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Non-negative continuous solutions of a functional inequality

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Abstract. The asymptotic behaviour for $x \to 0+0$ of the non-negative continuous solutions φ of the functional inequality (1) is studied under conditions (i)–(v) below, where G_n and A are given by (3) and (4), respectively. Some new results are derived concerning the existence of a (unique) continuous solution of the associated functional equation (2) and its asymptotic behaviour at the origin.

1. In the present paper we shall be concerned with the non-negative continuous solutions φ of the functional inequality in a single variable

$$\varphi(x) \leqslant g(x)\varphi[f(x)] + h(x) \tag{1}$$

in an interval $I = [0, a)$ or $[0, a]$, $0 < a \leqslant +\infty$. We put also $I^* = I \setminus \{0\}$.

Continuous solutions of functional inequalities in a single variable have recently been studied by D. Brydak [1], but from a different point of view. Also, Brydak did not assume the non-negativity of the solutions.

Problems concerning non-negative solutions may arise when we obtain an estimation of the absolute value of a function. Then φ in (1) can be interpreted as the absolute value of another function, and we may want to gain from (1) some informations about the behaviour of φ near zero (which will be assumed as the fixed point of the function f). Thus our main concern in this paper will be the asymptotic behaviour of the non-negative continuous solutions φ of (1) for $x \to 0+0$.

In this context it will be natural to assume that also the functions g, h appearing in (1) are continuous and non-negative. All the asymptotic symbols occurring in the present paper refer to $x \to 0+0$.

We shall deal also with the associated functional equation

$$\varphi(x) = g(x)\varphi[f(x)] + h(x). \tag{2}$$

The theory of continuous solutions of equation (2) has been developed in [3] (cf. also [4], Chapter II). The number and properties of continuous solutions of (2) depend heavily on the behaviour of the sequence

(3) $$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \quad n = 1, 2, \ldots; \; G_0(x) \equiv 1.$$

Here f^i denotes the i-th iterate of f. In the sequel also the function

(4) $$A(x) = \frac{h(x)}{1-g(x)}$$

will play an important role.

We shall make the following assumptions about the given functions f, g, h:

(i) f is continuous in I and we have $0 < f(x) < x$ in I^*.

(ii) g is continuous in I and we have $0 < g(x) < 1$ in I^*.

(iii) h is continuous and non-negative in I.

(iv) $\lim_{n\to\infty} G_n(x) = 0$ for $x \in I^*$.

(v) $A(x) = o(1)$.

Other possible assumptions will be specified at every instance. In particular, we shall often assume that function (4) is monotonic in I^*.

Let us note the following simple facts (cf., e.g., [4], Theorems 0.4 and 2.6).

Lemma 1. *Under condition* (i), *for every* $x \in I^*$ *the sequence* $f^n(x)$ *decreases to zero.*

Lemma 2. *Under conditions* (i)–(iv), *equation* (2) *may have at most one continuous solution in* I.

The solution spoken of in Lemma 2 exists only under further assumptions about the given functions f, g, h. On the other hand, for inequality (1) we have neither the problem of existence, nor of uniqueness of non-negative continuous solutions. In fact, let φ be an arbitrary continuous function in I fulfilling the condition

$$0 \leqslant \varphi(x) \leqslant h(x) \quad \text{for } x \in I.$$

Then, provided g is non-negative,

$$\varphi(x) \leqslant h(x) \leqslant h(x) + g(x)\varphi[f(x)],$$

i.e., φ satisfies (1). Thus we have always the existence and never (except when $h \equiv 0$) the uniqueness.

