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On a subclass of univalent functions I

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Abstract. Let $S^*(\lambda)$ denote the class of holomorphic functions f in the unit disc E, with $f(0) = 0 = f'(0)-1$, $f(z)f'(z)/z \neq 0$ for z in E and satisfying the condition

$$|\{(zf'(z)/f(z))-1\}/\{(zf'(z)/f(z))+1\}| < \lambda, \quad 0 < \lambda \leqslant 1,$$

$z \in E$. In this paper the class $M(\alpha, \lambda)$ consisting of functions f satisfying in E the condition $|(J(\alpha, f)-1)/(J(\alpha, f)+1)| < \lambda$, $0 < \lambda \leqslant 1$, where $J(\alpha, f) = \alpha\{1+zf''(z)/f'(z)\}+(1-\alpha)zf'(z)/f(z)$, $\alpha > 0$, is introduced and its properties are investigated. It is proved that $M(\alpha, \lambda) \subset S^*(\lambda)$ and the sharp radius r_0, such that $f \in S^*(\lambda)$ also satisfies the condition

$$|(J(\alpha, f)-1)/(J(\alpha, f)+1)| < \lambda, \quad 0 < \lambda \leqslant 1, \text{ for } |z| < r_0,$$

is determined. Further, a representation formula for $f \in M(\alpha, \lambda)$ and an inequality relating the coefficients of functions in $M(\alpha, \lambda)$ are obtained.

1. Introduction. Let f be analytic in the unit disc E, with $f(0) = 0$, $f'(0) = 1$, $f(z)f'(z)/z \neq 0$ in E. Denote by V the class of these functions.

Let $S^*(\lambda)$ denote the class of functions $f \in V$ satisfying in E the condition

$$\left|\left\{\frac{zf'(z)}{f(z)}-1\right\} \Big/ \left\{\frac{zf'(z)}{f(z)}+1\right\}\right| < \lambda, \quad 0 < \lambda \leqslant 1.$$

This class was introduced by the first author in [4]. For $\lambda = 1$, the class $S^*(\lambda)$ coincides with the well-known class of starlike functions.

Let $K(\lambda)$ denote the class of functions $f \in V$ satisfying in E the condition

$$\left|\frac{zf''(z)}{f'(z)} \Big/ \left\{2+\frac{zf''(z)}{f'(z)}\right\}\right| < \lambda, \quad 0 < \lambda \leqslant 1.$$

For $\lambda = 1$, the class $K(\lambda)$ coincides with the class of convex functions.

We now introduce the class $M(\alpha, \lambda)$ of functions $f \in V$ satisfying in E the condition

$$(1.1) \qquad |\{J(\alpha, f)-1\}/\{J(\alpha, f)+1\}| < \lambda, \quad 0 < \lambda \leqslant 1,$$

where $J(\alpha, f) = \alpha\{1+zf''(z)/f'(z)\}+(1-\alpha)zf'(z)/f(z)$ and α is any positive real number. For $\lambda = 1$, the class $M(\alpha, \lambda)$ coincides with the class of α-convex functions.

In this paper we investigate a few properties of the class $M(\alpha, \lambda)$.

2. It is well known that all α-convex functions are starlike [3]. We now prove an analogous theorem for the class $M(\alpha, \lambda)$.

THEOREM 1. *Let $f \in M(\alpha, \lambda)$, $\alpha > 0$. Then $f \in S^*(\lambda)$.*

Proof. Let

$$\frac{zf'(z)}{f(z)} = \frac{1-\lambda w(z)}{1+\lambda w(z)}.$$

Evidently, $w(0) = 0$ and $1+\lambda w(z) \neq 0$. We shall show that $|w(z)| < 1$ for z in E. For if not, by Jack's lemma [2], there exists z_0, $z_0 \in E$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$, $k \geqslant 1$,

$$J(\alpha, f(z_0)) = \frac{1-\lambda w(z_0)}{1+\lambda w(z_0)} - \frac{2\alpha\lambda k w(z_0)}{(1+\lambda w(z_0))(1-\lambda w(z_0))},$$

$$\left|\frac{J(\alpha, f(z_0)-1)}{J(\alpha, f(z_0)+1)}\right| = \lambda \left|\frac{1+\alpha k-\lambda w(z_0)}{1-(1+\alpha k)\lambda w(z_0)}\right|.$$

Now $|(J(\alpha, f)-1)/(J(\alpha, f)+1)| \lesseqgtr \lambda$, according as

$$|1+\alpha k-\lambda w(z_0)|^2 \lesseqgtr |1-(1+\alpha k)\lambda w(z_0)|^2 \quad \text{or} \quad (2\alpha k+\alpha^2 k^2)(1-\lambda^2) \lesseqgtr 0.$$

Since α and k are positive and $0 < \lambda \leqslant 1$, this last expression is positive. This means that $f(z) \notin M(\alpha, \lambda)$, a contradiction. Thus the proof is complete.

