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On convergence of iterates of the Frobenius–Perron operator for expanding mappings

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Abstract. It is shown that the sequence $P_\tau^n f$ of iterates of the Frobenius–Perron operator is convergent to its limit as rapidly as the geometric sequence.

1. Introduction. Lasota [4] and Jabłoński [1] have shown that the sequence $P_\tau^n f$ of iterates of the Frobenius–Perron operator P_τ: $L^1 \to L^1$ given by an expanding transformation τ: $M \to M$ of a differentiable manifold into itself is uniformly convergent for a certain class of functions f: $M \to R$. A similar theorem has also been stated by Krzyżewski [2]. But none of the theorems given by the above authors says how rapidly $P_\tau^n f$ is convergent. The purpose of this note is to estimate the rate of convergence of the sequence $P_\tau^n f$; namely, we shall show that $P_\tau^n f$ is convergent to its limit as rapidly as the geometric sequence.

2. The convergence theorem. Let (X, Σ, μ) be a measure space with a σ-finite measure μ. Denote by $L^1(X, \Sigma, \mu)$ the space of all integrable functions defined on X.

We say that a measurable transformation τ: $X \to X$ is *non-singular* if $\mu(\tau^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for any measurable set A.

For non-singular τ: $X \to X$ we define the Frobenius–Perron operator P_τ: $L^1 \to L^1$ by the formula

$$\int_A P_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu$$

which is valid for each measurable set $A \subset X$. It is well known that the operator P_τ is linear and satisfies the following conditions:

(a) P_τ is positive: $f \geqslant 0 \Rightarrow P_\tau f \geqslant 0$,
(b) P_τ preserves integrals:

$$\int_X P_\tau f d\mu = \int_X f d\mu, \quad f \in L^1,$$

(c) $P_{\tau^k} = P_\tau^k$ (τ^k denotes the n-th iterate of τ),

(d) $P_\tau f = f$ if and only if the measure $d\nu = f d\mu$ is invariant under τ, i.e., $\nu(\tau^{-1}(A)) = \nu(A)$ for each measurable A.

We shall not make a distinction between functions $f\colon X \to R$ defined on X and functions $f\colon X \to R$ taken as the elements of the space L^1. This difference will be clear from the context.

A transformation $\tau\colon [0, 1] \to [0, 1]$ will be called *piecewise* C^2 if there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, 2, \ldots, p$) the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function. The transformation τ need not be continuous at the points a_i.

Lasota and Yorke [5] have shown that for any $\tau\colon [0, 1] \to [0, 1]$ piecewise C^2 with $\inf|\tau'(x)| > 1$ there exists an absolutely continuous probabilistic measure invariant under τ, and the density of any such measure has bounded variation. Moreover, it is well known [3], [6] that if a transformation $\tau\colon [0, 1] \to [0, 1]$ is piecewise C^2 with $\inf|\tau'(x)| > 1$ and there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, 2, \ldots, p$) the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 bijective map of $[a_{i-1}, a_i]$ onto $[0, 1]$, then the absolutely continuous measure invariant under τ is unique.

Let $\bigvee_a^b f$ denote the variation of f over the interval $[a, b]$.

THEOREM. *Let the transformation* $\tau\colon [0, 1] \to [0, 1]$ *be a piecewise* C^2 *function such that*

(i) *there exists a partition* $0 = a_0 < a_1 < \ldots < a_p = 1$ *of the unit interval such that for each integer* i ($i = 1, 2, \ldots, p$) *the restriction* τ_i *of* τ *to the open interval* (a_{i-1}, a_i) *is a* C^2 *function which can be extended to the closed interval* $[a_{i-1}, a_i]$ *as a* C^2 *bijective map of* $[a_{i-1}, a_i]$ *onto* $[0, 1]$,

(ii) $$s = \sup_{i,x}\left|(\tau_i^{-1}(x))'\right| + \sup_{i,x}\left|(\tau_i^{-1}(x))''\right|\left(\inf_{i,x}\left|(\tau_i^{-1}(x))'\right|\right)^{-1} < 1.$$

Then for any $f \geqslant 0$ *with bounded variation*

$$\left|(P_\tau^n f)(x) - \|f\| f_0(x)\right| \leqslant s^n\left(\bigvee_0^1 f + \|f\| \bigvee_0^1 f_0\right),$$

where f_0 *is the density of the probabilistic measure invariant under* τ.

Proof. Set $\varphi_i = \tau_i^{-1}$. A simple computation shows that the Frobenius–Perron operator corresponding to τ may be written in the form

$$(P_\tau f)(x) = \sum_{i=1}^{p} f(\varphi_i(x))\,|\varphi_i'(x)|.$$

By its very definition the operator P_τ is a mapping from L^1 into L^1, but the last formula enables us to consider P_τ as a map from the space of functions defined on [0, 1] into itself.

Let f be a function with bounded variation such that $\int_0^1 f dx = 0$. We have

$$\begin{aligned}\bigvee_0^1 P_\tau f &\leqslant \sum_{i=1}^{p} \bigvee_0^1 f(\varphi_i)|\varphi_i'| = \sum_{i=1}^{p} \int_0^1 |d(f(\varphi_i)|\varphi_i'|)| \\ &\leqslant \sum_{i=1}^{p} \int_0^1 |f(\varphi_i)|\,|\varphi_i''|\,dx + \sum_{i=1}^{p} \int_0^1 |\varphi_i'|\,|df(\varphi_i)| \\ &\leqslant \sum_{i=1}^{p} K \int_0^1 |f(\varphi_i)|\,|\varphi_i'|\,dx + \sup_{x,i}|\varphi_i'| \sum_{i=1}^{p} \int_0^1 |df(\varphi_i)| \\ &= K \sum_{i=1}^{p} \int_{a_{i-1}}^{a_i} |f|\,dx + (\sup_{i,x}|\varphi_i'|) \sum_{i=1}^{p} \int_{a_{i-1}}^{a_i} df \\ &= K \int_0^1 |f|\,dx + \sup_{i,x}|\varphi_i'| \bigvee_0^1 f,\end{aligned}$$

where $K = \sup_{i,x}|\varphi_i''|/(\inf_{i,x}|\varphi_i'|)$. Therefore, since $\int_0^1 |f|\,dx \leqslant \bigvee_0^1 f$ (recall that $\int_0^1 f dx = 0$) we obtain

$$\bigvee_0^1 P_\tau f \leqslant s \bigvee_0^1 f$$

and, consequently, by induction

$$\bigvee_0^1 P_\tau^n f \leqslant s^n \bigvee_0^1 f. \tag{1}$$

Now, let $f \geqslant 0$ be a function with bounded variation. Since

$$\int_0^1 (f - \|f\| f_0)\,dx = 0,$$

from (1) we obtain

$$\bigvee_0^1 P_\tau^n (f - \|f\| f_0) \leqslant s^n \Big(\bigvee_0^1 f + \|f\| \bigvee_0^1 f_0\Big).$$

Since $P_\tau f_0 = f_0$, the last inequality gives the conclusion of the theorem.

References

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