2. The simplest asymptotic property of non-negative continuous solutions of (1) is described by the following

Theorem 1. *Let conditions* (i)–(v) *be fulfilled. If a continuous function* $\varphi \geqslant 0$ *satisfies inequality* (1) *in* I, *then*

(5) $$\varphi(x) = o(1).$$

Proof. Write $c = \varphi(0) = \lim_{x\to 0+0} \varphi(x)$ and suppose that $c > 0$. Then we have

(6) $$0 < c_1 < \varphi(x) < c_2 \quad \text{for } x \in (0, \delta),$$

with suitable c_1, c_2 and $\delta > 0$. Take an $x \in (0, \delta)$. In virtue of Lemma 1 $f^n(x) \in (0, \delta)$ for $n = 0, 1, 2, \ldots$ Inequality (1) together with (6) yield

$$\frac{\varphi(x)}{\varphi[f(x)]} \leqslant g(x) + \frac{h(x)}{\varphi[f(x)]} \leqslant g(x) + c_1^{-1} h(x).$$

Hence

$$\frac{\varphi(x)}{\varphi[f^n(x)]} = \prod_{i=0}^{n-1} \frac{\varphi[f^i(x)]}{\varphi[f^{i+1}(x)]} \leqslant \prod_{i=0}^{n-1} \big(g[f^i(x)] + c_1^{-1} h[f^i(x)]\big), \qquad n = 1, 2, \ldots$$

We may assume that δ is so small that (cf. (v))

$$h(x) \leqslant r c_1 \big(1 - g(x)\big) \qquad \text{for } x \in (0, \delta),$$

where $0 < r < 1$. Thus

$$\begin{aligned} g[f^i(x)] + c_1^{-1} h[f^i(x)] &\leqslant g[f^i(x)] + r\big(1 - g[f^i(x)]\big) \\ &= 1 - (1-r)\big(1 - g[f^i(x)]\big), \end{aligned}$$

and

$$\frac{\varphi(x)}{\varphi[f^n(x)]} \leqslant \prod_{i=0}^{n-1} \Big(1 - (1-r)\big(1 - g[f^i(x)]\big)\Big),$$

or else, letting $n \to \infty$,

(7) $$\varphi(x) \leqslant c \prod_{i=0}^{\infty} \Big(1 - (1-r)\big(1 - g[f^i(x)]\big)\Big).$$

Condition (iv) implies the divergence of the series $\sum\big(1 - g[f^i(x)]\big)$, and hence also of the series $(1-r)\sum\big(1 - g[f^i(x)]\big)$ in I^*. Consequently the right-hand side of (7) is zero, which contradicts relation (6). Consequently (5) must hold true.

Theorem 1 will not longer be true if we drop assumption (v). In particular, if $A(x) \geqslant c \geqslant 0$ in I^*, then $\varphi(x) \equiv c$ is a non-negative continuous solution of (1) in I.

For a given constant c write $h^*(x; c) = h(x) - c\big(1 - g(x)\big)$.

THEOREM 2. *Let conditions* (i)–(iv) *be fulfilled. If there exists a constant* c_0 *such that* $h^*(x; c_0) \geqslant 0$ *and* $h^*(x; c_0) = o\big(1 - g(x)\big)$, *then for every continuous solution* φ *of inequality* (1) *in* I *we have*

(8) $$\varphi(0) \leqslant c_0.$$

Proof. Suppose that a continuous function φ satisfies inequality (1) in I and we have $\varphi(0) > c_0$. Then there exists a $\delta > 0$, $\delta \in I$, such that $\varphi(x) > c_0$ in $[0, \delta)$, and thus the function $\psi(x) = \varphi(x) - c_0$ is a continuous and positive solution of the inequality

$$\psi(x) \leqslant g(x)\psi[f(x)] + h^*(x; c_0) \tag{9}$$

in $[0, \delta)$. We may apply Theorem 1 to inequality (9) in $[0, \delta)$, and we get $\psi(0) = 0$, a contradiction. Thus (8) must hold.

We may obtain more informations about the behaviour of φ at zero under further conditions.

