ANNALES
POLONICI MATHEMATICI
XXXI (1976)

On D-connected sets in the space V^q

by S. Topa (Kraków)

Abstract. Let be given an arbitrary topological space X and its arbitrary decomposition $D: = \{X_\lambda\}_{\lambda \in \Lambda}$ into disjoint sets. We take $\Lambda_0 \subset \Lambda$ arbitrarily and by Λ_0^* we denote the family of all $\lambda^* \in \Lambda_0$ such that for every $\lambda \neq \lambda^*$, $\lambda \in \Lambda_0$, we have $\mathrm{cl}(X_{\lambda^*}) \cap X_\lambda = \emptyset$ (cl being the operation of closure) and we put

$$A: = \bigcup_{\lambda \in \Lambda_0} X_\lambda \quad \text{and} \quad A^*: = \bigcup_{\lambda \in \Lambda_0^*} X_\lambda.$$

Any function $f: A \to R$ being constant on every $X_\lambda \subset A$ is called *D-function* on A.

Definition. The set A is said to be *D-connected in* X iff (1) $A^* \neq \emptyset$. (2) For every $\lambda \in \Lambda_0$ we have $\mathrm{cl}(X_\lambda) \cap A^* \neq \emptyset$. (3) For every D-function f on A the condition

$$f(\mathrm{cl}(X_\lambda) \cap A^*) \subset \mathrm{cl}(f(X_\lambda)) \quad \text{for all } \lambda \in \Lambda_0$$

implies the constancy of f on A.

In the paper a certain method of construction of D-connected sets in $X: = V^q$ for the decomposition D consisting of all $\mathrm{GL}(V)$-orbits in V^q is presented, $\mathrm{GL}(V)$ denoting the general linear group of a given topological vector space V and q an arbitrary natural number.

I. Introduction. In this paper we give a method for the determination of some families of D-connected sets in V^q for D being the decomposition of V^q into orbits of the linear group $\mathrm{GL}(V)$ acting on V^q in the natural way (called $\mathrm{GL}(V)$*-decomposition*), V is a given topological vector space and q a natural number (see Theorems 1 and 2). The problem of the determination of all D-connected sets in V^q is open.

The notion of the D-connected set (see Definition 6) is original. To give the definition let us suppose we are given a topological space X and an arbitrary decomposition D of X into disjoint sets. For example, D may be considered as the decomposition of X into orbits of an abstract group G acting on X arbitrarily (such decomposition D being called *G-decomposition*).

If A is a subspace of X, then we denote by D_A the decomposition of A induced by D.

Definition 1. We say that a set $A \subset X$ *reaches* a set $B \subset X$ iff $B \cap \mathrm{cl} A \neq \emptyset$ (cl being the operation of closure).

Definition 2. We say that a set $A \subset X$ is *concentrating* on a subset B of A iff each element of D_A reaches the set B.

Definition 3. A function $f\colon A \to R$, $A \subset X$, is said to be *D-continuous* iff for each subset C of each element of D_A there is $f(\mathrm{cl}\, C) \subset \mathrm{cl}\big(f(C)\big)$.

Definition 4. A function $f\colon A \to R$ is called *D-function* iff it is constant on each element of D_A.

Definition 5. A set $A \subset X$ is called *D-admissible* iff it is the union of a family of elements of D.

For $A \subset X$ we denote by A^* the union of all those elements of D_A which do not reach any other elements of D_A.

Now the basic

Definition 6. Let A be a given D-admissible set in X. Then A is called a *D-connected set* in X iff (1) $A^* \neq \emptyset$, (2) A is concentrating on A^*, (3) for each D-function $f\colon A \to R$, D-continuous on A^*, we obtain that f is constant.

A theory of D-connected sets has not yet been elaborated. We may only give a result concerning some sufficient conditions for a given family of D-connected sets, the union of which is also a D-connected set (see Proposition 1).

It may be observed that any element of D is a D-connected set in X. Such D-connected sets are called *trivial*.

