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An extremal arclength problem in some classes of univalent and p-symmetric functions

by WIESŁAW MAJCHRZAK (Łódź)

Abstract. Let λ, σ, p be arbitrarily fixed numbers, $\lambda, \sigma \in \langle 0, 1\rangle$, $p = 1, 2, \ldots$ Let $S_\sigma^{*(p)}$ denote the class of functions $g(z) = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1}$ which are holomorphic and univalent in the disc $K = \{z: |z| < 1\}$ and such that $\operatorname{Re}\{zf'(z)/f(z)\} \geqslant \sigma$ in this disc. Let $L^{(p)}(\lambda, \sigma)$ be the class of functions $f(z) = z + \sum_{n=1}^{\infty} c_{np+1} z^{np+1}$, holomorphic and univalent in K, for which there exists a function $\varepsilon g(z) \in S_\sigma^{*(p)}$ such that $\operatorname{Re}\{zf'(z)/g(z)\} \geqslant \lambda$, where $z \in K$ and $|\varepsilon| = 1$.

Let $L_r(f)$ denote the arclength of the image $f[C_r]$, where $C_r = \{z: |z| = r, 0 < r < 1\}$, and f belongs to some family of functions. The fundamental result of the present paper is an estimate from above of the functional $L_r(f)$ in the family $L^{(p)}(\lambda, \sigma)$. The result obtained implies analogous estimates in some subclasses of the family $L^{(p)}(\lambda, \sigma)$, among others in $S_\sigma^{*(p)}$. All the obtained estimates are sharp.

1. Let $S^{(p)}$, $p = 1, 2, \ldots$, be the class of functions

$$f(z) = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1} \tag{1}$$

holomorphic and univalent in the disc $K = \{z: |z| < 1\}$. Functions of this form are called *p-symmetric*; they satisfy the condition $f(e^{2\pi i/p} z) = e^{2\pi i/p} f(z)$ for $z \in K$.

Let $S_\sigma^{*(p)}$, $0 \leqslant \sigma \leqslant 1$, be the class of functions of the form (1) satisfying in K the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \sigma.$$

The functions belonging to the class $S_\sigma^{*(p)}$ are called *p-symmetric starlike functions of order σ*. For $\sigma = 0$ we obtain the known family $S^{*(p)}$ of p-symmetric starlike functions.

Let $C_\lambda^{(p)}$, $0 \leqslant \lambda \leqslant 1$, be the family of functions of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} a_{np} z^{np}$$

which are holomorphic and satisfy the condition $\operatorname{Re} h(z) \geqslant \lambda$ for $z \in K$.

For every function $f \in S_\sigma^{*(p)}$ there exists a function $h \in C^{(p)} = C_0^{(p)}$ such that

$$\frac{zf'(z)}{f(z)} = (1-\sigma)h(z) + \sigma. \tag{2}$$

A function f of the form (1) is said to be *p-symmetric convex of order* σ, $0 \leqslant \sigma \leqslant 1$, if for $z \in K$

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \sigma.$$

The class of such functions will be denoted by $\hat{S}_\sigma^{(p)}$. For $\sigma = 0$ we obtain the known family $\hat{S}^{(p)}$ of p-symmetric convex functions.

It is not difficult to verify that $f \in \hat{S}_\sigma^{(p)}$ if and only if $zf' \in S_\sigma^{*(p)}$.

Let $L^{(p)}(\lambda, \sigma)$, $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \sigma \leqslant 1$, denote the class of functions which are p-symmetric close-to-convex of order λ and of type σ (cf. [10]). We say that $f \in L^{(p)}(\lambda, \sigma)$ if and only if the function f has the form (1) and there exists a function g such that $e^{ia} g \in S_\sigma^{*(p)}$ for some real a and

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \geqslant \lambda \tag{3}$$

for $z \in K$.

