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Bounds for the set of solutions of functional-differential equations

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Dedicated to the memory of my friend Jacek Szarski

Abstract. Sets of systems of ordinary functional-differential equations with Volterra type functionals under sets of initial values are considered. Upper and lower bounds are constructed for the sets of all solutions. Classes of such problems are given where these bounds are optimal. The main tool is a Lemma of Max Müller on inequalities. Also ideas from interval mathematics are used.

1. Significance and explanation. If differential equations

$$u'(t) = f\big(t, u(t)\big), \qquad u(0) = a$$

appear in Applied Mathematics there is normally not just *one* right-hand side f. Instead of this a whole set $\{f\}$ of right-hand sides must be considered. This is due to many facts such as: data errors, data intervals obtained from measurements, approximation of f by a more suitable function, poor knowledge of the laws involved, etc. The same is true for the initial "value" a which is usually a set $\{a\}$. Hence the above initial value problem has to be replaced by the inclusion problem

$$u'(t) \in \big\{f\big(t, u(t)\big)\big\}, \qquad u(0) \in \{a\}.$$

It is normally completely impossible to solve *all* the real problems which are combined in this set of problems. The goal of the following paper is therefore to find at least lower and upper bounds to the set of all such solutions. This can always be done. Since these bounds are sometimes very pessimistic, classes of such problems are given where the bounds obtained are *optimal*.

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The main ideas of this paper are also valid in the more general case where f does depend as a functional upon the unknown solution u. This is written in the form $f(t, u(t), u)$. Therefore the theory of this paper also includes integro-differential equations and difference-differential equations. Sets of such problems do occur for example in Economics and in Biology.

An earlier version of this paper has been written while the author worked as a guest during the summer of 1977 at the Mathematics Research Center of the University of Wisconsin in Madison/Wisc., USA. It first appeared as Technical Summary Report No. 1782 of this institution and was sponsored by the U.S. Army under Contract No. DAAG29-75-C-0024. The author got inspired to work on this field by the papers of professor Jacek Szarski [9]–[14].

2. Introduction. In the following paper systems of functional-differential equations

$$u'(t) = f(t, u(t), u) \quad \text{for } 0 < t \leqslant T, \tag{1}$$

are considered under the initial conditions

$$u(0) = a. \tag{2}$$

Herein $u = (u_1, u_2, \ldots, u_n)$, $f = (f_1, f_2, \ldots, f_n)$ and $a = (a_1, a_2, \ldots, a_n)$ are n-vectors. As usual $u(t)$ means the value of the function u at the point t; moreover $u'(t) = du(t)/dt$. Opposite to this the notation u means that u is regarded as an element of the class of admissible functions. Hence $f(\cdot, \cdot, u)$ is a functional on u; in what follows only special "Volterra" functionals will be regarded.

If f is continuous then the system (1), (2) is equivalent to the system of functional-integral equations

$$u(t) = a + \int_0^t f(s, u(s), u)\,ds \quad \text{for } 0 \leqslant t \leqslant T. \tag{3}$$

It is the subject of the following paper to find bounding functions $v(t)$, $w(t)$ such that for every solution $\hat{u}(t)$ of (1), (2) or (3)

$$v(t) \leqslant \hat{u}(t) \leqslant w(t) \quad \text{for } 0 \leqslant t \leqslant T. \tag{4}$$

Hence the classical theory of maximal and minimal solutions for differential equations appears as a special case of these results.

If a solution $\hat{u}$ of (1), (2) is uniquely determined then it is trivial that (4) is satisfied for $w := v := \hat{u}$. It is therefore interesting to switch to a more general problem: Let $\{a\}$ be a set of initial values and let $\{f\}$ be a set of right-hand sides to (1). Then the more general initial value problem

$$u'(t) \in \{f(t, u(t), u)\} \quad \text{for } 0 < t \leqslant T, \tag{5}$$

$$u(0) \in \{a\} \tag{6}$$

is considered. Herein $\hat{u}$ is called a *solution* of (5), (6) if there is a right-hand side $\hat{f} \in \{f\}$ and an initial vector $\hat{a} < \{a\}$ such that $\hat{u}'(t) = \hat{f}(t, \hat{u}(t), \hat{u})$ on $0 < t \leqslant T$ and that $\hat{u}(0) = a$. Let $\{\hat{u}\}$ be the set of all solutions of (5), (6). Again two functions $v(t)$, $w(t)$ are looked for such that (4) is true for any solution $\hat{u} \in \{\hat{u}\}$. If one writes $[v, w]$ for the function interval from the two bound functions v and w then this can be written as

$$\{\hat{u}\} \subseteq [v, w]. \tag{7}$$

It is in general quite simple to find rough bounds v, w. In what follows special emphasis is therefore given to the look for "optimal" bounds. Here "optimality" means the following: let there exist the infimum and the supremum of the set $\{\hat{u}\}$ such that

$$v = \inf\{\hat{u}\}, \qquad w = \sup\{\hat{u}\}. \tag{8}$$

In that case one can call $[v, w]$ the "interval hull" of $\{\hat{u}\}$. It is the goal of this paper to find classes of sets $\{f\}$ and $\{a\}$ such that (7) and (8) are true.

In order to get such results a lemma of Max Müller [5] on differential inequalities is essential. This lemma has been published more than 50 years ago. For decades however, it remained widely unnoticed. In what follows this lemma will be extended to the case of functional-differential inequalities. This will be done by extending an old paper of the author (Nickel [7]). See also Adams–Spreuer [1] and the papers of Szarski.

It should finally be remarked that the problem of this paper and some of the formulations have been strongly influenced by the ideas of interval mathematics.

3. Notations and assumptions. Let $n \in \boldsymbol{N}$, $0 < T \in \boldsymbol{R}$, $I := [0, T]$, $I_0 := (0, T]$, $a \in \boldsymbol{R}^n$. The n-vectors a, u, f are written as $a = (a_1, a_2, \ldots, a_n)$, $u = (u_1, u_2, \ldots, u_n)$, $f = (f_1, f_2, \ldots, f_n)$. Together with the kth component u_k of the vector u also the n—1-vector ${}_k u = (u_1, u_2, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n)$ is used.