In the following we are only interested in non-trivial D-connected sets. Of course, not every decomposition D of X admits non-trivial D-connected sets. Such a situation is, for example, if X is of the form V^q with V being equipped with an inner product and D is a G-decomposition of V^q for G being the group of all isometries of V acting on V^q in the natural way. This may be shown easily by using, for example, the results contained in Topa's paper (1). We get the same situation if G is a unimodular or conformal group. But if $G = \mathrm{GL}(V)$, then non-trivial D-connected sets in V^q exist. Indeed, any D-admissible set $A \subset V^q$ containing the point $0 = (0, \ldots, 0) \in V^q$ such that $A \neq \{0\}$, is a non-trivial D-connected set in V^q. The main purpose of this paper is to give a method for the construction of other non-trivial D-connected sets in V^q.

Remark 1. The notion of the D-connected set has some connections with the following problem posed to me by Professor Gołąb:

Let there be given a Lie group G acting on a manifold X and a G-invariant decomposition of X into disjoint submanifolds $X_1, \ldots, X_m$.

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(1) S. Topa, *On complete linear, metric, conformal and unimodular classifications of space of all finite sequences of vectors in a given vector space*, Zeszyty Naukowe UJ, t. 17 (in press).

Consider the functional equations

(1) $$\varphi(Tx) = \varphi(x), \quad x \in X, \ T \in G,$$

and

(2) $$\varphi_j(Tx) = \varphi_j(x), \quad x \in X_j, \ T \in G \ (j = 1, \ldots, m),$$

$\varphi\colon X \to R$, $\varphi_j\colon X_j \to R$ being unknowns, and assume we are given families Φ $(j = 1, \ldots, m)$ of regular solutions of (2) (for example, the families of all regular solutions). The problem is to determine the family Φ of all regular solutions φ of (1) such that $\varphi | X_j$ belongs to Φ_j for each $j = 1, \ldots, m$.

If for each j, Φ_j contains all constant solutions of (2), then the problem has a positive solution; Φ contains all constant solutions of (1). From Definition 6 it directly follows that if X is a D-connected set with respect to our G-decomposition, then each continuous solution φ of (1) is constant. This means that in our case Φ is the family of all constant solutions of (1), e.g. the family of all constant functions $\varphi\colon X \to R$.

In other cases the problem has not been worked out.

Remark 2. The following fact does not require proof: If B_1 and B_2 are concentrating on A, then the set $B = B_1 \cup B_2$ also concentrates on A. This may be generalized for any family of sets.

Definition 7. A family $\{A_\lambda\}_{\lambda \in \Lambda}$ of sets in the space X is said to have the *property* of *finite connectivity* iff for each pair A', A'' of sets in the family there is a finite sequence $A_1, \ldots, A_r$ of its sets such that (1) $A_1 = A'$, $A_r = A''$ and (2) $(\varrho = 1, \ldots, r)$

$$A_{\varrho_0} \cap \bigcup_{\varrho \neq \varrho_0} A_\varrho \neq \emptyset \quad \text{for all } \varrho_0 \in \{2, \ldots, r\}.$$

We shall now prove the following:

Proposition 1. *Let a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of D-connected sets in the space X have the property of finite connectivity and satisfy the condition*

$$\Big(\bigcup_{\lambda \in \Lambda} A_\lambda\Big)^* = \bigcup_{\lambda \in \Lambda} A_\lambda^*.$$

Then $A = \bigcup_{\lambda \in \Lambda} A_\lambda$ is a D-connected set in X.

Proof. From the above condition the first and second properties of D-connected sets directly follow. For the third property let us take an arbitrary D-function f on A, D-continuous on A^*. Denoting $f_\lambda = f | A_\lambda$ for $\lambda \in \Lambda$ we may state that each function f_λ is constant. This fact immediately follows from the D-connectivity of the set A. The constancy of f is implied by the property of finite connectivity of the given family of sets. This completes the proof.

Remark 3. Let us assume that we are given n arbitrary decompositions $D_1, \ldots, D_n$ of X. Then we shall say that we are given an *n-decomposition* of X and denote it by $(D_1, \ldots, D_n)$.