It follows from the definition of $L^{(p)}(\lambda, \sigma)$ that $a \in \langle -\arccos\lambda, \arccos\lambda\rangle$. Moreover, if $\sigma_2 \leqslant \sigma_1$ or $\lambda_2 \leqslant \lambda_1$, then $L^{(p)}(\lambda, \sigma_1) \subset L^{(p)}(\lambda, \sigma_2)$ or $L^{(p)}(\lambda_1, \sigma) \subset L^{p}(\lambda_2, \sigma)$, respectively.

If $f \in L^{(p)}(\lambda, \sigma)$, then for $z \in K$

$$\frac{zf'(z)}{g(z)} = (\cos a - \lambda)h(z) + \lambda - i\sin a, \tag{4}$$

where $e^{ia} g \in S_\sigma^{*(p)}$ and $h \in C^{(p)}$.

For some values of the parameters λ and σ we obtain known subclasses of the family $S^{(p)}$, for example $L^{(p)}(0, 0)$ is the class of p-symmetric close-to-convex functions in K [7]. Z. Lewandowski has proved [9] that this class coincides with the family of linearly attainable functions which was introduced by Biernacki [1]. If $\sigma = 1$, then $g(z) = z$ and $\operatorname{Re}\{f'(z)\} \geqslant \lambda$. Hence $L^{(p)}(\lambda, 1) = R_\lambda^{(p)}$, where $R_\lambda^{(p)}$ is the class of functions of the form (1) whose derivatives belong to $C_\lambda^{(p)}$. Observe also that if $f \in L^{(p)}(1, \sigma)$, then it follows from (3) that $zf'(z) = g(z)$ in the disc K. Thus $L^{(p)}(1, \sigma) = \hat{S}_\sigma^{(p)}$. It can be also easily verified that $S_\sigma^{*(p)} \subset L^{(p)}(\sigma, \sigma)$.

The functions from the family $L^{(p)}(0, 0)$ are univalent [7]. Hence and from the relation between the classes introduced it follows that the functions from $S_\sigma^{*(p)}$, $\hat{S}_\sigma^{(p)}$, $R_\lambda^{(p)}$ and $L^{(p)}(\lambda, \sigma)$ are also univalent in K.

For a given function f on K let $L_r(f)$ denote the arclength of the image (under f) of the circle $|z| = r$. Let f be a function of class $S^{(p)}$ and let r be an arbitrarily fixed number from the interval $(0, 1)$. Then $L_r(f)$ is a functional defined on the class $S^{(p)}$ whose values are given by the formula

$$L_r(f) = \int_{|z|=r} |f'(z)|\,|dz|. \tag{5}$$

The natural problem to be solved is to find a sharp estimate from above of functional (5) in a given class.

The problem has been investigated in the family $S = S^{(1)}$ ([11], p. 215), but the estimate obtained there is not sharp. Also in the class $S^{(p)}$ the solution of the problem remains unknown.

The present paper gives the sharp estimate from above of the functional $L_r(f)$ in the class $L^{(p)}(\lambda, \sigma)$. Such an estimate exists, since $L_r(f)$ is a continuous functional and the class considered is normal and compact.

The result obtained for the family $L^{(p)}(\lambda, \sigma)$ gives rise to a solution of the problem in the classes $S_\sigma^{*(p)}$, $\hat{S}_\sigma^{(p)}$, $R_\lambda^{(p)}$.

2. We shall need the following well-known lemmas.

LEMMA 1. *$g \in S_\sigma^{*(p)}$ if and only if for $z \in K$*

$$g(z) = z\exp\left\{-\frac{2(1-\sigma)}{p}\int_0^{2\pi} \log(1-z^p e^{-it})\,d\mu(t)\right\},$$

where $\mu(t)$ is a non-decreasing function for $t \in \langle 0, 2\pi\rangle$ and $\mu(2\pi)-\mu(0) = 1$.

