorname{proj}_{1-\mu}^0(\Delta)$ *and for almost every* (*in* $|\mu|$-*dimensional Lebesgue measure*) $\mu x \in \operatorname{proj}_\mu^0(\Delta)$.

This means, roughly speaking, that $U_\mu(x)$ $(\mu \in I_n)$ may be different from zero $(0 \in R^m)$ only at those points of Δ whose coordinates x_i, for all i such that $\mu_i = 1$, belong to some set of linear measure zero, this set being a subset of $\operatorname{proj}_{(0,\ldots,1,\ldots,0)}^0(\Delta)$ (1 on the i-th place).

In the sequel we note this fact shortly as follows

$$U_\mu(x) = 0 \quad \text{for every } (1-\mu)x \in \operatorname{proj}_{1-\mu}^0(\Delta), \text{for a.e. } \mu x \in \operatorname{proj}_\mu^0(\Delta), \quad \mu \in I_n;$$

we do not distinguish the case when $\mu = (0, \ldots, 0)$.

In this part of the paper we consider the following

PROBLEM ($\tilde{\mathrm{Q}}$). *For given* $\lambda = (\lambda_\mu)_{\mu\in I_n}$ *and* $\psi = (\psi_\mu)_{\mu\in I_n}$, $\psi_{(0,\ldots,0)} \in C_0^m(V_{(0,\ldots,0)})$, $\psi_\mu \in L_1^m(V_\mu)$ *for* $\mu \in I_n \setminus \{(0,\ldots,0)\}$, *find a function*

$$Z = (Z_\mu)_{\mu\in I_n}$$

such that for $\mu \in I_n$ *the equality*

$$Z_\mu(x) = \lambda_\mu(x) + \int\limits_{\Delta_{x,\mu}} f\big((1-\mu)t+\mu x, (A\hat{Z})\big((1-\mu)t+\mu x\big)\big)(1-\mu)\,dt \tag{1.5}$$

is satisfied for every $(1-\mu)x \in \mathrm{proj}^0_{1-\mu}(\Delta)$ *and for a.e.* $\mu x \in \mathrm{proj}^0_\mu(\Delta)$.

In this formula we put

$$A\hat{Z} = (A_\nu \hat{Z}_\nu)_{\nu\in I_n},$$

where

$$\hat{Z}_\nu = Z_\nu \cup \psi_\nu \quad \text{for } \nu \in I_n. \tag{1.6}$$

Under the notation

$$\Phi^\Psi\colon \mathscr{E} \to \mathscr{E},$$

where

$$\Phi^\Psi = (\Phi^\Psi_\mu)_{\mu\in I_n}$$

and

$$\Phi^\Psi_\mu\colon \mathscr{E} \to E^m_\mu(\Delta)$$

is given by the formula

$$\Phi^\Psi_\mu(x, Z) = \int\limits_{\Delta_{x,\mu}} f\big((1-\mu)t+\mu x, (A\hat{Z})\big((1-\mu)t+\mu x\big)\big)(1-\mu)\,dt, \tag{1.7}$$

we can write shortly equation (1.5) in the form

$$Z = \lambda + \Phi^\Psi(Z). \tag{1.8}$$

2. Auxiliary lemmas

LEMMA 2.1. *Assume that* Z *and* λ *belong to* $\mathscr{E}$ *and that* $\psi = (\psi_\mu)_{\mu\in I_n}$ *is such that* $\psi_{(0,\ldots,0)} \in C_0^m(V_{(0,\ldots,0)})$, $\psi_\mu \in L_1^m(V_\mu)$.

Then the function W *given by the formula*

$$W = \lambda + \Phi^\Psi(Z) \tag{2.1}$$

belongs to $\mathscr{E}(g+\lambda_{(1,\ldots,1)})$, *where* g *is given by* (1.2).