A given n-decomposition $(D_1, \ldots, D_n)$ of X is called *quasi-cartesian* iff $\inf\{D_1, \ldots, D_n\} = X$ (²).

It can be easily seen that: If $(D_1, \ldots, D_n)$ is a quasi-cartesian n-decomposition of X and $f\colon X \to R$ is a D_i-function for each $i = 1, \ldots, n$, then f is necessarily a constant function.

II. D-connected sets in V^q. We have in mind $\mathrm{GL}(V)$-decomposition of V^q (see Example 1). We shall use the terminology of $\mathrm{GL}(V)$-orbits instead of D-components, $\mathrm{GL}(V)$-admissible sets instead of D-admissible sets, and so on. D-connected sets are called *$\mathrm{GL}(V)$-connected sets*. For the construction of certain families of such sets in V^q we introduce some denotations and give some lemmas.

Let us write $K = (1, \ldots, q)$, $p = \min(q, \dim V)$, N — the set of all natural numbers and $X = V^q$.

By $K(p)$ we mean the family of all subsequences of the sequence K consisting of at most p elements; we also treat the empty sequence, denoted by $\emptyset$, as an element of $K(p)$.

We introduce the mapping $\iota\colon X \to K(p)$ (see footnote (¹)), being defined in the following way:

Let $x = (x_k) \in X$. By $n(x)$ we denote the dimension of the subspace $V(x)$ of V, generated by the system of vectors $x_1, \ldots, x_q$, and $\iota(x)$ we define as follows:

1. In the case $x = 0$ we put $\iota(x) = \emptyset$;

2. If $x \neq 0$, then we put $\iota(x) = (i_1, \ldots, i_n)$, where $n = n(x)$ and $i_1, \ldots, i_n$ are obtained by use of the following recurrent formula:

1° i_1 is the smallest element in K such that $x_{i_1} \neq 0$;

2° Suppose that $i_1, \ldots, i_\nu$ are determined and that $\nu < n$. Then by $i_{\nu+1}$ we mean the smallest element in the sequence $i_\nu + 1, \ldots, q$ such that the vector $x_{i_{\nu+1}}$ is linearly independent of the system of vectors $x_{i_1}, \ldots, x_{i_\nu}$.

The mapping ι is a surjection.

For a given $I \in K(p)$ we denote by X_I the *coimage* of I by the mapping ι. For example, we have $X_\emptyset = \{0\}$.

It can be easily seen that each set X_I is a $\mathrm{GL}(V)$-admissible set. If $I = \emptyset$ or $I = K$ (the case of which may appear only in the situation when $q \leqslant \dim V$), then X_I is trivial, e.g. consists of one orbit only. If $I \neq \emptyset, K$, then X_I is non-trivial. In this case let us denote $J = K \setminus I$ (a subsequence of K) and introduce the family $\mathrm{Mat}(I, J)$ of all matrices

(²) The family of all decompositions of X into disjoint sets forms a lattice partially ordered by refinements. A quasi-cartesian n-decomposition $(D_1, \ldots, D_n)$ of X is called *cartesian* iff $\sup\{D_1, \ldots, D_n\} = \{X\}$.

$\|a^i_j\|$, $i \in I$, $j \in J$, such that the condition

$$j < i \Rightarrow a^i_j = 0 \tag{3}$$

is valid.

Furthermore, let $n = |I|$, and by E_n let us denote the family of all sequences $(e_1, \ldots, e_n)$ of linearly independent vectors of the space V.