LEMMA 2. *If $\mu(t)$ satisfies the assumptions of Lemma 1 and if $\varphi(t)$ is positive and integrable with respect to $\mu(t)$ in the interval $\langle 0, 2\pi\rangle$, then*

$$\exp\left\{\int_0^{2\pi} \log\varphi(t)\,d\mu(t)\right\} \leqslant \int_0^{2\pi} \varphi(t)\,d\mu(t).$$

LEMMA 3. *$f \in L^p(\lambda, \sigma)$ if and only if there exist functions $g \in S_\sigma^{*(p)}$ and h, with $e^{i\alpha}h \in C_\lambda^{(p)}$ for some real α, such that*

$$zf'(z) = e^{i\alpha}g(z)h(z).$$

LEMMA 4. *$e^{ia}h \in C_\lambda^{(p)}$ with some $a \in \langle -\arccos\lambda, \arccos\lambda\rangle$, if and only if*

$$h(z) = e^{-ia}\int_0^{2\pi} \frac{1+z^p e^{-it}(e^{2ia}-2\lambda e^{ia})}{1-z^p e^{-it}}\, d\mu(t) \tag{6}$$

for $z \in K$, where $\mu(t)$ satisfies the assumptions of Lemma 1.

Lemmas 1 and 4 follow from (2) and (4), respectively, and from the Hergoltz formula for the family $C^{(p)}$; Lemma 2 can be found in [6], p. 156. Lemma 3 is an immediate consequence of the definition of the class $L^{(p)}(\lambda, \sigma)$.

Let $F(x)$ be a non-negative measurable function of the real variable x such that the measure $M(y)$ of the set $\{x\colon F(x) \geqslant y\}$ is the finite and decreasing function of y for all positive y. Hence we can define an even function $F^*(x)$ by the condition

$$F^*[\tfrac{1}{2}M(y)] = y.$$

The function $F^*(x)$ increases for $x \leqslant 0$ and decreases for $x \geqslant 0$. In general $F^*(x)$ may tend to $+\infty$ if x tends to 0.

The function $F^*(x)$ is called ([6], p. 278) the *rearrangement of $F(x)$* in symmetrical decreasing order.

In [3] the following lemma has been proved:

LEMMA 5. *If $F(x)$, $G(x)$, and $H(x)$ are non-negative and integrable in $\langle -a, a\rangle$ functions, and if $F^*(x)$, $G^*(x)$ and $H^*(x)$ are their rearrangement in symmetrical decreasing order in this interval, then*

$$\int_{-a}^{a} F(x)G(x)H(x)\,dx \leqslant \int_{-a}^{a} F^*(x)G^*(x)H^*(x)\,dx.$$

3. Let us put ([5], p. 962 and 1054)

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt, \tag{7}$$

where $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$ and

$$F(a, \beta, \gamma, z) = \frac{1}{B(\beta, \gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-a}\,dt, \tag{8}$$

where $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, $z \in K$.

We shall now prove the following fundamental

THEOREM 1. *If $f \in L^{(p)}(\lambda, \sigma)$, then for $r \in (0, 1)$ there is a sharp estimate of the form*

$$L_r(f) \leqslant \int_0^{2\pi} \frac{r\,|1+r^p e^{ip\vartheta}(1-2\lambda)|}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}}\, d\vartheta. \tag{9}$$

The equality in (9) *is realized by the functions*

$$(10)\quad f(z) = \frac{z}{p} B\left(\frac{1}{p}, 1\right) F\left(1+\frac{2(1-\sigma)}{p}, \frac{1}{p}, 1+\frac{1}{p}, z^p\right) + $$
$$+\frac{(1-2\lambda)z^{p+1}}{p} B\left(1+\frac{1}{p}, 1\right) F\left(1+\frac{2(1-\sigma)}{p}, 1+\frac{1}{p}, 2+\frac{1}{p}, z^p\right),$$

where B *is Euler's beta function* (7) *and* F *is the hypergeometric function* (8).