THEOREM 3. *Let conditions* (i)–(iv) *be fulfilled and suppose that equation* (2) *has a continuous solution* φ_0 *in* I. *If* φ *is a continuous solution of inequality* (1) *in* I, *then*

$$\varphi(x) \leqslant \varphi_0(x) \quad \textit{for } x \in I. \tag{10}$$

Proof. By induction we get from (1) and (2)

$$\varphi(x) \leqslant G_n(x)\varphi[f^n(x)] + \sum_{i=0}^{n-1} G_i(x)\varphi[f^i(x)]$$

and

$$\varphi_0(x) = G_n(x)\varphi_0[f^n(x)] + \sum_{i=0}^{n-1} G_i(x)\varphi_0[f^i(x)],$$

respectively. Letting $n \to \infty$ we obtain hence in view of (iv) $\varphi(x) \leqslant \varphi_0(x)$ for $x \in I^*$. Relation (10) now follows by the continuity of φ and φ_0 at $x = 0$.

Results similar to Theorems 2 and 3 above are also found in [1].

COROLLARY. *Under conditions of Theorem* 3, *if* φ *is a continuous non-negative solution of inequality* (1) *in* I, *then*

$$\varphi(x) = O\big(\varphi_0(x)\big). \tag{11}$$

In particular, if $\varphi_0(x) = o(1)$ (which is certainly the case if (v) is fulfilled, since φ_0 is a particular solution of (1)), then relation (11) gives us more information about the asymptotic behaviour of φ at zero, than relation (5).

In the next section we shall deduce the existence of φ_0 from other conditions.

3. The following estimation turns out useful.

THEOREM 4. *Let conditions* (i)–(v) *be fulfilled and suppose that the function* A *is monotonic in* I^*. *If a continuous function* $\varphi \geqslant 0$ *satisfies inequality* (1) *in* I, *then*

$$\varphi(x) \leqslant A(x) \quad \textit{for } x \in I^*. \tag{12}$$

Proof. Suppose that we have

$$\varphi(x_0) > A(x_0) \tag{13}$$

for an $x_0 \in I^*$, and put $x_n = f^n(x_0)$. We shall show that

$$\varphi(x_n) > A(x_n), \quad n = 0, 1, 2, \ldots, \tag{14}$$

and

$$\varphi(x_{n+1}) > \varphi(x_n), \quad n = 0, 1, 2, \ldots \tag{15}$$

First observe that the monotonicity assumption about A, condition (v), and Lemma 1 imply that

$$A(x_{n+1}) \leqslant A(x_n), \quad n = 0, 1, 2, \ldots \tag{16}$$

For $n = 0$ relation (14) reduces to (13). Now assume that (14) holds for an $n \geqslant 0$. Then we have by (1), (14) and (4)

$$\begin{aligned}\varphi(x_{n+1}) - \varphi(x_n) &= \varphi[f(x_n)] - \varphi(x_n) \geqslant \frac{\varphi(x_n) - h(x_n)}{g(x_n)} - \varphi(x_n) \\ &= \frac{\varphi(x_n)\big(1 - g(x_n)\big) - h(x_n)}{g(x_n)} > 0,\end{aligned}$$

i.e., (15) holds. Relations (14), (15) and (16) yield now

$$\varphi(x_{n+1}) > \varphi(x_n) > A(x_n) \geqslant A(x_{n+1}),$$

i.e., (14) for $n+1$. Induction completes the proof of (14) and (15). But relation (15), in view of Lemma 1, contradicts Theorem 1. Thus we must have (12).

THEOREM 5. *Let conditions* (i)–(v) *be fulfilled and suppose that the function A is monotonic in I^*. Then equation* (2) *has in I a unique continuous solution φ_0. This solution is given by the formula*

$$\varphi_0(x) = \sum_{n=0}^{\infty} G_n(x) h[f^n(x)], \tag{17}$$

and fulfils the inequality

$$h(x) \leqslant \varphi_0(x) \leqslant A(x) \quad \textit{for } x \in I^*. \tag{18}$$