Let $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_m)$. For a given $\underset{0}{x} \in X$, $\underset{0}{x} = (\underset{0}{x}_i, \underset{0}{x}_j)$, $i \in I$, $j \in J$, the $\mathrm{GL}(V)$-orbit $O(\underset{0}{x})$ of $\underset{0}{x}$ is given by the equations

$$\begin{aligned} x_{i_\nu} &= e_\nu, \\ x_{j_\mu} &= \underset{0}{a}{}^{i_\nu}_{j_\mu} e_\nu, \end{aligned} \qquad (e_\nu) \in E_n, \tag{4}$$

where $\underset{0}{a}{}^{i_\nu}_{j_\mu}$ are given by the decompositions

$$\underset{0}{x}_{j_\mu} = \underset{0}{a}{}^{i_\nu}_{j_\mu} \underset{0}{x}_{i_\nu}.$$

Furthermore, the set $S(\underset{0}{x})$ given by the equations

$$\begin{aligned} x_{i_\nu} &= \underset{0}{x}_{i_\nu}, \\ x_{j_\mu} &= a^{i_\nu}_{j_\mu} \underset{0}{x}_{i_\nu}, \end{aligned} \qquad \|a^{i_\nu}_{j_\mu}\| \in \mathrm{Mat}(I, J), \tag{5}$$

is a section (passing through $\underset{0}{x}$) of the space of $\mathrm{GL}(V)$-orbits in X (see footnote ([1])).

We shall prove the following

LEMMA 1. *Let $I \in K(p)$, $I \neq K$, and denote $B = X_I$. Then a given $\mathrm{GL}(V)$-admissible set $A \subset X$ $(X = V^q)$ is concentrating on the set B iff the condition*

$$\iota(x) \supset I \quad \textit{for each } x \in A \tag{6}$$

is satisfied.

Proof. For sufficiency let us take an arbitrary $\bar{x} \in A$. We have to show that the $\mathrm{GL}(V)$-orbit $O(\bar{x})$ of $\bar{x}$ reaches the set $B = X_I$. By assumption we have $\bar{I} \supset I$, where $\bar{I} = \iota(\bar{x})$. The case $\bar{I} = I$ is trivial. Let $\hat{I} \neq \emptyset$, where by definition $\hat{I} = \bar{I} \setminus I$. Denoting $\bar{J} = K \setminus \bar{I}$ we get for K the decomposition $K = I \cup \hat{I} \cup \bar{J}$. It can be verified that the sequence $\underset{n}{\bar{x}} = (\underset{n}{\bar{x}}_k) = (\underset{n}{\bar{x}}_i, \underset{n}{\bar{x}}_{\hat{i}}, \underset{n}{\bar{x}}_{\bar{j}}) \in V^q$ given by the formulas

$$\begin{aligned} \underset{n}{\bar{x}}_i &= \bar{x}_i, & i &\in I, \\ \underset{n}{\bar{x}}_{\hat{i}} &= \frac{1}{n} \bar{x}_{\hat{i}}, & \hat{i} &\in I, \\ \underset{n}{\bar{x}}_{\bar{j}} &= \bar{a}^i_{\bar{j}} \bar{x}_i + \bar{a}^{\hat{i}}_{\bar{j}} \left(\frac{1}{n} \bar{x}_{\hat{i}} \right), & \bar{j} &\in \bar{J}, \end{aligned}$$

where $\bar{a}^{i}_{\bar{j}}$ and $\bar{a}^{\hat{i}}_{\bar{j}}$ are given by the equalities $\bar{x}_{\bar{j}} = \bar{a}^{\bar{i}}_{\bar{j}}\bar{x}_{\bar{i}} + \bar{a}^{\hat{i}}_{\bar{j}}\bar{x}_{\hat{i}}$, has the properties that $\underset{n}{\bar{x}} \in O(\bar{x})$ and $\lim_{n|\infty} \underset{n}{\bar{x}}$ exists and belongs to the set B. This means nothing but the fact that the orbit $O(\bar{x})$ reaches the set B.

To prove the necessity let us assume that the $\mathrm{GL}(V)$-admissible set $A \subset V^q$ is concentrating on the set X_I. We have to show that for each $\bar{x} \in A$ we get $\bar{I} \supset I$, where $\bar{I} = \iota(\bar{x})$. On the contrary, let us suppose that there exists a point $\underset{0}{\bar{x}} \in A$ such that $\underset{0}{\bar{I}} \not\supset I$, where $\underset{0}{\bar{I}} = \iota(\underset{0}{\bar{x}})$. Then we can show that the orbit $O(\underset{0}{\bar{x}})$ does not reach the set X_I and this will lead us to a contradiction of the assumption. Indeed, by the supposition that $\underset{0}{\bar{I}} \not\supset I$ we obtain that $I \cap \underset{0}{\bar{J}} \neq 0$. Let us denote $k_0 = \inf(I \cap \underset{0}{\bar{J}})$ and distinguish the two cases: (a) $k_0 = 1$, (b) $k_0 > 1$.