Proof. If $f \in L^{(p)}(\lambda, \sigma)$, then by Lemma 3

$$(11)\qquad zf'(z) = e^{i\alpha} g(z) h(z),$$

where $g(z) \in S_\sigma^{*(p)}$, $e^{i\alpha} h(z) \in C_\lambda^{(p)}$. From Lemmas 1 and 2

$$|g(z)| = |z| \exp\left\{-\frac{2(1-\sigma)}{p} \int_0^{2\pi} \log|1-z^p e^{-it}|\, d\mu(t)\right\},$$

and hence

$$(12)\qquad |g(z)| \leqslant |z| \int_0^{2\pi} |1-z^p e^{-it}|^{-2(1-\sigma)/p}\, d\mu(t).$$

From (11) and (12) we have

$$|zf'(z)| \leqslant |z|\,|h(z)| \int_0^{2\pi} |1-z^p e^{-it}|^{-2(1-\sigma)/p}\, d\mu(t),$$

and thus for $z = re^{i\vartheta}$

$$L_r(f) = \int_0^{2\pi} r|f'(re^{i\vartheta})|\, d\vartheta \leqslant \int_0^{2\pi}\int_0^{2\pi} \frac{r|h(re^{i\vartheta})|}{|1-r^p e^{ip\vartheta} e^{-it}|^{2(1-\sigma)/p}}\, d\mu(t)\, d\vartheta.$$

Putting $\vartheta = \varphi + t/p$ and interchanging the order of integration, we obtain

$$L_r(f) \leqslant \int_0^{2\pi}\int_0^{2\pi} \frac{r|h(e^{it/p}\zeta)|}{|1-\zeta^p|^{2(1-\sigma)/p}}\, d\varphi\, d\mu(t),$$

where $\zeta = re^{i\varphi}$. Observe that if $e^{i\alpha}h(\zeta) \in C_\lambda^{(p)}$, then $e^{i\alpha}h(e^{it/p}\zeta) \in C_\lambda^{(p)}$. Hence

$$\max_{h \in C_\lambda^{(p)}} \int_0^{2\pi}\int_0^{2\pi} \frac{r|h(e^{it/p}\zeta)|}{|1-\zeta^p|^{2(1-\sigma)/p}} d\varphi\, d\mu(t) \leqslant \max_{h \in C_\lambda^{(p)}} \int_0^{2\pi}\int_0^{2\pi} \frac{r|h(\zeta)|}{|1-\zeta^p|^{2(1-\sigma)/p}}\, d\varphi\, d\mu(t).$$

Observe that the integrand does not depend on t and $\mu(2\pi)-\mu(0) = 1$, and consequently

$$(13)\qquad L_r(f) \leqslant \max_{h \in C_\lambda^{(p)}} \int_0^{2\pi} \frac{r|h(\zeta)|\, d\varphi}{|1-\zeta^p|^{2(1-\sigma)/p}}.$$

Let h_0 denote the function for which the maximum in (13) is attained. By Lemma 4 there exists a function $\mu_0(t)$ such that

$$h_0(\zeta) = e^{ia}\int_0^{2\pi} \frac{1+\zeta^p e^{-it}(e^{2ia}-2\lambda e^{ia})}{1-\zeta^p e^{-it}}\, d\mu_0(t).$$

Hence

$$L_r(f) \leqslant \int_0^{2\pi}\int_0^{2\pi} \frac{r\,|1+\zeta^p e^{-it}(e^{2ia}-2\lambda e^{ia})|}{|1-\zeta^p|^{2(1-\sigma)/p}\,|1-\zeta^p e^{-it}|}\, d\mu_0(t)\,d\varphi .$$

Interchanging the order of integration, we obtain

$$L_r(f) \leqslant \int_0^{2\pi} \left\{\max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi} \frac{r\,|1+\zeta^p e^{-it}(e^{2ia}-2\lambda e^{ia})|}{|1-\zeta^p|^{2(1-\sigma)/p}\,|1-\zeta^p e^{-it}|}\, d\varphi\right\} d\mu_0(t).$$