Proof. Write

$$\varphi_k(x) = \sum_{n=0}^{k-1} G_n(x) h[f^n(x)].$$

We have by (3)

$$G_{n+1}(x) = g(x) G_n[f(x)], \quad n = 0, 1, 2, \ldots,$$

whence

$$h(x)+g(x)\varphi_k[f(x)] = h(x)+\sum_{n=0}^{k-1} g(x)G_n[f(x)]\, h[f^{n+1}(x)]$$

$$= h(x)+\sum_{n=1}^{k} G_n(x)h[f^n(x)],$$

i.e.,

(19) $$h(x)+g(x)\varphi_k[f(x)] = \varphi_{k+1}(x).$$

Since $\varphi_{k+1}(x) \geqslant \varphi_k(x)$, it follows from (19) that φ_k is a continuous and non-negative solution of inequality (1) in I, whence by Theorem 4

$$\varphi_k(x) \leqslant A(x) \quad \text{in } I^*.$$

This implies that series (17) converges in I^* and its sum φ_0 fulfils inequalities (18). The convergence of series (17) at $x = 0$ is trivial, since condition (v) implies that $h(0) = 0$.

It follows by (19) that φ_0 satisfies equation (2). Let $\omega(x)$ denote the oscillation of φ_0 at x. We get by (2)

$$\omega[f(x)] = \frac{\omega(x)}{g(x)} \geqslant \omega(x) \quad \text{for } x \in I^*,$$

since $g < 1$. If we had $\omega(x_0) > 0$ for an $x_0 \in I^*$, then we would get

(20) $$\omega(x_n) \geqslant \omega(x_0), \quad n = 0, 1, 2, \ldots,$$

where $x_n = f^n(x_0)$. On the other hand, we have by (18) $\lim\limits_{x\to 0+0} \omega(x) = 0$, which is incompatible with (20). Consequently φ_0 is continuous in I^*. The continuity of φ_0 at $x = 0$ follows from (18), and the uniqueness of φ_0 from Lemma 2.

4. Condition (12) may be written (for non-negative φ) as

(21) $$\varphi(x) = O\big(A(x)\big).$$

It follows from (18) that estimation (21) is less sharp than (11). However, since formula (4) is much simpler than (17), condition (21) may be much more convenient to use. Moreover, it may still yield quite good results. This may be seen, e.g., from the following example. Consider the equation [1]

(22) $$\varphi(x) = (1-x)\varphi(x-x^3)+x^3+2x^4-2x^5-x^6+x^7$$

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[1] Equations (22), (23) and (28) are considered in an interval $I = [0, a] \subset [0, 1)$, where $a > 0$ is so small that the function A is monotonic in I.

with the continuous solution $\varphi_0(x) = x^2$. Here $A(x) = x^2+2x^3-2x^4-x^5+ +x^6$, so it differs from $\varphi_0(x)$ only in the terms of orders higher than two.

Similarly, in the case of the equation

$$\varphi(x) = \left(1+\frac{1}{\log x}\right)\varphi\left(\frac{x}{x+1}\right)+\frac{x^2\log x-x}{(x+1)\log x} \tag{23}$$

(where at $x = 0$ the functions are assigned their limit values) we have the solution $\varphi_0(x) = x$, whereas

$$A(x) = \frac{x-x^2\log x}{x+1} = x-x^2\log x-x^2+x^3\log x+x^3-\ldots$$

Again $A(x)$ differs from $\varphi_0(x)$ only in the terms of higher orders.

Moreover, it may be generally proved that in many cases the estimation (21) is almost as good as (11). Namely, we have the following

THEOREM 6. *Let conditions* (i)–(v) *be fulfilled and assume that the function A is increasing and the function g is decreasing in I^*. Assume, moreover, that $h > 0$ I^* and*

$$1-p(x) = o\big(1-g(x)\big), \tag{24}$$

where

$$p(x) = \inf_{(0,x]} \frac{[h[f(t)]}{h(t)}. \tag{25}$$

Then the functions A and φ_0 (given by (4) and (17), respectively) *are asymptotically equal, i.e.*

$$\lim_{x\to 0+0} \frac{\varphi_0(x)}{A(x)} = 1. \tag{26}$$

Proof. If $x \in I^*$, then, by Lemma 1, $f^n(x) \in (0, x]$ for $i = 0, 1, 2, \ldots,$ and we have by (25)

$$\frac{h[f^n(x)]}{h(x)} = \prod_{i=0}^{n-1} \frac{h[f^{i+1}(x)]}{h[f^i(x)]} \geqslant [p(x)]^n.$$