Let the class $\mathfrak{Z}$ of the admissible functions be defined as the set of all function vectors u: $I \to \boldsymbol{R}^n$, continuous on I such that the derivate u' exists in I_0.

The notation u means that u is an *element* of the class $\mathfrak{Z}$. Opposite to this $u(t)$, $u'(t)$ mean the *values* of these functions at the point t.

Let f be a mapping f: $I_0 \times \boldsymbol{R}^n \times \mathfrak{Z} \to \boldsymbol{R}^n$. Let the dependence of each component $f_k(t, y, z)$ of z be that of a "Volterra" functional. Here a functional $g(t, z)$ with g: $I_0 \times \mathfrak{Z} \to \boldsymbol{R}$ is called "Volterra", if

$$g(t, v) = g(t, w) \quad \text{for all } t \in I_0$$

is true for all functions v, $w \in \mathfrak{Z}$ which satisfy the equality $v(s) = w(s)$ on $0 < s \leqslant t$. Hence the value of $g(t, z)$ depends only upon "the past" of the function z.

EXAMPLES of Volterra functionals are

$$g(t, z) := \int_0^t K(t, s, z(s))\,ds,$$

(9) $$g(t, z) := z(\tau \cdot t) \qquad \text{with } 0 \leqslant \tau \leqslant 1,$$

(10) $$g(t, z) := \begin{cases} z(t-s) & \text{for} \quad 0 \leqslant s \leqslant t, \\ a(t-s) & \text{for} \quad 0 \leqslant t < s \end{cases}$$

with some function $a(t)$
given for $-s \leqslant t < 0$.

These examples show that the theory given in this paper can be applied to: differential equations, (Volterra) integro-differential equations, difference-differential equations with retarded argument and naturally also to combinations of these equations.

Inequalities $v(t) \leqslant w(t)$ are always meant componentwise as $v_k(t) \leqslant w_k(t)$ for $k = 1(1)n$. Inequalities of the kind $v \leqslant w$ are meant both componentwise and pointwise for all points in the definition set. Inclusions $\in$ are defined analogously.

To a set $\{a\}$ the inequality $z < \{a\}$ means that $z < \hat{a}$ for all $\hat{a} \in \{a\}$. Similarly the inequalities $z > \{a\}$, $z \leqslant \{a\}$, $z \geqslant \{a\}$ are defined. For $v \leqslant w$ the interval $[v(t), w(t)]$ is defined as the set $[v(t), w(t)] := \{z \in \mathbf{R}^n \mid v(t) \leqslant z \leqslant w(t)\}$. Similarly $[v, w] := \{z \in \mathfrak{Z} \mid v \leqslant z \leqslant w\}$.

In order to simplify the results the following notation will be used for the components f_k of $f(t, y, z)$: the second argument $y \in \mathbf{R}^n$ of f is broken up in the component y_k with the same index k as f_k and in the rest vector ${}_ky$:

$$f_k = f_k(t, y_k, {}_ky, z).$$

Furthermore it is suitable to have a special notation for the set of all functions f_k if the arguments lie in certain intervals. This will be denoted by

$$\{f_k(t, x_k, [{}_ky, {}_kz], [v, w]\} := \{f(t, x_k, {}_kp, q) \mid {}_kp \in [{}_ky, {}_kz], q \in [v, w]\}.$$

From Section 8 to the rest of this paper only functions f will be regarded which are partially monotone. The corresponding definitions will be given in Section 8.

There is, however, one more notation to be introduced already here: The Volterra functional $g(t, z) = g(t, z_k, {}_kz)$ is called partially strictly monotone increasing (decreasing) with respect to the component z_k if for all $t \in I_0$ the following is true:

$$v_k(s) < w_k(s) \quad \text{in } 0 < s \leqslant t$$
implies
$$g(t, v_k, {}_kv) \ (\lesseqgtr) \ g(t, w_k, {}_kv) \quad \text{for all functions } (v_k, {}_kv), (w_k, {}_kv) \in \mathfrak{Z}.$$

See Nickel [7].

4. Existence. The following theorem is the extension of the well known Peano existence theorem for systems of differential equations:

THEOREM. *Let f be defined and continuous on $I \times \mathbf{R}^n \times \mathfrak{Z}$. Then* (1), (2) *and* (3) *are equivalent. If f is bounded there exists* (*at least*) *one solution $\hat{u} \in \mathfrak{Z}$ of* (1), (2). *If f is not bounded then there exists a solution of* (1), (2) *at least in a largest interval $0 \leqslant t < T_1 \leqslant T$.*

Proof. The equivalence is trivial. For the existence the fixed point theorem of Schauder is applied to equation (3). The main ideas are exactly the same as in the case of differential equations. They are described in the book of Walter [15], p. 23–25.

For more results on functional-differential equations see for example the book of Myschkis [6].

5. The lemma of Max Müller.

LEMMA. *Let the functions $v, w \in \mathfrak{Z}$ with $v \leqslant w$ satisfy the following inequalities:*

$$v(0) < a < w(0), \tag{11}$$

$$v_k'(t) < \{f_k(t, v_k(t), [{}_k v(t), {}_k w(t)], [v, w])\}, \tag{12}$$

$$w_k'(t) > \{f_k(t, w_k(t), [{}_k v(t), {}_k w(t)], [v, w])\} \tag{13}$$

for $t \in I_0$ and $k = 1(1)n$.