Since $\mu_0(2\pi)-\mu_0(0) = 1$, we have

$$L_r(f) \leqslant r\max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi} \frac{|1+\zeta^p e^{-it}(e^{2ia}-2\lambda e^{ia})|}{|1-\zeta^p|^{2(1-\sigma)/p}\,|1-\zeta^p e^{-it}|}\, d\varphi .$$

Let us put $\varphi = \vartheta + t/p$, $\zeta = re^{i\varphi}$. Then

$$L_r(f) \leqslant r\max_{0\leqslant t\leqslant 2\pi}\int_0^{2\pi} \frac{|1+r^p e^{i(p\vartheta+2a)}(1-2\lambda e^{-ia})|}{|1-r^p e^{ip\vartheta}|}\,\frac{1}{|1-r^p e^{i(p\vartheta+t)}|^{2(1-\sigma)/p}}\, d\vartheta .$$

Now put $1-2\lambda e^{-ia} = \varrho(a)e^{i\tau(a)}$, where $\varrho(a) = \sqrt{1-4\lambda\cos a+4\lambda^2}$ for $a \in \langle -\arccos\lambda, \arccos\lambda\rangle$. It can be easily seen that $\varrho(a) \in \langle |1-2\lambda|, 1\rangle$. Then

$$\begin{aligned} F(\vartheta) &= |1+r^p e^{i(p\vartheta+2a)}(1-2\lambda e^{-ia})| \\ &= \sqrt{1+2r^p\varrho(a)\cos\big(p\vartheta+2a+\tau(a)\big)+r^{2p}\varrho^2(a)}. \end{aligned}$$

Observe that the rearrangement of $F(\vartheta)$ in symmetrical decreasing order in the interval $\langle -\pi/p, \pi/p\rangle$ is the function $F^*(\vartheta) = \big(1+2r^p\varrho(a)\cos p\vartheta + r^{2p}\varrho^2(a)\big)^{1/2}$; indeed, this function is even, increasing for $\vartheta\leqslant 0$ and decreasing for $\vartheta \geqslant 0$ and both the functions $F(\vartheta)$ and $F^*(\vartheta)$ have identical sets of values.

Similarly we show that the rearrangement in symmetrical decreasing order of the function $G(\vartheta) = 1/|1-r^p e^{i(p\vartheta+t)}|^{2(1-\sigma)/p}$ in the interval $\langle -\pi/p, \pi/p\rangle$ is the function $G^*(\vartheta) = 1/|1-r^p e^{ip\vartheta}|^{2(1-\sigma)/p}$. Finally, notice that the function $H(\vartheta) = 1/|1-r^p e^{ip\vartheta}|$ is itself its own rearrangement in symmetrical decreasing order in the interval $\langle -\pi/p, \pi/p\rangle$. Since

$$I(a, t, \lambda, \sigma) = \int\limits_0^{2\pi} \frac{|1+r^p e^{i(p\vartheta+2a)}(1-2\lambda e^{-ia})|\,d\vartheta}{|1-r^p e^{ip\vartheta}|\,|1-r^p e^{i(p\vartheta+t)}|^{2(1-\sigma)/p}}$$

$$= p \int\limits_{-\pi/p}^{\pi/p} \frac{|1+r^p e^{i(p\vartheta+2a)}(1-2\lambda e^{-ia})|\,d\vartheta}{|1-r^p e^{ip\vartheta}|\,|1-r^p e^{i(p\vartheta+t)}|^{2(1-\sigma)/p}},$$

we obtain by Lemma 5 that for any values of a and t

$$I(a, t, \lambda, \sigma) \leqslant \int\limits_0^{2\pi} \frac{[1+2r^p \varrho(a)\cos p\vartheta + r^{2p}\varrho^2(a)]^{1/2}\,d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}}.$$