Also, by (3),

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] \geqslant [g(x)]^n.$$

Hence, in view of (17) and (4), we have for $x \in I^*$

$$\begin{aligned} \frac{\varphi_0(x)}{A(x)} &= [1-g(x)]\sum_{n=0}^{\infty} G_n(x)\frac{h[f^n(x)]}{h(x)} \\ &\geqslant [1-g(x)]\sum_{n=0}^{\infty}[g(x)p(x)]^n = \frac{1-g(x)}{1-p(x)g(x)}. \end{aligned} \tag{27}$$

Condition (24) implies that $\lim_{x\to 0+0}[1-g(x)]/[1-p(x)g(x)] = 1$. Relation (26) follows now from (27) and (18).

On the other hand, in the case of the equation

$$(28)\qquad \varphi(x) = \left(1+\frac{1}{\log x}\right)\varphi\left(\frac{1}{2}x\right)+\frac{x}{2}\left(1+\frac{1}{\log x}\right)$$

we have $\varphi_0(x) = x$ and $A(x) = -\frac{1}{2}x(\log x+1) = O(x\log x)$. Thus in this case estimation (11) is essentially better than (21). This is again a particular case of a more general situation.

THEOREM 7. *Let conditions* (i)–(v) *be fulfilled and assume that the function A is monotonic in I^*, and $g(0) = 1$. Assume, moreover, that $h > 0$ in I^* and*

$$(29)\qquad \limsup_{x\to 0+0}\frac{h[f(x)]}{h(x)} < 1.$$

Then the functions A and φ_0 (given by (4) and (17), respectively) *fulfil the condition*

$$(30)\qquad \varphi_0(x) = o\big(A(x)\big).$$

Proof. In virtue of (29) we can find a $\delta \in I^*$ and a $p < 1$ such that

$$\frac{h[f(x)]}{h(x)} < p \quad \text{in } (0, \delta).$$

Hence we get in view of (17) and (4), for $x \in (0, \delta)$,

$$\frac{\varphi_0(x)}{A(x)} = [1-g(x)]\sum_{n=0}^{\infty}G_n(x)\frac{h[f^n(x)]}{h(x)} < [1-g(x)]\sum_{n=0}^{\infty}p^n = \frac{1-g(x)}{1-p},$$

which proves (30).

Remark. Since the values of a function far apart from zero are irrelevant for its asymptotic behaviour at zero, and since a continuous solution of equation (2) in a neighbourhood of zero may be uniquely extended onto I (cf. [4], Theorem 3.2), the results of the present paper remain valid if the function A (and, in Theorem 6, the function g) is assumed monotonic only in a neighbourhood of the origin, except that inequalities (10), (12) and (18) need not hold in the whole I resp. I^*, but only in a neighbourhood of the origin.

5. As an illustration of the strength of the results obtained we shall prove two theorems which improve on a result of B. Choczewski ([2], Theorem 4.7).

THEOREM 8. *Suppose that the functions f, g, h are continuous in I, $0 < f(x) < x$ in I^* and $g \neq 0$ in I. Further assume that*

$$f(x) = x - x^{m+1}u(x), \quad g(x) = 1 - x^k v(x), \quad h(x) = cv(x)x^k + x^q w(x),$$

where the functions u and w are bounded, $v_0 = \liminf_{x\to 0+0} v(x) > 0$, c is a real constant, and k, m, q are positive constants such that $k \leqslant m \leqslant q$, $k < q$. Then equation (2) has in I a unique continuous solution φ_0. This solution fulfils the condition