For the proof it suffices to observe that the function

$$W_{(1,\ldots,1)}(x) = \lambda_{(1,\ldots,1)}(x) + f\big(x, (A\hat{Z})(x)\big)$$

is summable on Δ and then to apply the estimation (1.1).

LEMMA 2.2. *If $U \in \mathscr{E}_*$, $U_{(0,\ldots,0)}(x) = 0$ for $x \in \Delta \cap V_{(0,\ldots,0)}$, then $V = \Omega(U)$ belongs to $\mathscr{E}_*(g_\omega)$, where g_ω is as in Assumption $(\tilde{\mathrm{A}}_1)$ (α) (iii).*

The simple proof follows directly from the formulae

$$V_\mu(x) = \int\limits_{\Delta_{x,\mu}} \omega\big((1-\mu)t+\mu x, (BU)\big((1-\mu)t+\mu x\big)\big)(1-\mu)\,dt \qquad (\mu \in I_n)$$

and from Assumption $(\tilde{\mathrm{A}}_1)$–(α).

LEMMA 2.3. *Assume that $\{U^k\}_{k=1,2,\ldots} \subset \mathscr{E}_*$ and*

$$U^{k+1}_{(1,\ldots,1)}(x) \leqslant U^k_{(1,\ldots,1)}(x) \quad \text{a.e. in } \Delta. \tag{2.2}$$

Then

1° *there exists $\lim\limits_{k\to\infty} U^k$, which will be denoted by U,*

2° $U \in \mathscr{E}_*(U^1_{(1,\ldots,1)})$.

More precisely, in 1° *we have for $\mu \in I_n$*

$$U^k_\mu(x) \to U_\mu(x)$$

uniformly with respect to $(1-\mu)x$ in the set $\mathrm{proj}^0_{1-\mu}(\Delta)$, for a.e. μx in the set $\mathrm{proj}^0_\mu(\Delta)$, and the function

$$\mathrm{proj}^0_{1-\mu}(\Delta) \ni (1-\mu)x \to U^k_\mu\big((1-\mu)x+\mu x\big) \in R^m$$

is absolutely continuous for a.e. $\mu x \in \mathrm{proj}_\mu(\Delta)$.

Proof. Inequality (2.2) implies that there exists a set $F \subset \Delta$ of measure zero, such that for $x \in \Delta \setminus F$ the function

$$U_{(1,\ldots,1)}(x) = \lim_{k\to\infty} U^k_{(1,\ldots,1)}(x)$$

is well defined and we have

$$0 \leqslant U_{(1,\ldots,1)}(x) \leqslant U^1_{(1,\ldots,1)}(x) \quad \text{for } x \in \Delta\setminus F.$$

Hence, by the Lebesgue theorem, we obtain for $\mu \in I_n$

$$\begin{aligned} U_\mu(x) &= \int\limits_{\Delta_{x,\mu}} U_{(1,\ldots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt \\ &= \lim_{k\to\infty} \int\limits_{\Delta_{x,\mu}} U^k_{(1,\ldots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt = \lim_{k\to\infty} U^k_\mu(x). \end{aligned}$$

Thus $U = (U_\mu)_{\mu\in I_n}$ is the required (in 1° and 2°) function. For the proof of the uniform convergence of $\{U^k_\mu\}$, observe that

$$\begin{aligned} |U^k_\mu(x) - U^k_\mu(\tilde{x})|_0 = \Big| \int\limits_{\Delta_{x,\mu}} & U^k_{(1,\ldots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt - \\ & - \int\limits_{\Delta_{\tilde{x},\mu}} U^k_{(1,\ldots,1)}\big((1-\mu)t+\mu\tilde{x}\big)(1-\mu)\,dt\Big|_0. \end{aligned}$$