Thus

$$\max_{a,t} I(a, t, \lambda, \sigma) = \max_{a} \int\limits_0^{2\pi} \frac{[1+2r^p \varrho(a)\cos p\vartheta + r^{2p}\varrho^2(a)]^{1/2}\,d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}}.$$

We shall show that

$$(16) \qquad \max_{a} \int\limits_0^{2\pi} \frac{[1+2r^p \varrho(a)\cos p\vartheta + r^{2p}\varrho^2(a)]^{1/2}\,d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}}$$

$$= \int\limits_0^{2\pi} \frac{[1+2r^p|1-2\lambda|\cos p\vartheta + r^{2p}(1-2\lambda)^2]^{1/2}}{|1-r^p e^{ip\vartheta}|^{1-2(1-\sigma)/p}}\,d\vartheta$$

for $a \in \langle -\arccos\lambda, \arccos\lambda\rangle$ and for arbitrary values of λ and σ.

S. S. Miller has proved [13] that $I(a, t, 0, 0) \leqslant I(0, 0, 0, 0)$ for $t \in \langle 0, 2\pi\rangle$ and $a \in \langle -\pi/2, \pi/2\rangle$. This result can be easily generalized, by using the same method, to the case of $\lambda = 0$ and $\sigma \in \langle 0, 1\rangle$. Thus

$$(17) \qquad I(a, t, 0, \sigma) \leqslant I(0, 0, 0, \sigma)$$

for $t \in \langle 0, 2\pi\rangle$ and $a \in \langle -\pi/2, \pi/2\rangle$. The function $I(a, t, \lambda, \sigma)$ is continuous with respect to λ and thus inequality (17) holds true also for λ which are sufficiently close to 0. So for small values of λ

$$(18) \qquad \max_{a,t} \int\limits_0^{2\pi} \frac{|1+r^p e^{i(p\vartheta+2a)}(1-2\lambda e^{-ia})|\,d\vartheta}{|1-r^p e^{ip\vartheta}|\,|1-r^p e^{i(p\vartheta+t)}|^{2(1-\sigma)/p}}$$

$$= \int\limits_0^{2\pi} \frac{[1+2r^p|1-2\lambda|\cos p\vartheta + r^{2p}(1-2\lambda)^2]^{1/2}\,d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}}.$$

From (18) and (15) we obtain (16) for λ sufficiently close to 0. Hence it follows that the function $W(\varrho) = 1+2r^p\cos p\vartheta\varrho(a)+r^{2p}\varrho^2(a)$ defined in the interval $\langle |1-2\lambda|, 1\rangle$ attains the maximal value for $\varrho = |1-2\lambda|$, if λ is sufficiently close to 0. This shows that the function $W(\varrho)$ decreases in $\langle|1-2\lambda|, 1\rangle$ and attains the maximal value for $\varrho = |1-2\lambda|$ for any $\lambda \in \langle 0, 1\rangle$.

Thus (16) is valid for any admissible λ and σ. From (14), 15), (16) and (18) we have

$$L_r(f) \leqslant \int_0^{2\pi} \frac{r\left|1+r^p e^{ip\vartheta}|1-2\lambda|\right|}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}} d\vartheta \tag{19}$$

for any values of λ and σ and $r \in (0, 1)$.

If $\lambda \leqslant \frac{1}{2}$, then (19) yields (9). For $\lambda > \frac{1}{2}$

$$\int_0^{2\pi} \frac{r\left|1+r^p e^{ip\vartheta}|1-2\lambda|\right| d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}} = \int_0^{2\pi} \frac{r|1+r^p e^{i(p\vartheta+\pi)}(1-2\lambda)|}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}} d\vartheta,$$

and by Lemma 5

$$\int_0^{2\pi} \frac{r|1+r^p e^{i(p\vartheta+\pi)}(1-2\lambda)| d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}} \leqslant \int_0^{2\pi} \frac{r|1+r^p e^{ip\vartheta}(1-2\lambda)| d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}},$$

which proves that estimate (9) holds for $\lambda > \frac{1}{2}$ as well.