THEOREM 2. *For $0 \leqslant \beta < \alpha$, $M(\alpha, \lambda) \subset M(\beta, \lambda)$.*

Proof. If $\beta = 0$, then $M(\alpha, \lambda) \subset M(0, \lambda)$, by Theorem 1. Assume therefore that $\beta \neq 0$ and $f \in M(\alpha, \lambda)$. Then there exist functions w_i, $i = 1, 2$, analytic in E with $w_i(0) = 0$ and $|w_i(z) < 1|$, $i = 1, 2$, such that

$$\frac{zf'(z)}{f(z)} = \frac{1-\lambda w_1(z)}{1+\lambda w_1(z)} \quad \text{and} \quad J(\alpha, f) = \frac{1-\lambda w_2(z)}{1+\lambda w_2(z)}.$$

Now,

$$J(\beta, f) = \frac{\beta}{\alpha} J(\alpha, f) + (1-\beta/\alpha)\frac{zf'(z)}{f(z)}.$$

Since $\beta < \alpha$, one can show that

$$J(\beta, f) = \frac{1-\lambda w(z)}{1+\lambda w(z)},$$

for some w analytic in E, $w(0) = 0$ and $|w(z)| < 1$.

COROLLARY. *For $\alpha \geqslant 1$, $M(\alpha, \lambda) \subset K(\lambda)$.*

Remark. For $\lambda = 1$, the above corollary yields the established result that α-convex functions are convex for $\alpha \geqslant 1$ [3].

3. In view of Theorem 2, given a function in $S^*(\lambda)$ we can find the largest possible value of α such that $f \in M(\alpha, \lambda)$, $\alpha \geqslant 0$.

DEFINITION. Let $f \in S^*(\lambda)$ and

$$\alpha = \alpha(f) = \text{l.u.b.}\{\beta / f \in M(\beta, \lambda),\ \beta \geqslant 0\}.$$

Then we say that f is *starlike of order* λ *and type* α and we write $f \in M^*(\alpha, \lambda)$. Clearly α is non-negative and may be infinite.

If $f \in M^*(\alpha, \lambda)$, then $f \in M(\beta, \lambda)$ for all β, $0 \leqslant \beta \leqslant \alpha$. That is,

$$J(\beta, f) = \frac{1 - \lambda w(z)}{1 + \lambda w(z)}, \quad 0 \leqslant \beta \leqslant \alpha,$$

where w is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ in E. Allow $\beta \nearrow \alpha$. Then $J(\alpha, f) = (1 - \lambda w(z))/(1 + \lambda w(z))$ or $f \in M(\alpha, \lambda)$. Hence $f \in M^*(\alpha, \lambda)$ for $\alpha < \infty$ if and only if $f \in M(\beta, \lambda)$ for $0 \leqslant \beta \leqslant \alpha$ and $f \notin M(\beta, \lambda)$ for $\beta > \alpha$. Thus we can write $S^*(\lambda)$ as a disjoint union

$$S^*(\lambda) = \bigcup_{\alpha \geqslant 0} M^*(\alpha, \lambda).$$

THEOREM 3. *Let* $f \in M^*(\alpha, \lambda)$, $\alpha > 0$. *For* $0 < \beta < \alpha$, *choose the branch of* $\{zf'(z)/f(z)\}^\beta$ *which takes the value* 1 *at the origin. Then the function*

$$F_\beta(z) = f(z)\{zf'(z)/f(z)\}^\beta$$

belongs to $S^*(\lambda)$.