Then any solution $\hat{u} \in \mathfrak{Z}$ of (1), (2) *is bounded by*

$$v(t) < \hat{u}(t) < w(t) \quad \text{for } t \in I. \tag{14}$$

COROLLARIES. 1. *If $u \in \mathfrak{Z}$ is a solution of the inequalities*

$$u(0) \leqslant a, \quad u'(t) \leqslant f(t, u(t), u) \quad \text{in } I_0$$

then

$$u(t) < w(t) \quad \text{for } t \in I.$$

2. *Similarly*

$$\bar{u}(t) > v(t) \quad \text{for } t \in I$$

for any solution

$$\bar{u} \in \mathfrak{Z} \text{ of the inequalities } u(0) \geqslant a,$$
$$u'(t) \geqslant f(t, u(t), u) \quad \text{in } I_0.$$

3. *If all f_k are strictly monotone* (*increasing or decreasing*) *with respect to* (*at least*) *one of the components of u, it then suffices to have the $\geqslant$ - and $\leqslant$-signs in* (12) *and* (13) *instead of the $>$ - and $<$-signs.*

Remarks. 1. This lemma has been formulated and proven by Max Müller [5] as Theorem 5 on the pages 13 to 15 for the special case where f does not depend upon u. See W. Walter [15], p. 93–94.

2. The original notation of M. Müller was very inconvenient. It has here been replaced by the interval notation.

3. Kindly note that there are no assumptions to be made with respect to be function f, such as continuity, monotonicity, etc.

Proof. The original proof of M. Müller carries right over to this case.

6. Example. In the following example $n = 2$. The functional (9) is used for $\tau = 1/2$ together with the other functional

$$\int_0^t \left(u_1^2(s) + u_2^2(s)\right) ds.$$

Let

$$f_1\left(t, u(t), u\right) := -2u_1^2(t) + u_2(t)/(1+t^2) + \left(u_1(t/2) + u_2(t/2)\right)/2,$$

$$f_2\left(t, u(t), u\right) := 1 - \sin\left(\pi u_1(t)/2\right) + 2u_2(t)\left(1 - 2u_2(t)\right) + $$

$$+ (1/2t) \int_0^t \left(u_1^2(s) + u_2^2(s)\right) ds,$$

$$0 < a_1, a_2 < 1.$$

Define $v_1 = v_2 := 0$, $w_1 = w_2 := 1$ for $t \geqslant 0$. Then one easily verifies for $t > 0$:

$$f_1\left(t, v_1(t), [v_2(t), w_2(t)], [v_1, w_1], [v_2, w_2]\right)$$
$$= [0, 1]/(1+t^2) + [0, 1] \geqslant 0,$$
$$f_1\left(t, w_1(t), [v_2(t), w_2(t)], [v_1, w_1], [v_2, w_2]\right)$$
$$= -2 + [0, 1]/(1+t^2) + [0, 1] \leqslant 0,$$
$$f_2\left(t, v_2(t), [v_1(t), w_1(t)], \ldots\right) = [0, 1] + [0, 1] \geqslant 0,$$
$$f_2\left(t, w_2(t), [v_1(t), w_1(t)], \ldots\right) = [0, 1] - 2 + [0, 1] \leqslant 0.$$

Since f_1 and f_2 are strictly monotone increasing with respect to both components u_1 and u_2, the third corollary to the lemma can be used. This gives the a priori estimate $0 < \hat{u}_1(t), \hat{u}_2(t) < 1$ for $t \geqslant 0$ for any solution $\hat{u}$. By the existence theorem of Chapter 4 there exists at least one solution $\hat{u}$ in a certain interval $0 \leqslant t < T_1 \leqslant T$. Because of the bounds obtained it does exist for all $t \geqslant 0$. It is furthermore uniquely determined because of the third corollary of the Lemma. Hence there is exactly one solution for all $t \geqslant 0$ and to all initial values $0 < a_1, a_2 < 1$ and it satisfies the inequalities $0 < \hat{u}_1, \hat{u}_2 < 1$.

7. Uniqueness and error bounds. For ordinary differential equations all uniqueness theorems can be derived from the Lemma of M. Müller. In his original paper he did use the lemma exactly for that purpose. Similarly probably all known a posteriori bounds can be proven with its help.

Therefore a very large number of such theorems can immediately be proven also for the case of functional-differential equations. It is possible to translate all the results from the books of J. Szarski [8] and of W. Walter [15] to these extended systems of equations. Some first results have already been given in the old paper by K. Nickel [7]. It is, however, not the purpose of this paper to publish such theorems.

8. Monotonicity conditions.

DEFINITION (unconditionally partially isotone/antitone/monotone). Let $g(x_1, x_2, \ldots, x_m)$ be a mapping $g\colon D \to \boldsymbol{R}$ with $D \subseteq \boldsymbol{R}^m$. Again the notation $g(x_1, \ldots, x_m) = g(x_k, {}_kx)$ is used. The function g is called *unconditionally partially isotone* or *antitone* on D with respect to the variable x_k if

$$g(y_k, {}_kx) \geqslant g(x_k, {}_kx) \quad \text{ for } y_k \geqslant x_k \text{ or } y_k \leqslant x_k$$

and for all $(x_k, {}_kx)$, $(y_k, {}_kx) \in D$. A function which is either unconditionally partially isotone or unc. part. antitone is called *unconditionally partially monotone*.

Kindly note that the function $g(x_1, x_2) := x_1 \cdot x_2$ is unconditionally partially monotone with respect to x_1 and x_2 on $D := [0, \infty) \times [0, \infty)$, but *not* on $D := \boldsymbol{R}^2$.

DEFINITION (monotonicity class $\mathfrak{M}$). Let the class $\mathfrak{M}$ consist on all functions $f(t, x, y)$ for which the following is true: Each function $f_k(t, x_k, {}_kx, y)$ is unconditionally partially monotone on $I_0 \times \boldsymbol{R} \times \boldsymbol{R}^{n-1} \times \mathfrak{Z}$ with respect to any component of ${}_kx$ and y, but not necessarily with respect to x_k.