The equality in (9) takes place for the function

$$f(z) = \int_0^z \frac{1+(1-2\lambda)\zeta^p}{(1-\zeta^p)^{1+2(1-\sigma)/p}} d\zeta, \tag{20}$$

which belongs to the class $L^{(p)}(\lambda, \sigma)$. It turns out that function (20) can be expressed by formula (10) in terms of Euler's special beta functions and a hypergeometric series.

4. We have already observed that $S_\sigma^{*(p)} \subset L^{(p)}(\sigma, \sigma)$. Moreover, for $\lambda = \sigma$, function (20) is of the from

$$f(z) = \frac{z}{(1-z^p)^{2(1-\sigma)/p}} \tag{21}$$

and belongs to the class $S_\sigma^{*(p)}$. Thus from Theorem 1 we obtain

COROLLARY 1. *If $f \in S_\sigma^{*(p)}$, then for $r \in (0, 1)$*

$$L_r(f) \leqslant \int_0^{2\pi} \frac{r|1+r^p e^{ip\vartheta}(1-2\sigma)| d\vartheta}{|1-r^p e^{ip\vartheta}|^{1+2(1-\sigma)/p}}, \tag{22}$$

and the equality is realized by function (21).

Since $\hat{S}_\sigma^{(p)} = L^{(p)}(1, \sigma)$ and $L^{(p)}(\lambda, 1) = R_\lambda^{(p)}$, we obtain the following corollaries of Theorem 1.

COROLLARY 2. *If $f \in \hat{S}_\sigma^{(p)}$, then for $r \in (0,1)$*

$$L_r(f) \leqslant \int_0^{2\pi} \frac{r\,d\vartheta}{|1-r^p e^{ip\vartheta}|^{2(1-\sigma)/p}}, \tag{23}$$

where the equality is realized by the functions $f_p \in \hat{S}_\sigma^{(p)}$ given by the formulas

$$f_1(z) = \begin{cases} \dfrac{1-(1-z)^{2\sigma-1}}{2\sigma-1} & \text{for } \sigma \neq \tfrac{1}{2}, \\ -\log(1-z) & \text{for } \sigma = \tfrac{1}{2}, \end{cases} \tag{24}$$
$$f_2(z) = \begin{cases} \dfrac{z}{2} B(\tfrac{1}{2},1) F(1-\sigma, \tfrac{1}{2}, \tfrac{3}{2}, z^2) & \text{for } \sigma \neq 0, \\ \tfrac{1}{2}\log\dfrac{1+z}{1-z} & \text{for } \sigma = 0, \end{cases}$$

$$f_p(z) = \frac{z}{p} B\left(\frac{1}{p}, 1\right) F\left(\frac{2(1-\sigma)}{p}, \frac{1}{p}, 1+\frac{1}{p}, z^p\right), \tag{25}$$

where $p = 3, 4, \ldots$

COROLLARY 3. *If $f \in R_\lambda^{(p)}$, then for $r \in (0,1)$*

$$L_r(f) \leqslant \int_0^{2\pi} \frac{r|1+r^p e^{ip\vartheta}(1-2\lambda)|}{|1-r^p e^{ip\vartheta}|} d\vartheta \tag{26}$$

and the equality is realized by the functions $f_p \in R_\lambda^{(p)}$ given by the formulas

$$f_1(z) = 2(\lambda-1)\log(1-z) + (2\lambda-1)z,$$
$$f_p(z) = \frac{z}{p} B\left(\frac{1}{p}, 1\right) F\left(1, \frac{1}{p}, 1+\frac{1}{p}, z^p\right) + \frac{(1-2\lambda)z^{p+1}}{p} B\left(1+\frac{1}{p}, 1\right) \times$$
$$\times F\left(1, 1+\frac{1}{p}, 2+\frac{1}{p}, z^p\right),$$

where $p = 2, 3, \ldots$

5. For some values of the parameters λ, σ, p one can easily express the integrals in the obtained estimates in terms of hypergeometric function (8) (e.g., for the integral in (9) for $\lambda = \sigma = 0$ and $p = 1$ cf. [4]).