$$\varphi_0(x) = c + O(x^{q-k}). \tag{31}$$

Proof. Write $h^*(x; c) = h(x) - c\big(1-g(x)\big) = x^q w(x)$, and consider the series

$$\varphi^*(x) = \sum_{n=0}^{\infty} G_n(x) h^*[f^n(x); c]. \tag{32}$$

Put $w_0 = \limsup_{x\to 0+0} |w(x)|$. Given ε, $0 < \varepsilon < v_0$, we may find a $\delta > 0$ such that in the interval $[0, \delta] \subset I$ we have

$$0 < g(x) \leqslant 1 - v_1 x^k, \quad |h^*(x, c)| \leqslant w_1 x^q,$$

where $v_1 = v_0 - \varepsilon$, $w_1 = w_0 + \varepsilon$. Thus the terms of series (32) are majorized in $[0, \delta]$ by those of the series

$$\tilde{\varphi}(x) = \sum_{n=0}^{\infty} \tilde{G}_n(x) \tilde{h}[f^n(x)], \tag{33}$$

where $\tilde{G}_n(x) = \prod_{i=0}^{n-1} \tilde{g}[f^i(x)]$, $\tilde{g}(x) = 1 - v_1 x^k$, $\tilde{h}(x) = w_1 x^q$. Series (33) is associated with the functional equation

$$\tilde{\varphi}(x) = \tilde{g}(x)\tilde{\varphi}[f(x)] + \tilde{h}(x).$$

The functions $f, \tilde{g}, \tilde{h}$ clearly fulfil conditions (i)–(iii). The function

$$\tilde{A}(x) = \frac{\tilde{h}(x)}{1-\tilde{g}(x)} = \frac{w_1}{v_1} x^{q-k} \tag{34}$$

fulfils condition (v) and is increasing in $[0, \delta]$. We shall verify that the sequence $\tilde{G}_n$ fulfils condition (iv).

By a theorem of Thron [5], for every $x \in (0, \delta]$ there exists a positive integer N and a positive constant d such that

$$f^n(x) \geqslant dn^{-1/m} \quad \text{for } n \geqslant N.$$

This proves that the series

$$\sum_{n=0}^{\infty}\left(1-\tilde{g}[f^n(x)]\right) = v_1\sum_{n=0}^{\infty}[f^n(x)]^k$$

diverges, and hence $\lim_{n\to\infty}\tilde{G}_n(x) = 0$. This implies that also

$$\lim_{n\to\infty} G_n(x) = 0. \tag{35}$$

Now, in virtue of Theorem 5 series (33) converges in $[0, \delta]$ to a continuous function $\tilde{\varphi}$ such that

$$\tilde{\varphi}(x) \leqslant \tilde{A}(x) \quad \text{in } (0, \delta]. \tag{36}$$

Since the terms of series (33) are positive, the convergence is uniform in $[0, \delta]$. Consequently also series (32) uniformly converges in $[0, \delta]$, whence its sum φ^* is continuous in $[0, \delta]$ and we have by (36)

$$|\varphi^*(x)| \leqslant \tilde{\varphi}(x) \leqslant \tilde{A}(x) \quad \text{in } (0, \delta]. \tag{37}$$

The function φ^* is a continuous solution of the functional equation

$$\varphi^*(x) = g(x)\varphi^*[f(x)] + h^*(x; c),$$

whence it follows that the function

$$\varphi_0(x) = c + \varphi^*(x) \tag{38}$$

is a continuous solution of equation (2) in $[0, \delta]$. This solution may be uniquely extended onto I ([4], Theorem 3.2).

It follows from (35) that the continuous solution of equation (2) in I is unique (cf. [3], or [4], Theorem 2.6). Relation (31) results from (38), (37) and (34).