Since the condition

$$x_i = \tilde{x}_i \quad \text{for } i \text{ such that } \mu_i = 1$$

implies that

$$\mu x = \mu\tilde{x},$$

we may write the right-hand side of the equality under consideration in the form

$$\Big| \int\limits_{R(x,\tilde{x})} U^k_{(1,\ldots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt\Big|_0,$$

where

$$R(x, \tilde{x}) = (\Delta_{x,\mu} \setminus \Delta_{\tilde{x},\mu}) \cup (\Delta_{\tilde{x},\mu} \setminus \Delta_{x,\mu}).$$

In view of assumption (2.2) and of the absolute continuity of the integral, we obtain for any $\varepsilon > 0$ ($\varepsilon = (\varepsilon, \ldots, \varepsilon) \in R^n$) the estimation

$$\Big| \int\limits_{R(x,\tilde{x})} U^k_{(1,\ldots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt\Big|_0 \leqslant \Big| \int\limits_{R(x,\tilde{x})} U^1_{(1,\ldots,1)}\big((1-\mu)t+\mu x\big)(1-\mu)\,dt\Big|_0 < \varepsilon$$

for $k = 1, 2, \ldots$ and $|(1-\mu)x-(1-\mu)\tilde{x}|_0 < \delta$ when $\delta = \delta(\varepsilon)$ ($\delta = (\delta, \ldots, \delta) \in R^n$) is sufficiently small.

Hence the limit function U_μ, defined for almost every $\mu x \in \operatorname{proj}^0_\mu(\Delta)$, satisfies also the above inequality, and this shows the absolute continuity of U_μ with respect to $(1-\mu)x \in \operatorname{proj}^0_{1-\mu}(\Delta)$. Since monotone convergence to a continuous function is, by Dini's theorem, uniform, the proof of the lemma is complete. In particular, the function $U_{(0,\ldots,0)}$ is absolutely continuous in Δ and the convergence

$$U^k_{(0,\ldots,0)}(x) \to U_{(0,\ldots,0)}(x) \quad \text{as } k \to \infty$$

is uniform in Δ.

Lemma 2.4. *If the sequence* $\{U^k\}_{k=1,2,\ldots} \subset \mathscr{E}_*$ *satisfies condition* (2.2) *and the operators* B_ν, $\nu \in I_n$, *satisfy Assumption* $(\tilde{\mathrm{A}}_2)$, (d), *then for* $\mu \in I_n$ *the sequence*

$$\{\Omega_\mu(x, U^k)\}_{k=1,2,\ldots}$$

is decreasing and converges to $\Omega_\mu(x, U)$ *for a.e.* $\mu x \in \operatorname{proj}^0_\mu(\Delta)$, *where* U *is the function considered in Lemma* 2.3.

For the proof it suffices to observe that for $\mu \in I_n$ we have

$$\Omega_\mu(x, U^k) = \int\limits_{\Delta_{c,\mu}} \omega\Big((1-\mu)t+\mu x, (B\tilde{U}^k)\big((1-\mu)t+\mu x\big)\Big)(1-\mu)\,dt$$

and that, in view of Lemma 2.3, Assumption $(\tilde{\mathrm{A}}_2)$ (d) and Assumption $(\tilde{\mathrm{A}}_1)$ (α) (ii), the integral on the right-hand side converges in a monotone way to

$$\int_{\Delta_{x,\mu}} \omega\big((1-\mu)t+\mu x,\ (B\tilde{U})\big((1-\mu)t+\mu x\big)\big)(1-\mu)\,dt = \Omega_\mu(x,\ U).$$

3. The main result

THEOREM 3.1. *Assume that*

(a) *f satisfies Condition* (W), *ω satisfies Assumption* $(\tilde{\mathrm{A}}_1)$, *A_μ, B_μ* $(\mu \in I_n)$ *satisfy Assumption* $(\tilde{\mathrm{A}}_2)$, *$\lambda \in \mathscr{E}$, $\psi = (\psi_\mu)_{\mu\in I_n}$ is such that $\psi_{(0,\ldots,0)} \in C_0^m(V_{(0,\ldots,0)})$, $\psi_\mu \in L_1^m(V_\mu)$, $\mu \in I_n\setminus\{(0,\ldots,0)\}$;*