In case (a) we obtain the fact that

$$x_1 \neq 0 \quad \text{for each } x = (x_1, \ldots, x_q) \in X_I \tag{7}$$

which follows from the fact that $k_0 \in I$. Simultanously, we have that $k_0 \in \underset{0}{\bar{J}}$, which implies that

$$\bar{x}_1 = 0 \quad \text{for each } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_q) \in X_{\underset{0}{\bar{I}}}. \tag{8}$$

If we now assume the existence of a sequence $\underset{n}{\bar{x}} = (\underset{n}{\bar{x}_1}, \ldots, \underset{n}{\bar{x}_q}) \in O(\underset{0}{\bar{x}})$, convergent and having the property $\lim_{n|\infty} \underset{n}{\bar{x}} \in X_J$, then by virtue of (8) we obtain $\lim_{n|\infty} \underset{n}{\bar{x}_1} = 0$, which is a contradiction of (7).

In case (b) we follow the same pattern and obtain, instead of (7) and (8), the following two facts:

(9) x_{k_0} is linearly independent of $x_1, \ldots, x_{k_0-1}$

for each $x = (x_1, \ldots, x_{k_0-1}, x_{k_0}, \ldots, x_q) \in X_I$,

(10) $\bar{x}_{k_0}$ is linearly dependent on $\bar{x}_1, \ldots, \bar{x}_{k_0-1}$

for each $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{k_0-1}, \bar{x}_{k_0}, \ldots, \bar{x}_q) \in X_{\underset{0}{\bar{I}}}$.

By these facts the supposition that the orbit $O(\underset{0}{\bar{x}})$ reaches the set X_I leads us to a contradiction.

Lemma 1 is therefore proved.

We shall now give an illustration of this lemma.

Example 1. Let $\dim V = 3$, $q = 7$. We get $p = 3$ and $K = (1, \ldots, 7)$. Putting $I = (2, 5)$, let us form the set X_I. We have

$$X_{(2,5)} = \{(0, e_1, a^1_3 e_1, a^1_4 e_1, e_2, a^1_6 e_1 + a^2_6 e_2, a^1_7 e_1 + a^2_7 e_2):\ a^i_j \in R,\ (e_1, e_2) \in E_2\},$$

where E_2 is the family of all linearly independent pairs of vectors in V.

If, for example, we take

$$A = \{(0, e_1, b_3^1 e_1, b_4^1 e_1, e_2, e_3, b_7^1 e_1 + b_7^2 e_2 + b_7^3 e_3)\colon b_j^i \in R,\ (e_1, e_2, e_3) \in E_3\},$$

where E_3 is the family of all linearly independent triples of vectors in V, then it may be seen that the set A concentrates on the set $B = X_{(2,5)}$. However, for example, the set

$$C = \{(e_1, e_2, c_3^1 e_1 + c_3^2 e_2, e_3, 0, 0, 0)\colon c_j^i \in R,\ (e_1, e_2, e_3) \in E_3\}$$

is not concentrating on the set $X_{(2,5)}$. Moreover, any orbit in C does not reach the set $X_{(2,5)}$, the fact of which is a consequence of the following:

LEMMA 2. *For arbitrary $I, \bar{I} \in K(p)$ such that $I \subset \bar{I}$ and $I \neq \bar{I}$, we have the following two facts*: 1. *Each orbit in $X_{\bar{I}}$ reaches the set X_I*. 2. *Any orbit in X_I does not reach the set $X_{\bar{I}}$*

This lemma is a direct consequence of Lemma 1.

Remark 4. Any orbit in X_I does not reach any other orbit in X_I.