Proof. $f \in M^*(\alpha, \lambda)$ implies that $f \in M(\beta, \lambda)$ for all $\beta < \alpha$. The result immediately follows from the relation

$$zF'_\beta(z)/F_\beta(z) = J(\beta, f).$$

Conversely, assume that $F \in S^*(\lambda)$ and $\alpha > 0$. Define f by the differential equation

$$F(z) = f(z)\{zf'(z)/f(z)\}^\alpha. \tag{3.1}$$

Obviously,

$$f(z) = \left\{\frac{1}{\alpha}\int_0^z F^{1/\alpha}(t)\, t^{-1} dt\right\}^\alpha \tag{3.2}$$

is a solution of the differential equation (3.1) with the initial condition $f(0) = 0$. We now show that this formal solution is indeed a function in $M(\alpha, \lambda)$.

THEOREM 4. *Let* $F \in S^*(\lambda)$ *and* $\alpha > 0$. *Then* f *defined by* (3.2) *belongs to* $M(\alpha, \lambda)$.

Proof. Let γ be a path in E connecting 0 and z. We assign a value to lim arg t as $t \to 0$ on γ and the same value to lim arg $F(t)$ as $t \to 0$ on γ. Now define $t^{1/\alpha}$ and $[F(t)]^{1/\alpha}$ by continuation. Since $F(t) = t + A_2 t^2 + \ldots = t(1 + A_2 t + \ldots)$ belongs to $S^*(\lambda)$, the bracketed series has no zeros in E. Hence

$$[F(t)]^{1/\alpha} = t^{1/\alpha}(1 + A_2 t + \ldots)^{1/\alpha} = t^{1/\alpha}(1 + b_1 t + \ldots),$$

where $(1 + b_1 t + \ldots)$ is the branch of $(1 + A_2 t + \ldots)^{1/\alpha}$ which equals 1 when $t = 0$,

$$\int_0^z F^{1/\alpha}(t) t^{-1} dt = \alpha z^{1/\alpha}\left[1 + \frac{b_1}{\alpha+1} z + \ldots\right].$$

Let

$$g(z) = \frac{1}{z^{1/\alpha}} \int_0^z F^{1/\alpha}(t) t^{-1} dt = \alpha\left[1 + \frac{b_1}{\alpha+1} z + \ldots\right].$$

We now show that $g(z)$ has no zeros in E.

Let $t = H(u)$ be the inverse of $u = F(t)$ and let $p = F(z)$, $z = H(p)$. Further let Γ be a line segment joining 0 and p. Then Γ lies in the image of E under F, since $F \in S^*(\lambda)$ is also starlike. Let γ denote the preimage of Γ, in E. Then

$$g(z) = \frac{1}{z^{1/\alpha}} \int_\gamma F(t)^{1/\alpha} t^{-1} dt = \frac{1}{z^{1/\alpha}} \int_\Gamma u^{1/\alpha} \frac{H'(u)}{H(u)} du.$$

Let $u = pe^{i\beta}$ and $p = Re^{i\beta}$. Then

$$|g(z)| = \frac{1}{|z|^{1/\alpha}} \left| \int_0^R p^{\frac{1}{\alpha}-1} \frac{uH'(u)}{H(u)} dp \right| \geqslant \frac{1}{|z|^{1/\alpha}} \int_0^R p^{\frac{1}{\alpha}-1} \operatorname{Re}\left\{\frac{uH'(u)}{H(u)}\right\} dp.$$

Since $F \in S^*(\lambda)$, there exist constants M, $N > 0$ such that

$$\operatorname{Re}\left\{\frac{tF'(t)}{F(t)}\right\} \geqslant M \quad \text{and} \quad \left|\frac{tF'(t)}{F(t)}\right| \leqslant N$$

on γ. Hence on Γ,

$$\operatorname{Re}\left\{\frac{uH'(u)}{H(u)}\right\} = \operatorname{Re}\left\{\frac{1}{tF'(t)/F(t)}\right\} \geqslant \frac{M}{N^2}.$$

Therefore

$$|g(z)| \geqslant \frac{1}{|z|^{1/\alpha}} \cdot \frac{M}{N^2} \alpha R^{1/\alpha} = \alpha \frac{M}{N^2} \left|\frac{F(z)}{z}\right|^{1/\alpha} > 0.$$

Now choose the branch of $\left[\frac{1}{\alpha} g(z)\right]^\alpha$ which takes the value 1 at the origin.