(b) *there exist $\overset{0}{Z} \in \mathscr{E}$ and $U^0 \in \mathscr{E}_*$ such that the inequality*

$$(3.1)\qquad \big|f\big(x,\ A(\overset{0}{Z}+\lambda)(x)\big)-\overset{0}{Z}_{(1,\ldots,1)}(x)\big|_0 \leqslant U^0_{(1,\ldots,1)}(x)-\omega\big(x,\ (BU^0)(x)\big)$$

is satisfied a.e. in Δ;

(c) *Assumption $\big(\tilde{\mathrm{A}}_3(h)\big)$ is satisfied for $h = U^0_{(1,\ldots,1)}$.*

Then there exists $Z^* \in \mathscr{E}(U^0_{(1,\ldots,1)}+\overset{0}{Z}_{(1,\ldots,1)}+\lambda_{(1,\ldots,1)})$ which is a solution of Problem $(\tilde{\mathrm{Q}})$ and

$$Z^* = \lim_{k\to\infty} Z^k,$$

where Z^k are given by the formulae

$$(3.2)\qquad Z^0 = \lambda+\overset{0}{Z},\qquad Z^{k+1} = \lambda+\Phi^\psi(Z^k)\qquad \textit{for } k = 0,1,2,\ldots$$

Remark 3.1. Assumption (b) may be formulated in another form. Namely, it may be replaced by the condition:

(b′) the set

$$\tilde{E}(\psi,\lambda) = \{\overset{0}{Z} \in \mathscr{E}\colon \text{there exists } U^0 \text{ such that (3.1) is satisfied}\}$$

is not empty.

Contrary to the case considered in the first part of the paper, the more general condition: the set

$$E(\psi,\lambda) = \{\overset{0}{Z} \in \mathscr{E}\colon \text{there exists } U^0 \text{ such that } |\Phi^\psi(\overset{0}{Z}+\lambda)-\overset{0}{Z}|_0 \leqslant U^0-\Omega(U^0)\}$$

is not empty, is not sufficient for the proof presented below.

COROLLARY 3.1. *Theorem* 3.1 *can be used in particular to differential functional equations with regular right-hand members corresponding to the case: $\lambda_{(1,\ldots,1)}(x) = 0$ a.e. in Δ* (Problem $(\tilde{\mathrm{P}})$ in Part I of the paper).

Remark 3.2. Corollary 3.1 generalizes the result of Shanahan [27].

Proof of Theorem 3.1. Starting with U^0 (see assumption (b)) we put

$$U^{k+1} = \Omega(U^k), \qquad k = 0, 1, 2, \ldots \tag{3.3}$$

By induction we show that the sequence so obtained is decreasing and is contained in $\mathscr{E}_*(U^0_{(1,\ldots,1)})$.

Indeed, from (3.1) it follows that

$$\omega\big(x, (B\tilde{U}^0)(x)\big) \leqslant U^0_{(1,\ldots,1)}(x) \quad \text{a.e. in } \Delta.$$

Hence, for $\mu \in I_n$ we obtain

$$\Omega_\mu(U^0)(x) = \int_{\Delta_{x,\mu}} \omega\big((1-\mu)t + \mu x, (B\tilde{U}^0)\big((1-\mu)t + \mu x\big)\big)(1-\mu)\,dt$$
$$\leqslant \int_{\Delta_{x,\mu}} U^0_{(1,\ldots,1)}\big((1-\mu)t + \mu x\big)(1-\mu)\,dt = U^0_\mu(x).$$

Thus we have $U^1 = \Omega(U^0) \leqslant U^0$.