This fact follows from formula (4).

From Remark 4 and Lemma 2 there immediately follows:

LEMMA 3. *If $I, \bar{I} \in K(p)$ such that $I \subset \bar{I}$ and $I \neq \bar{I}$, then denoting $A = X_I \cup X_{\bar{I}}$ we get* 1° $A^* = X_I$, 2° *A is concentrating on A^*.*

Now we pass on to the very important

LEMMA 4. *Assume that $I, \bar{I} \in K(p)$ and $I \subset \bar{I}$, $I \neq K$, $\bar{I} \neq K$. Then we can prove the following two facts*:

(a) *For each $x = (x_i, x_j) \in X_I$ there exists $\bar{x} = (\bar{x}_{\bar{i}}, \bar{x}_{\bar{j}}) \in X_{\bar{I}}$ such that*

$$\bar{x}_k = x_k \quad \text{for } k \in I \cup \bar{J}; \tag{11}$$

(b) *For a given $x \in X_I$ and $\bar{x} \in X_{\bar{I}}$ satisfying condition* (11), *and for any* GL(V)*-function f on $B = X_I \cup X_{\bar{I}}$ being* GL(V)*-continuous on X_I we have*

$$f(\bar{x}) = f(x). \tag{12}$$

Proof. For proof of (a) let us take an arbitrary $x \in X$ and introduce a point of the form $(x_i, v_{\hat{i}}, x_{\bar{j}})$ denoted by $\bar{x}$, where $v_{\hat{i}} \in V$ are fixed arbitrarily, provided that $(x_i, v_{\hat{i}})$ is a linearly independent system of vectors and, as already denoted, $\hat{I} = \bar{I} \setminus I$. It is evident that the point $\bar{x}$ belongs to $X_{\bar{I}}$ and together with the point x satisfies condition (11).

For proof of (b) let us fix $x \in X_I$ and $\bar{x} \in X_{\bar{I}}$ according to condition (11). We consider the sequence $\underset{n}{\bar{x}} = (\underset{n}{\bar{x}}_i, \underset{n}{\bar{x}}_{\hat{i}}, \underset{n}{\bar{x}}_{\bar{j}})$ given by the formulas

$$\begin{aligned} \underset{n}{\bar{x}}_i &= \bar{x}_i && \text{for } i \in I, \\ \underset{n}{\bar{x}}_{\hat{i}} &= \frac{1}{n}\bar{x}_{\hat{i}} + x_{\hat{i}} && \text{for } \hat{i} \in \hat{I}, \\ \underset{n}{\bar{x}}_{\bar{j}} &= \bar{x}_{\bar{j}} && \text{for } \bar{j} \in \bar{J}. \end{aligned} \tag{13}$$

We easily observe that 1° $\underset{n}{\bar{x}} \in O(\bar{x})$ for $n \in N$ and 2° $\lim_{n|\infty} \underset{n}{\bar{x}} = x$. From the assumed GL($V$)-continuity of f on X_I and by equality 2° we obtain that

$$\lim_{n|\infty} f(\underset{n}{\bar{x}}) = f(x), \tag{14}$$

and from the constancy of f on each GL(V)-orbit (we have assumed that f is a GL(V)-function) we find that

$$f(\underset{n}{\bar{x}}) = f(\bar{x}). \tag{15}$$

Equalities (14) and (15) imply (12). The lemma is proved.

For the formulation of fundamental results we introduce the following additional notions:

DEFINITION 8. A given family $K_\Omega = \{I_\omega\}_{\omega\in\Omega}$ of subsequences I_ω of the sequence K is called a *covering* of K iff $\bigcup_{\omega\in\Omega} I_\omega = K$.

By $K_\Omega(p)$ we denote families of subsequences such that $I_\omega \in K(p)$ for each $\omega \in \Omega$.

DEFINITION 9. We say that a given family $K_\Omega = \{I_\omega\}_{\omega\in\Omega}$ of subsequences of K is *closed from below* (or simply *closed*) iff for $I_\Omega = \bigcap_{\omega\in\Omega} I_\omega$ we get $I_\Omega \neq \emptyset$ and $I_\Omega \in K_\Omega$. We call I_Ω the *lower bound* of K_Ω.