Then $f(z) = z\left[\frac{1}{\alpha}g(z)\right]^{\alpha}$ is regular, has its only zero at the origin and $f'(0) = 1$. Since $f(z)$ is a solution of the differential equation (3.1), $f'(z_0) = 0$ for some z_0, $0 < |z_0| < 1$, would imply that $F(z_0) = 0$, which is impossible. Thus $f'(z) \neq 0$ for $z \in E$. Also from (3.1), $J(\alpha, f) = zF'(z)/F(z)$. This completes the proof.

Remark. Theorems 3 and 4 yield a representation formula (3.2) for functions in $M(\alpha, \lambda)$.

If we denote by $B(\alpha, \lambda)$ the subclass of Bazilevič functions f defined by

$$f(z) = \{\alpha \int_0^z F(t)^{\alpha} t^{-1}\, dt\}^{1/\alpha},$$

where $F \in S^*(\lambda)$ and $\alpha > 0$, then it can be easily seen that

$$B\left(\frac{1}{\alpha}, \lambda\right) \equiv M(\alpha, \lambda).$$

4. In this section we obtain an inequality for the coefficients of functions in $M(\alpha, \lambda)$.

THEOREM 5. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(\alpha, \lambda)$ *and let* $s_1 = 0$, $s_m = (1-\alpha)(\beta_m - \alpha_m) + \alpha\gamma_{m-1}$, $t_1 = 2$, $t_m = (1-\alpha)\beta_m + (1+\alpha)\alpha_m + \alpha\gamma_{m-1}$, $m = 2, 3, \ldots$, *where* α_m, β_m *and* γ_m *are defined by*

$$\alpha_m = \sum_{k=1}^{m} (m-k+1)\, a_k a_{m-k-1},$$

$$(4.1) \qquad \beta_m = \sum_{k=1}^{m} k(m-k+1)\, a_k a_{m-k+1},$$

$$\gamma_m = \sum_{k=1}^{m} k(k+1)\, a_{k+1} a_{m-k+1}.$$

Then the coefficients a_n *satisfy the following inequality*:

$$\sum_{m=1}^{n} |s_m|^2 \leqslant \lambda^2 \sum_{m=1}^{n-1} |t_m|^2, \qquad n = 2, 3, \ldots$$

Equality holds for the function

$$f_{\alpha}(z) = \left\{\frac{1}{\alpha}\int_0^z t^{1/\alpha - 1}(1+\varepsilon\lambda t)^{-2/\alpha}\, dt\right\}^{\alpha}, \qquad |\varepsilon| = 1.$$

Proof. Since $f \in M(\alpha, \lambda)$,

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)} = \frac{1-\lambda w(z)}{1+\lambda w(z)},$$

where w is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ in E. This gives

$$(4.2)\quad (1-\alpha)\{z(f'(z))^2 - f(z)f'(z)\} + \alpha z f(z) f''(z) \\ = -\lambda w(z)\{(1-\alpha)z(f'(z))^2 + (1+\alpha)f(z)f'(z) + \alpha z f(z) f''(z)\}.$$

Given $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we note that

$$f(z)f'(z) = \sum_{m=1}^{\infty} \alpha_m z^m, \quad (f'(z))^2 = \sum_{m=1}^{\infty} \beta_m z^{m-1}, \quad f(z)f''(z) = \sum_{m=1}^{\infty} \gamma_m z^m,$$

where α_m, β_m and γ_m are defined in (4.1). Thus (4.2) becomes

$$(1-\alpha)\sum_{m=1}^{\infty} (\beta_m - \alpha_m) z^m + \alpha \sum_{m=1}^{\infty} \gamma_m z^{m+1} \\ = -\lambda w(z)\{(1-\alpha)\sum_{m=1}^{\infty} \beta_m z^m + (1+\alpha)\sum_{m=1}^{\infty} \alpha_m z^m + \alpha \sum_{m=1}^{\infty} \gamma_m z^{m+1}\}$$

which simplifies to

$$\sum_{m=1}^{\infty} s_m z^m = -\lambda w(z)\{\sum_{m=1}^{\infty} t_m z^m\}.$$

Now

$$\left|\sum_{m=1}^{n} s_m z^m + \sum_{m=n+1}^{\infty} h_m z^m\right| \leqslant \lambda \left|\sum_{m=1}^{n-1} t_m z^m\right|,$$