Consider (9) for $\lambda = \frac{1}{2}$, $\sigma = 0$, $p = 1$. If $p = 1$, then the extremal function (20) has the form

$$f(z) = \begin{cases} \dfrac{z(1-\sigma)(1-2\lambda) + (\lambda-\sigma)|1-(1-z)^{2(1-\sigma)}|}{(1-\sigma)(1-2\sigma)(1-z)^{2(1-\sigma)}} & \text{for } \sigma \neq \dfrac{1}{2}, 1, \\ (1-2\lambda)\log(1-z) + \dfrac{2(1-\lambda)z}{1-z} & \text{for } \sigma = \dfrac{1}{2}, \\ 2(\lambda-1)\log(1-z) + (2\lambda-1)z & \text{for } \sigma = 1. \end{cases}$$

Then making use of [2], 110.08, 111.06, 160.02, and [5], 8.114, we obtain

$$L_r(f) = \frac{2\pi r}{(1-r)^3}\left[\frac{1-r^2}{1+4r-r^2}\right]^{1/2} F\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{4r}{1+4r-r^2}\right).$$

If in (22) $\sigma = 0$ and $p = 2$, then from [2], 110.06, 160.02, and [5], 8.113, we have

$$L_r(f_2) = \frac{2\pi r}{1+r^2} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{4r^2}{(1+r^2)^2}\right).$$

6. Observe that functions (24) and (25) map the disc K onto a domain with a boundary whose arclength is given by the formula

$$L_1(f_p) = \int_0^{2\pi} \frac{dt}{|1-e^{itp}|^{2(1-\sigma)/p}} = 2^{2(\sigma-1)/p} \int_0^{2\pi} \left(\sin\frac{\vartheta}{2}\right)^{2(\sigma-1)/p} d\vartheta$$

for $p \geqslant 2$; and this yields

$$L_1(f_p) = 2^{2+4(\sigma-1)/p} \int_0^{\frac{\pi}{2}} (\sin\varphi)^{2(\sigma-1)/p} (\cos\varphi)^{2(\sigma-1)/p} d\varphi.$$

Since for function (7) we have

$$B(z, w) = 2\int_0^{\frac{\pi}{2}} \sin^{2z-1}\vartheta \cos^{2w-1}\vartheta \, d\vartheta,$$

where $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$, it follows that

$$(24) \qquad L_1(f_p) = 2^{1+4(\sigma-1)/p} B\left(\frac{\sigma-1}{p}+\frac{1}{2}, \frac{\sigma-1}{p}+\frac{1}{2}\right),$$

where $0 < \sigma \leqslant 1$ when $p = 2$ and $0 \leqslant \sigma \leqslant 1$ when $p = 3, 4, \ldots$

One can easily verify that the functional $L_r(f)$ increases with respect to r, $r \in (0, 1\rangle$, and thus from Corollary 2 and (27) we obtain

Corollary 4. *For any function $f \in \hat{S}_\sigma^{(p)}$, $p = 2, 3, \ldots$, and for every $r \in (0, 1\rangle$ we have*

$$(28) \qquad L_r(f) \leqslant 2^{1+4(\sigma-1)/p} B\left(\frac{\sigma-1}{p}+\frac{1}{2}, \frac{\sigma-1}{p}+\frac{1}{2}\right),$$

and $\sigma > 0$ when $p = 2$ and $\sigma \geqslant 0$ when $p = 3, 4, \ldots$

Note that in the particular case $\lambda = \sigma = 0$ from (9), (22), (23) and (28) we obtain Miller's results [13]. If, moreover, $p = 1$, then from (9), (22) and (23) we obtain the results of Duren, Clunie [3], [4], Marx [12] and Keogh [8].

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