If $f \in \mathfrak{M}$ then it is convenient to write

$$f_k(t, x_k, {}_kx, y) = f_k(t, x_k, {}_kx\uparrow, {}_kx\downarrow, y\uparrow, y\downarrow).$$

This clearly means that the vectors ${}_kx$ and y are divided in the two sets of components for which f_k is isotone ($\uparrow$) and antitone ($\downarrow$).

Let $f \in \mathfrak{M}$. Then in the lemma of M. Müller the (rather inconvenient) inequalities of sets can be replaced by real inequalities (which are much simpler to handle). In this case the lemma reduces to the

SPECIAL CASE OF THE LEMMA. *Let $f \in \mathfrak{M}$. Assume that* (11) *is true for functions $v, w \in \mathfrak{Z}$ with $v < w$ and that furthermore*

$$v_k'(t) < f_k\big(t, v_k(t), {}_kv(t)\uparrow, {}_kw(t)\downarrow, v\uparrow, w\downarrow\big),$$
$$w_k'(t) > f_k\big(t, w_k(t), {}_kw(t)\uparrow, {}_kv(t)\downarrow, w\uparrow, v\downarrow\big),$$
$$\text{for } t \in I_0 \text{ and } k = 1(1)n.$$

Then any solution $\hat{u} \in \mathfrak{Z}$ of (1), (2) *is bounded by* (14); *furthermore Corollaries* 1 *to* 3 *hold.*

The proof comes immediately from the Lemma together with the definition of the class $\mathfrak{M}$.

9. Construction of bounds. In what follows the special case of the Lemma will be used to construct bound functions v and w. The above problem (1), (2), (4) will be replaced by a more general since no additional difficulties are generated by the extension:

The initial data a in the initial conditions (2) are to be replaced by a set $\{a\}$. For simplicity assume $\{a\} \subseteq [\underline{a}, \bar{a}]$ with two vectors $\underline{a}, \bar{a} \in \boldsymbol{R}^n$ and $\underline{a} \leqslant \bar{a}$ such that

$$\underline{a}, \bar{a} \in \{a\} \subseteq [\underline{a}, \bar{a}]. \tag{15}$$

Hence the initial conditions (2) are to be replaced by (6). Let $\{\hat{u}\}$ be the set of all solutions of (1) under the set of all initial conditions (6). Wanted are bounds v, w such that (7) is true.

Assume that f is continuous and bounded on $I \times \boldsymbol{R}^n \times \mathfrak{Z}$. The following sequence of problems for $\nu \in \boldsymbol{N}$ is considered:

$$\begin{aligned} v_k'(t) &= f_k\big(t, v_k(t), {}_kv(t)\uparrow, {}_kw(t)\downarrow, v\uparrow, w\downarrow\big) - 1/\nu, \\ w_k'(t) &= f_k\big(t, w_k(t), {}_kw(t)\uparrow, {}_kv(t)\downarrow, w\uparrow, v\downarrow\big) + 1/\nu, \\ &\qquad \text{for } t \in I_0 \text{ and } k = 1(1)n, \end{aligned} \tag{16}$$

$$v(0) = \underline{a} - 1/\nu, \qquad w(0) = \bar{a} + 1/\nu. \tag{17}$$

By the existence theorem of Section 4 (with n replaced by $2n$) there exists for every $\nu \in \boldsymbol{N}$ at least one solution $v, w \in \mathfrak{Z}$ of the coupled system (16), (17). An arbitrary solution is picked up and called (v^ν, w^ν). Then

$$v^\nu < v^{\nu+1} < \hat{u} < w^{\nu+1} < w^\nu$$

for all $\nu \in \boldsymbol{N}$ and for any solution $\hat{u} \in \mathfrak{Z}$ of (1), (2). This follows by the special case of the Lemma and by the definition of the right-hand sides in (16) and in (17). Hence the sequences $\{v^\nu\}$ and $\{w^\nu\}$ are monotone and bounded. As sequences of measurable functions they have measurable limit functions $\underline{v}(t) := \sup\{v^\nu(t)\}$ and $\overline{w}(t) := \inf\{w^\nu(t)\}$. Furthermore

$$\underline{v} \leqslant \hat{u} \leqslant \overline{w}. \tag{18}$$

One can show as usual that these sequences are uniformly convergent on I and that $\underline{v}$, $\overline{w} \in \mathfrak{Z}$ (see W. Walter [15], p. 68). Furthermore the pair $(\underline{v}, \overline{w})$ satisfies the following functional-differential system in I_0 consisting of $2n$ equations

$$\begin{aligned} \underline{v}_k'(t) &= f_k\big(t, \underline{v}_k(t), {}_k\underline{v}(t)\uparrow, {}_k\overline{w}(t)\downarrow, \underline{v}\uparrow, \overline{w}\downarrow\big), \\ \overline{w}_k'(t) &= f_k\big(t, \overline{w}_k(t), {}_k\overline{w}(t)\uparrow, {}_k\underline{v}(t)\downarrow, \overline{w}\uparrow, \underline{v}\downarrow\big) \end{aligned} \tag{19}$$

under the $2n$ initial conditions

$$\underline{v}(0) = \underline{a} \quad \overline{w}(0) = \overline{a}. \tag{20}$$

If $\hat{u} \in \mathfrak{Z}$ is a solution of (1), (2), then the pair $(\hat{u}, \hat{u})$ is also a solution of (19), (2). In general the reverse is however, not true, i.e. the functions $\underline{v}$, $\overline{w}$ are in general no solutions of (1), (6). Hence they are not minimal or maximal solutions to (1), (6) in the usual sense (see however, Section 12). In what follows the interval $[\underline{v}, \overline{w}]$ will be called *maximal interval solution* of (1), (6).