where h_m's are some complex numbers. This yields

$$\sum_{m=1}^{n} |s_m|^2 + \sum_{m=n+1}^{\infty} |h_m|^2 \leqslant \lambda^2 \sum_{m=1}^{n-1} |t_m|^2$$

or

$$\sum_{m=1}^{n} |s_m|^2 \leqslant \lambda^2 \sum_{m=1}^{n-1} |t_m|^2.$$

5. We now determine the radius of the largest disc where the converse of Theorem 1 holds.

THEOREM 6. *Let $f \in S^*(\lambda)$. Then f satisfies condition* (1.1) *for $|z| < r_0$, where r_0 is the smallest positive root of the equation*

$$1 - (1+\alpha)(1+\lambda) r + \lambda r^2 = 0.$$

The bound r_0 is sharp.

Proof. Since $f \in S^*(\lambda)$,

$$\frac{zf'(z)}{f(z)} = \frac{1-\lambda w(z)}{1+\lambda w(z)}, \tag{5.1}$$

where w is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ in E,

$$J(\alpha, f) = \frac{1-\lambda w(z)}{1+\lambda w(z)} - \frac{2\alpha\lambda z w'(z)}{(1-\lambda w(z))(1+\lambda w(z))},$$

$$\left|\frac{J(\alpha, f)-1}{J(\alpha, f)+1}\right| = \lambda \left|\frac{w(z)+\alpha z w'(z)/(1-\lambda w(z))}{1-\alpha\lambda z w'(z)/(1-\lambda w(z))}\right| < \lambda$$

provided

$$|w(z)+\alpha z w'(z)/(1-\lambda w(z))| < |1-\alpha\lambda z w'(z)/(1-\lambda w(z))|.$$

This, in turn, is true if

$$|w(z)|+\alpha |z|\,|w'(z)|/(1-\lambda |w(z)|) < 1-\alpha\lambda |z|\,|w'(z)|/(1-\lambda |w(z)|). \tag{5.2}$$

Using the following well-known estimate

$$|w'(z)| \leqslant \frac{1-|w(z)|^2}{1-|z|^2},$$

inequality (5.2) reduces to

$$t^2 \{\lambda+\alpha(1+\lambda)r-\lambda r^2\} - t(1+\lambda)(1-r^2)+1-r^2-\alpha r-\alpha\lambda r \geqslant 0, \tag{5.3}$$

where $|w(z)| = t$ and $|z| = r$. Denoting the left-hand member of (5.3) by $E(t)$, we see that $E'(t)$ vanishes when

$$t = t_1 = \frac{(1+\lambda)(1-r^2)}{2[\lambda+\alpha(1+\lambda)r-\lambda r^2]}.$$

Evidently t_1 is positive. Also $E''(t)$ is positive. Now $t_1 \gtreqless r$ according as $Q(r) \equiv 1+\lambda-2\lambda r-(1+\lambda)(1+2\alpha)r^2+2\lambda r^3 \gtreqless 0$. The equation $Q(r) = 0$ has at least one root in (0, 1). Call the smallest positive root r_1. Thus for $0 \leqslant r < r_1$, $Q(r) > 0$. This means that for $0 \leqslant r < r_1$, $t_1 > r$ and $E(t)$ attains its minimum at $t = r$ for $0 \leqslant t \leqslant r < r_1$. Also $E(r) > 0$ would imply $E(t) > 0$, $0 \leqslant t \leqslant r$. This condition becomes

$$P(r) \equiv 1-(1+\alpha)(1+\lambda)r+\lambda r^2 > 0.$$

The equation $P(r) = 0$ has at least one root in (0, 1) and let r_0 be the smallest positive root. Hence for $0 \leqslant r < r_0$, $P(r) > 0$. Also $P(r_1) < 0$, implying that $r_0 < r_1$. Thus

$$|(J(\alpha, f)-1)/(J(\alpha, f)+1)| < \lambda \quad \text{for } |z| < r_0.$$

If $|z| = r_0$, then for the function f corresponding to $w(z) = z$ in (5.1), we see that

$$\left|\frac{J(\alpha, f)-1}{J(\alpha, f)+1}\right| = \lambda.$$

This shows that the bound r_0 is sharp.

Remark. For $\lambda = 1$, Theorem 6 gives the radius of α-convexity for starlike functions [1].

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