Assume that $U^k \leqslant U^{k-1}$ for a fixed $k \geqslant 1$. Hence, by the monotonicity assumptions for B_ν, ω and by the monotonicity of the integral we get

$$U^{k+1} = \Omega(U^k) \leqslant \Omega(U^{k-1}) = U^k.$$

Notice that for $k = 0, 1, 2, \ldots$

$$U^{k+1}_{(1,\ldots,1)}(x) = \omega\big(x, (B\tilde{U}^k)(x)\big) \quad \text{a.e. in } \Delta$$

and that the inequalities

$$\omega\big(x, (B\tilde{U}^{k+1})(x)\big) \leqslant \omega\big(x, (B\tilde{U}^k)(x)\big) \leqslant \ldots \leqslant U^0_{(1,\ldots,1)}(x)$$

are fulfilled almost everywhere in Δ.

Thus the assumptions of Lemma 2.3 and Lemma 2.4 are satisfied. Consequently, there exists

$$U^* = \lim_{k\to\infty} U^k,$$

which is an element of $\mathscr{E}_*(U^0_{(1,\ldots,1)})$, and we have

$$\Omega(U^*) = \lim_{k\to\infty} \Omega(U^k).$$

Hence by (3.3) we obtain

$$U^* = \Omega(U^*),$$

which, in view of Assumption $(\tilde{A}_3(h))$ (with $h = U^0_{(1,\ldots,1)}$), implies that

$$U^* = 0 \quad \text{in } \Delta$$

(this means that $U_\mu(x) = 0$ for every $(1-\mu)x \in \text{proj}^0_{1-\mu}(\Delta)$ for a.e. $\mu x \in \text{proj}^0_\mu(\Delta)$, $\mu \in I_n$).

Therefore, in view, of Lemma 2.3, we get for $\mu \in I_n$

$$(3.4) \qquad U_\mu^k(x) \to 0, \quad \text{as } k \to \infty, \text{ uniformly with respect to } (1-\mu)x \in \operatorname{proj}_{1-\mu}^0(\Delta), \text{ for a.e. } \mu x \in \operatorname{proj}_\mu^0(\Delta).$$

Now, again by induction we show the validity of the following estimations:

$$(3.5) \qquad |Z^k - Z^0|_0 \leqslant U^0 \quad (k = 0, 1, 2, \ldots)$$

and

$$(3.6) \qquad |Z^{k+p} - Z^k|_0 \leqslant U^k \quad (k = 0, 1, 2, \ldots,\ p = 1, 2, \ldots).$$

Inequality (3.5) for $k = 0$ is trivial and for $k = 1$ it immediately follows from condition (3.1).

Suppose that (3.5) is true for a fixed integer $k \geqslant 0$. Notice that

$$(3.7) \qquad |Z^{k+1} - Z^0|_0 \leqslant |\Phi^\varphi(Z^k) - \Phi^\varphi(Z^0)|_0 + |\Phi^\varphi(Z^0) - \overset{0}{Z}|_0.$$

Owing to (3.1), the second member of the right-hand side of (3.7) satisfies the inequality

$$(3.8) \qquad |\Phi^\varphi(Z^0) - Z|_0 \leqslant U^0 - \Omega(U^0).$$

For an estimation of the first memeber of the right-hand side of (3.7), observe that for $\mu \in I_n$ we have:

$$|\Phi_\mu^\varphi(Z^k) - \Phi_\mu^\varphi(Z^0)|_0 \leqslant \int_{\Delta_{x,\mu}} \Big| f\big((1-\mu)t + \mu x,\ (A\hat{Z}^k)((1-\mu)t + \mu x)\big) - f\big((1-\mu)t + \mu x,\ (A\hat{Z}^0)((1-\mu)t + \mu x)\big)\Big|_0 \cdot (1-\mu)\, dt.$$