The reason for this notation comes from the following: Define the interval operator $F[v, w] = (F_1, F_2, \ldots, F_n)$ by its kth component as follows:

$$F_k[v, w](t) := \Big[\underline{a}_k + \int_0^t f_k\big(s, v_k(s), {}_kv(s)\uparrow, {}_kw(s)\downarrow, v\uparrow, w\downarrow\big)\,ds,$$
$$\overline{a}_k + \int_0^t f_k\big(s, w_k(s), {}_kw(s)\uparrow, {}_kv(s)\downarrow, w\uparrow, v\downarrow\big)\,ds\Big].$$

Any solution $\hat{u}$ of (1), (6) satisfies

$$\hat{u} \in F[\hat{u}, \hat{u}].$$

Moreover, the interval $[\underline{v}, \overline{w}]$ is a fixed interval of the operator F by (19) and (20). By construction $[\underline{v}, \overline{w}]$ is the smallest fixed interval of F for which (18) is true. If $\underline{a} = \overline{a} = a$ and if maximal and minimal solutions of (1), (2) exist then they are equal to $\underline{v}$ and $\overline{w}$.

This idea consists therefore in replacing the usual ordering relation $\leqslant$ (componentwise with respect to k and pointwise with respect to t) by the inclusion $\subseteq$ as a new ordering relation (also componentwise and pointwise).

10. Bounds for the solutions of sets of functional-differential equations. Equations (1), (2) are now being replaced by inclusions (5), (6).

THEOREM. *Let* (15) *be satisfied by the set of initial values* $\{a\}$. *Assume that there exist two right-hand sides* $\underline{f}, \overline{f} \in \{f\}$ *with* $\underline{f} \leqslant \overline{f}$ *such that for all solutions* $\hat{u} \in \mathfrak{Z}$ *of* (5), (6)

$$\underline{f}\big(t, \hat{u}(t), \hat{u}\big), \overline{f}\big(t, \hat{u}(t), \hat{u}\big) \in \big\{f\big(t, \hat{u}(t), \hat{u}\big)\big\}$$
$$\subseteq \big[\underline{f}\big(t, \hat{u}(t), \hat{u}\big), \overline{f}\big(t, \hat{u}(t), \hat{u}\big)\big] \quad \text{in } I_0.$$

Assume furthermore that the two functions $\underline{f}, \overline{f} \in \mathfrak{M}$ *are continuous and bounded on* $I \times \mathbf{R}^n \times \mathfrak{Z}$. *Construct the maximal interval solutions to* $\underline{a}, \underline{f}$ *and* $\overline{a}, \overline{f}$ *to the problem* (1), (2) *and call them* $[\underline{v}, \underline{w}]$ *and* $[\overline{v}, \overline{w}]$. *Then* $\underline{v} \leqslant \overline{w}$ *and for the set* $\{\hat{u}\}$ *of all solutions* $\hat{u} \in \mathfrak{Z}$ *of* (5), (6) *the inclusion*

$$\{\hat{u}\} \subseteq [\underline{v}, \overline{w}] \tag{21}$$

is true.

Remarks. 1. There is *nothing* assumed for one of the right-hand sides $f \in \{f\}$ if $f \neq \underline{f}, \bar{f}$, only the existence of that function. If f for example is not continuous, *no* solution $\hat{u}$ of (1), (2) may exist. Kindly note that the theorem deals only with existing solutions.

2. In order to find the two bounds $\underline{v}$, $\overline{w}$ for all (in general ∞ many) solutions of (5), (6) one has to determine the maximal interval solutions of two coupled systems with $2n$ equations each. The functions $\bar{v}$ and $\underline{w}$ are a "side effect" of this procedure, they are not needed for the inclusion (21). They do have however, a meaning as "inner" bounds to $\{\hat{u}\}$ in the sense of interval mathematics.

Proof. Let $\hat{u}$ be a solution of (5), (6). Then

$$\hat{u}'(t) = f\left(t, \hat{u}(t), \hat{u}\right) \leqslant \bar{f}\left(t, \hat{u}(t), \hat{u}\right) \quad \text{in } I_0.$$

Then $\hat{u} \leqslant \overline{w}$ in I by Corollary 1 of the lemma and by the construction of $(\bar{v}, \overline{w})$. The inequality $\hat{u} \geqslant \underline{v}$ is shown similarly which finishes the proof.

11. Two examples. 1. Let $n = 1$, $a = 0$,

$$f\left(t, u(t), u\right) := 2\left(\sqrt{|u(t)|} + u(\tau \cdot t)\right) \tag{22}$$

and

$$\{f\} := \{f \mid 0 \leqslant \tau \leqslant 1\}.$$

The functional used is $u(\tau \cdot t)$ by (9). I do not know if the problem (1), (2) with f by (22) can explicitly be solved for $\tau \neq 0, 1$.

Since f is isotone in u, the two inequalities (12) and (13) of the Lemma are decoupled. By putting $v(t) := -\varepsilon$ with $0 < \varepsilon < 1$ one gets $v(0) = -\varepsilon < 0 = a$ and

$$0 = v'(t) < 2\left(\sqrt{|v(t)|} + v(\tau \cdot t)\right) = 2\sqrt{\varepsilon}(1 - \sqrt{\varepsilon}).$$

Hence by the Lemma $\hat{u}(t) > -\varepsilon$ for any solution $\hat{u} \in \mathfrak{Z}$ of (1), (2). For $\varepsilon \to 0$ one gets $\hat{u}(t) \geqslant 0$ and therefore by (22) there is also $\hat{u}'(t) \geqslant 0$ for any solution u. Hence $0 \leqslant \hat{u}(\tau \cdot t) \leqslant \hat{u}(t)$ for any solution, therefore one can define

$$\underline{f}\left(t, u(t), u\right) := 0, \quad \bar{f}\left(t, u(t), u\right) := 2\left(\sqrt{|u(t)|} + u(t)\right).$$

The maximal interval solution is found easily as $\underline{v} := 0$, $\overline{w} := (e^t - 1)^2$; hence

$$\hat{u}(t) \in [0, (e^t - a)^2] \quad \text{for } t \geqslant 0 \tag{23}$$

for all solutions $\hat{u}$ of (1), (2) with (22).