Choczewski's theorem gives only the weaker estimation

$$\varphi_0(x) = c + O(x^{q-m_1}),$$

under the additional assumptions that $m < q$, the function f is strictly increasing in I and the function v is bounded in I. (His proof requires also the assumption that $\liminf_{x\to 0+0} u(x) > 0$, which is not mentioned in the formulation of the theorem.) Thus, e.g., for the continuous solution $\varphi_0(x) = x^2$ of equation (22) Choczewski's theorem yield only $\varphi_0(x) = O(x)$, whereas our Theorem 8 furnishes a fairly sharp estimation $\varphi_0(x) = O(x^2)$. However, for equations like (22) we may deduce a still better asymptotic formula making use of Theorem 6.

THEOREM 9. *Let hypotheses of Theorem 8 be fulfilled and assume, moreover, that* $k < m$ *and*

$$\lim_{x\to 0+0} v(x) = v_0 > 0, \qquad \lim_{x\to 0+0} w(x) = w_0 > 0.$$

Then the continuous solution φ_0 of equation (2) *fulfils the condition*

$$(39) \qquad \varphi_0(x) = c + \frac{w_0}{v_0} x^{q-k} + o(x^{q-k}).$$

Proof. Fix an arbitrary ε, $0 < \varepsilon < \min(v_0, w_0)$, and choose a $\delta > 0$ such that

$$0 < 1 - v_2 x^k \leqslant g(x) \leqslant 1 - v_1 x^k, \qquad w_2 x^q \leqslant h^*(x; c) \leqslant w_1 x^q \quad \text{in } [0, \delta],$$

where $v_1 = v_0 - \varepsilon$, $v_2 = v_0 + \varepsilon$, $w_1 = w_0 + \varepsilon$, $w_2 = w_0 - \varepsilon$. The argument in the proof of Theorem 8 (cf., in particular, relations (34) and (37)) gives for function (32)

$$(40) \qquad \varphi^*(x) \leqslant \frac{w_1}{v_1} x^{q-k} \quad \text{in } [0, \delta].$$

Now, write $\hat{G}_n(x) = \prod_{i=1}^{n-1} \hat{g}[f^i(x)]$, $\hat{g}(x) = 1 - v_2 x^k$, $\hat{h}(x) = w_2 x^q$, and consider the series

$$(41) \qquad \hat{\varphi}(x) = \sum_{n=0}^{\infty} \hat{G}_n(x) \hat{h}[f^n(x)]$$

associated with the functional equation

$$(42) \qquad \hat{\varphi}(x) = \hat{g}(x)\hat{\varphi}[f(x)] + \hat{h}(x).$$

It follows from Theorem 5 (like for series (33) in the proof of Theorem 8) that series (41) converges in $[0, \delta]$ to the unique continuous solution $\hat{\varphi}$ of equation (42). We have

$$(43) \qquad \hat{\varphi}(x) \leqslant \varphi^*(x) \quad \text{in } [0, \delta].$$

Moreover,

$$\frac{\hat{h}[f(x)]}{\hat{h}(x)} = \big(1 - u(x)x^m\big)^q = 1 - O(x^m),$$

whence the function

$$\hat{p}(x) = \inf_{(0,x]} \frac{\hat{h}[f(t)]}{\hat{h}(t)} = 1 - O(x^m)$$

fulfils the condition $1 - \hat{p}(x) = o\big(1 - \hat{g}(x)\big)$. Thus all assumptions of Theorem 6 are fulfilled for equation (42), whence

$$(44) \qquad \hat{\varphi}(x) = \hat{A}(x) + o\big(\hat{A}(x)\big),$$

where

$$\hat{A}(x) = \frac{\hat{h}(x)}{1-\hat{g}(x)} = \frac{w_2}{v_2} x^{q-k}. \tag{45}$$

Relations (40), (43), (44) and (45) give the estimation

$$\frac{w_0-\varepsilon}{v_0+\varepsilon} x^{q-k} + o(x^{q-k}) \leqslant \varphi^*(x) \leqslant \frac{w_0+\varepsilon}{v_0-\varepsilon} x^{q-k} \quad \text{in } [0, \delta],$$

where ε may be arbitrarily small provided δ is sufficiently small. This together with (38) implies (39).

References

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