In virtue of assumptions $(\tilde{A}_1)$–(δ), $(\tilde{A}_2)$–(a), $(\tilde{A}_2)$–(b) and $(\tilde{A}_1)$–(β), the last integral may be estimated by the integral

$$\int_{\Delta_{x,\mu}} \omega\big((1-\mu)t + \mu x,\ (B\,\widetilde{|Z^k - Z^0|_0})((1-\mu)t + \mu x)\big)(1-\mu)\, dt,$$

which, in turn, under the assumption that (3.5) is true for our k, may be estimated by

$$\int_{\Delta_{x,\mu}} \omega\big((1-\mu)t + \mu x,\ (B\tilde{U}^0)((1-\mu)t + \mu x)\big)(1-\mu)\, dt = \Omega_\mu(U^0).$$

Hence, in virtue of (3.8) and (3.7), we obtain

$$|Z^{k+1} - Z^0|_0 \leqslant \Omega(U^0) + U^0 - \Omega(U^0) = U^0.$$

This completes, by the induction principle the proof of inequality (3.5).

The proof of inequality (3.6) is quite similar (for $k = 0$ we have simply (3.5)) and it will be omitted.

From (3.6) and (3.4) it follows that the sequence

$$\{Z^k\}_{k=0,1,2,\dots}$$

given by (3.2) converges in Δ to a function Z^*.

More precisely, we have for $\mu \in I_n$:

$$Z^k_\mu(x) \to Z_\mu(x) \quad \text{as } k \to \infty,$$

uniformly with respect to $(1-\mu)x \in \operatorname{proj}^0_{1-\mu}(\Delta)$, for a.e. $\mu x \in \operatorname{proj}^0_\mu(\Delta)$. Hence, in virtue of Assumption $(\tilde{\mathrm{A}}_2)$–(c) and condition (W), we get

$$f\big(x, (A\hat{Z}^k)(x)\big) \to f\big(x, (A\hat{Z}^*)(x)\big) \quad \text{a.e. in } \Delta$$

and the limit function is summable.

This convergence, together with the second equality in (3.2), imply by the Lebesgue theorem that

$$Z^* = \lambda + \Phi^\varphi(Z^*),$$

which means exactly that for $\mu \in I_n$ we have

$$Z^*_\mu(x) = \lambda_\mu(x) + \Phi^\varphi_\mu(x, Z^*) \quad \text{for a.e. } \mu x \in \operatorname{proj}^0_\mu(\Delta).$$

Thus the function Z^* is the requiered solution of Problem $(\tilde{\mathrm{Q}})$, and, moreover (in virtue of (3.5)), the estimation

$$|Z^* - Z^0|_0 \leqslant U^0$$

holds true. Hence

$$Z^* \in \mathscr{E}(U^0_{(1,\dots,1)} + Z^0_{(1,\dots,1)}),$$

which means that

$$Z^* \in \mathscr{E}(U^0_{(1,\dots,1)} + \overset{0}{Z}_{(1,\dots,1)} + \lambda_{(1,\dots,1)}).$$

The proof of the theorem is thus complete.

Final remarks. In a similar way we can obtain the analogues of other theorems of Part I of the paper, concerning Carathéodory's theory of differential functional equations.

In particular, we formulate the following theorem concerning the uniqueness of solutions.

THEOREM 3.2. *Suppose assumption* (a) *of Theorem* 3.1. *Suppose, moreover, that for a function* $U^0 \in \mathscr{E}_*$ *Assumption* $\big(\tilde{\mathrm{A}}_3(h)\big)$ *is satisfied with* h *equal to* $U^0_{(1,\dots,1)}$ *and, in addition, that the inequality*

$$\Omega(U^0) \leqslant U^0 \tag{3.9}$$

is fulfilled.

Under these assumptions, if Z and W are two solutions of Problem ($\tilde{Q}$), *such that*

$$(3.10) \qquad |Z-W|_0 \leqslant U^0,$$

then

$$Z = W.$$

The proof of this theorem is quite similar to that of Theorem 10.2 in Part I.

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