If one now changes the functional (9) in (22) to (10) one gets the functional

$$f\left(t, u(t), u\right) := \begin{cases} 2\left(\sqrt{|u(t)|} + u(t-s)\right) & \text{for } 0 \leqslant s \leqslant t, \\ 2\sqrt{|u(t)|} & \text{for } 0 \leqslant t < s, \end{cases} \tag{24}$$

where $0 \leqslant s < \infty$. With this right-hand side (24) one gets a whole set of difference-differential equations with retarded argument. The same ideas as above give *exactly the same* bound functions $\underline{v}$ and $\overline{w}$. Hence also in this case (23) is true.

The same can be said for the third different right-hand side

$$f(t, u(t), u) := 2\left(\sqrt{|u(t)|} + \left(\int_0^t |u(s)|^p ds\right)^{1/p}\right) \tag{25}$$

with $1 \leqslant p \leqslant \infty$. The functional in this case is the Volterra p-norm with the sup norm for $p = \infty$.

Since in all three cases the bounds $\underline{v}$, $\overline{w}$ are solutions itself to $u' = \underline{f}$, $u' = \bar{f}$ one gets in addition the optimality condition

$$\underline{v}, \overline{w} \in \{\hat{u}\} \subseteq [\underline{v}, \overline{w}]. \tag{26}$$

This result is highly surprising. The three problems with the different right-hand sides (23), (24), and (25) most certainly have completely different solutions and therefore also different solution sets. In spite of this fact all three sets have the same bounds and furthermore these bounds are optimal.

This example shows also that it is very often *simpler* to look for bounds $\underline{v}$, $\overline{w}$ such that (26) is true than to try to solve the equations.

2. Let $n = 2$, the given system is

$$u_1' = u_2, \quad u_1(0) = a_1, \quad u_2' = -u_1, \quad u_2(0) = a_2.$$

The uniquely determined solution is

$$\hat{u}_1(t) = a_1 \cos t + a_2 \sin t, \quad \hat{u}_2(t) = -a_1 \sin t + a_2 \cos t.$$

Let $a_1 \in [0, 1]$, $a_2 \in [0, 1]$. Then by rules of interval arithmetic (see R.E. Moore [4])

$$\hat{u}_1(t) \in [0, 1]\cos t + [0, 1]\sin t,$$
$$\hat{u}_2(t) \in [-1, 0]\sin t + [0, 1]\cos t.$$

The set of solutions is hatched in Figure 1. In the picture also the "main" solution for $a_1 = a_2 = 1/2$ is shown. The extended system (19) reads here as

$$\begin{aligned}
\underline{v}_1' &= \underline{v}_2, & \underline{v}_1(0) &= 0,\\
\underline{v}_2' &= -\overline{w}_1, & \underline{v}_2(0) &= 0,\\
\overline{w}_1' &= \overline{w}_2, & \overline{w}_1(0) &= 1,\\
\overline{w}_2' &= -\underline{v}_1, & \overline{w}_2(0) &= 1.
\end{aligned}$$

The solution to this system is unique, hence the maximal interval solution

$$\begin{aligned} 2\underline{v}_1(t) &= -e^t+\sin t+\cos t, \\ 2\underline{v}_2(t) &= -e^t+\cos t-\sin t, \\ 2\overline{w}_1(t) &= e^t+\sin t+\cos t, \\ 2\overline{w}_2(t) &= e^t+\cos t+\sin t. \end{aligned}$$

Hence the functions $\underline{v}$, $\overline{w}$ "back away" from the "main" solution (see Figure 1) as fast as $e^t/2$ to below and to above.

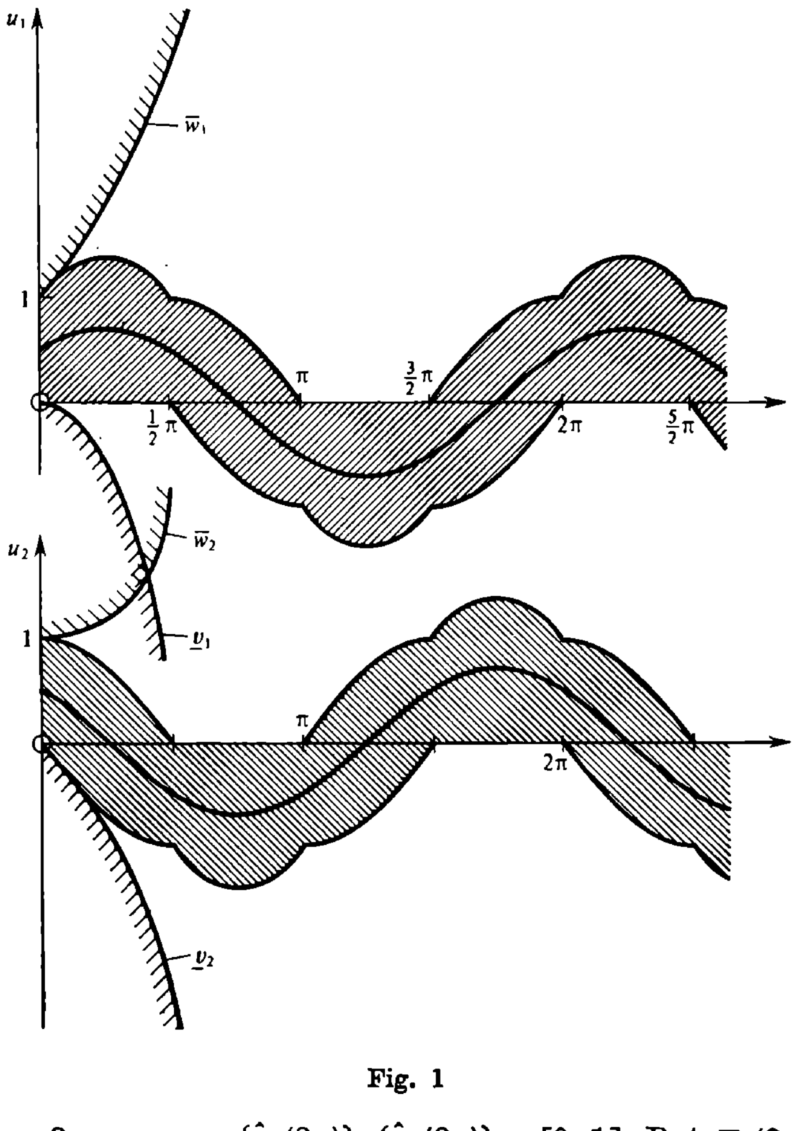

Fig. 1

For $t = 2\pi$ one sees $\{\hat{u}_1(2\pi)\}$, $\{\hat{u}_2(2\pi)\} \subseteq [0, 1]$. But $\overline{w}_k(2\pi)-\underline{v}_k(2\pi) = e^{2\pi} = 535.4 \ldots$ for $k = 1, 2$. The real set of solutions $\{\hat{u}\}$ is therefore surpassed at $t = 2\pi$ by $[\underline{v}, \overline{w}]$ by a factor of more than 500 and this grows rapidly worse for larger values of t!!!

This example was first discussed by R. E. Moore [4] by using geometrical reasoning.

12. Optimal bounds for the set of solutions: maximal and minimal solutions. By Section 10 one sees that it is enough to restrict the survey to the solution of two real functional differential equations if one wishes to bound the solutions of sets of such equations. Hence the following optimality considerations are given only for systems of the type (1). Since it is no aggravation the inital inclusion (6) has, however, been used instead of the initial condition (2). It is always assumed that (15) is true.

As is shown by the second example of Section 10 the maximal interval solution $[\underline{v}, \overline{w}]$ gives in general not optimal bounds to the set of solutions $\{\hat{u}\}$. In this and in the next section classes of problems will be given such that there is optimality either in the sense of (26) or at least of (8).

Let $f \in \mathfrak{M}$. Assume furthermore that all functions $f_k(t, x_k, {}_kx, y)$ are unconditionally partially isotone with respect to any of the components of ${}_kx$ and y. This is called "quasimonotone increasing" by W. Walter [15] in the case of differential equations. With this condition the equations for v and w in (16) and (19) are decoupled from each other. Therefore the functions $\underline{v}$ and $\overline{w}$ are even solutions of (1). In the case of differential equations these are the well known minimal and maximal solutions (see W. Walter [15], p. 95). Hence (26) is true which implies (8), therefore the bounds $\underline{v}$ and $\overline{w}$ are optimal.

A simple example for this case was given in Section 11.1. Another example for $n = 2$ is the system of differential equations

$$\begin{aligned} u_1' &= 2\sqrt{|u_1|}, & u_1(0) &= 0,\\ u_2' &= +2\sqrt{|u_1|}, & u_2(0) &= 0. \end{aligned} \tag{27}$$

One finds easily $\underline{v}_1(t) = \underline{v}_2(t) = 0$, $\overline{w}_1(t) = \overline{w}_2(t) = t^2$.

13. Further cases with optimal bounds. If one changes in system (27) the sign in the second equation one gets

$$\begin{aligned} u_1' &= 2\sqrt{|u_1|}, & u_1(0) &= 0,\\ u_2' &= -2\sqrt{|u_1|}, & u_2(0) &= 0. \end{aligned}$$

Now the right-hand sides are not anymore "quasimonotone increasing". The set of all solutions can be described quite easily, one finds $\{\hat{u}_1(t)\} \subseteq [0, t^2]$, $\{\hat{u}_2(t)\} \subseteq [-t^2, 0]$. The maximal interval solution of (19), (20) produces the bounds

$$\underline{v}_1(t) := 0, \quad \underline{v}_2(t) := -t^2, \quad \overline{w}_1(t) := t^2, \quad \overline{w}_2 := 0.$$

They are again optimal bounds. Opposite to the results of Section 12 the functions $\underline{v}$ ans $\overline{w}$ are *not* anymore solutions of (1) (but certainly of (19)).

By inspection one sees however that the "crossed" couples $(\underline{v}_1, \overline{w}_2)$ and $(\overline{w}_1, \underline{v}_2)$ each are a solution to (1). This is responsible for the optimality. Because if $\underline{v}$ and $\overline{w}$ are at least componentwise solutions of (1), then there exist no smaller intervals $[v, w] \subseteq [\underline{v}, \overline{w}]$ such that (4) is valid. Hence in this case (8) is true which means optimality.

The classes of problems (1), (6) considered in this section are extensions of this example. In the case of pure differential equations they have already been discussed by Burton–Whyburn [2] — with somewhat different notations.

Definition (monotonicity matrices). Let $f \in \mathfrak{M}$.

Define for $i = 1(1)n$

$$a_{ii} := 1,$$

$$a_{ik} := \begin{cases} 0 & \text{if } f_i \text{ does not depend upon } x_k, \\ +1 & \text{if } f_i \text{ depends isotone upon } x_k, \\ -1 & \text{if } f_i \text{ depends antitone upon } x_k \end{cases}$$

$$\text{for } i \neq k = 1(1)n,$$

$$b_{ik} := \begin{cases} 0 & \text{if } f_i \text{ does not depend upon } y_k, \\ +1 & \text{if } f_i \text{ depends isotone upon } y_k, \\ -1 & \text{if } f_i \text{ depends antitone upon } y_k \end{cases}$$

$$\text{for } k = 1(1)n.$$

The matrices $A = (a_{ik})$ and $B = (b_{ik})$ are called the *monotonicity matrices to* f.

Definition (monotonicity condition (M)). Let $f \in \mathfrak{M}$.

Assume the existence of associate[1] matrices $A' = (a'_{ik})$, $B' = (b'_{ik})$ to the monotonicity matrices A and B such that for all $i, k = 1(1)n$

$$\begin{aligned} & a'_{ik} \in \{+1, -1\}, \qquad b'_{ik} \in \{+1, -1\}, \\ & a'_{ik} = a_{ik} \quad \text{for } a_{ik} \neq 0, \\ & b'_{ik} = b_{ik} \quad \text{for } b_{ik} \neq 0, \\ & a'_{ik} = b'_{ik}, \\ & a'_{ik} = a'_{1i} \cdot a'_{1k}. \end{aligned} \tag{28}$$

Then f is said to satisfy condition (M).

Remark. By (28) one sees immediately

$$a'_{ik} = a'_{li} \cdot a'_{lk} \quad \text{for all } l = 1(1)n.$$

[1] These need not be uniquely determined.

THEOREM. *Assume* (15). *Let* $f \in \mathfrak{M}$ *be continuous and bounded on* $I \times \boldsymbol{R}^n \times \mathfrak{Z}$. *Assume that f satisfies condition* (M). *Let* $[\underline{v}, \overline{w}]$ *be the maximal interval solution to* (1), (6).

Then $[\underline{v}, \overline{w}]$ *is even the interval hull of the set of all solutions* $\{\hat{u}\}$ *of* (1), (6), *i.e.* (8) *is true.*

Proof. By construction of $[\underline{v}, \overline{w}]$ inclusion (18) is true. Define the two function vectors $p = (p_1, p_2, \ldots, p_n) = p(t)$, $q = (q_1, q_2, \ldots, q_n) = q(t)$ by

$$p_k := \begin{cases} \underline{v}_k & \text{for } a'_{1k} = +1, \\ \overline{w}_k & \text{for } a'_{1k} = -1, \end{cases}$$

$$q_k := \begin{cases} \overline{w}_k & \text{for } a'_{1k} = +1, \\ \underline{v}_k & \text{for } a'_{1k} = -1. \end{cases}$$

Certainly $p_k(0) \in \{a\}$ and $q_k(0) \in \{a\}$. Moreover, the vectors p and q are by construction and by (28) both solution of (1). Hence $\underline{v}$ and $\overline{w}$ are componentwise composed of solutions of the problem (1), (20). Hence (8) is true.

EXAMPLES. 1. The monotonicity matrices A and B to the system (28) are

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence a possible choice for A', B' is

$$A' = B' := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and with this the function f of (28) satisfies condition (M).

2. In the cases $n = 2$ and $n = 3$ the following matrices $A' = B'$ are all such matrices which guarantee condition (M):

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Remark. In the literature for systems of differential equations nearly always the case of "quasimonotone increasing" right-hand sides is considered if optimal bounds are looked for. In this case minimal and maximal solutions do exist to the problem (1), (6). By introducing the maximal interval solution and the class $\mathfrak{M}$ one finds a much larger number — namely 2^{n-1} — of favorable cases where the set of solutions of (1), (6) can be optimally bounded in a constructive manner. Moreover, not only differential equations but also systems of functional-differential equations can now be treated.

14. Constructing the interval hull. Consider the system of Moore [4] in Section 11.2. In this case the monotonicity matrix reads

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

i.e. the function f does certainly *not* satisfy the monotonicity condition (M). By Figure 1 and by the results of Section 11.2 the maximal interval solution from (19) and (20) does not give the interval hull of the set of all solutions. This does, however, not mean that this interval hull cannot be computed quite easily. It says only that the special method of Section 9 does not lead to optimality in this case.

In what follows a new and nearly trivial method will be given which produces *always* the interval hull to the set of all solutions of (1), (6), provided that the systems (1) are *linear*. There is no other restriction. The result is therefore true also if f is *not* in the monotonicity class $\mathfrak{M}$. This result in the case of pure differential equations has been found independently of the author by R. Lohner/Karlsruhe, Germany, see Lohner–Adams [3].

THEOREM. *Let the right-hand side f in* (1) *be continuous and linear. Let the set $\{a\}$ of the initial values in* (6) *be spanned by m points $a^1, a^2, \ldots$ $\ldots, a^m \in \{a\}$. Let the uniquely determined solutions of* (1) *under the m initial conditions*

$$(2') \qquad u(0) = a^k \quad \text{for } k = 1(1)m$$

be denoted by $\hat{u}^k$. Then the set $\{\hat{u}\}$ of solutions of (1), (6) *is spanned by $\{\hat{u}^1, \hat{u}^2, \ldots$ $\ldots, \hat{u}^m\}$.*

Proof. Each problem (1), (2′) has exactly one solution. Let $a = \sum_{k=1}^{m} c_k a^k$ with $\sum_{k=1}^{m} c_k = 1$ and $c_k \geqslant 0$. Because of the linearity of f the function $\hat{u}:$ $:= \sum_{k=1}^{m} c_k \hat{u}^k$ is then the uniquely determined solution of (1), (2). This means that the space of all initial conditions is transformed affine to the new space for each $t > 0$ by the solutions of (1), (2).

COROLLARY. *Let f be continuous and linear and assume* (15). *Let the $m := 2^n$ corners of the box $[\underline{a}, \bar{a}]$ be denoted by a^k for $k = 1(1)2^n$. Call $\hat{u}^k$ the solutions of* (1), (2′). *Define*

$$v(t) := \operatorname*{Min}_{k=1(1)m} \hat{u}^k(t), \qquad w(t) := \operatorname*{Min}_{k=1(1)m} \hat{u}^k(t).$$

Clearly $v, w \in C(I)$.

Then (4) *and* (8) *are true, i.e. the interval $[v, w]$ is the interval hull of the set $\{\hat{u}\}$ of all solutions of* (1), (6).

Remarks. 1. The computing effort of this new method is considerably larger compared with that of the previous method. In Section 9 only *one* $2n$-system (19), (20) has to be solved. Here 2^n solutions of a n-system have to be computed.

2. By looking at Figure 1 one sees that during the construction of v and w the minimal or maximal value is attained for different functions $\hat{u}^k$ if t varies. Hence the limiting functions v and w are indeed continuous but in general not differentiable.

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