For a given closed covering $K_\Omega(p)$ of K let us form the sets

$$X_\Omega = \{x \in V^q\colon\ \iota(x) \in K_\Omega(p)\} \tag{16}$$

and

$$X_{I_\Omega} = \{x \in V^q\colon\ \iota(x) = I_\Omega\}. \tag{17}$$

Now we may give the first fundamental

THEOREM 1. *If $K_\Omega(p)$ is a closed covering of K such that $|\Omega| > 1$, then the set X_Ω, defined in* (16), *is a* GL(V)*-connected set in V^q.*

Proof. From Lemma 2 and Lemma 3 it immediately follows that if we put $A = X_\Omega$, then $A^* = X_{I_\Omega}$, X_{I_Ω} having been defined in (17). Next, from Lemma 3 and Remark 2 we easily obtain that the set A is concentrating on the set A^*. Thus we see that for our set A the second condition in Definition 6, of D-connectivity (in our case D is determined by the group GL(V)), is valid. To show that also the third condition is valid we distinguish the following three cases:

1° $I_\Omega = \emptyset$, 2° $I_\Omega = K$, 3° $I_\Omega \neq \emptyset, K$.

Case 1°. This case is trivial.

Case 2°. In this case the covering $K_\Omega(p)$ of K is trivial (it consists only of one element, equal to the sequence K) and the assumption $|\Omega| > 1$ in our theorem is not satisfied.

Case 3°. At first we show that each GL(V)-function f on A, GL(V)-continuous on A^*, is constant on A^*. For this purpose let us take an arbitrary point $\underset{0}{x} \in X_{I_\Omega}$ and consider the section $S(\underset{0}{x})$ of the space of GL(V)-orbits in X_{I_Ω}, given by formulas (5), where I is replaced by I_Ω. By the assumption that f is constant on each GL(V)-orbit in X_Ω it is sufficient to show that f is constant on $S(\underset{0}{x})$. To show this we shall make use of Remark 3 and Lemma 4.

Let us make the denotations

$$J_\Omega = K \setminus I_\Omega, \quad I_\Omega = (i_1, \ldots, i_n), \quad J_\Omega = (j_1, \ldots, j_m)$$

and

$$a_{j_\mu} = \|a^i_{j_\mu}\| \quad (i \in I_\Omega), \quad j_\mu \in J_\Omega,$$

and define the function

$$F(a_{j_1}, \ldots, a_{j_m}) = f(\underset{0}{x}_{i_1}, \ldots, \underset{0}{x}_{i_n}, a^i_{j_1}\underset{0}{x}_i, \ldots, a^i_{j_m}\underset{0}{x}_i), \tag{18}$$

(e.g. F is the restriction of f to $S(\underset{0}{x})$). It remains to show that F is constant.

For this purpose, denoting by U the domain of the function F (remember that $\|a^i_j\| \in \mathrm{Mat}(I_\Omega, J_\Omega)$, see (3)), we shall define a quasi-cartesian s-decomposition of U, where s is equal to $|\Omega| - 1$. If in $K_\Omega(p)$ we fix an arbitrary element $\bar{I}$ different from I_Ω, then we may associate with it the following equivalence relation in U, denoted by $R_{\bar{I}}$ ($I_\Omega \subset \bar{I}$ and $I_\Omega \neq \bar{I}$ imply that $J_\Omega \supset \bar{J}$ and $J_\Omega \neq \bar{J}$),

$$(a'_{j_1}, \ldots, a'_{j_m}) R_{\bar{I}} (a''_{j_1}, \ldots, a''_{j_m}) \Leftrightarrow a'_{\bar{j}} = a''_{\bar{j}} \quad \text{for } \bar{j} \in \bar{J}. \tag{19}$$

Let us denote by $D_{\bar{I}}$ the obtained decomposition of U. We shall show that the function F is constant on each element from $D_{\bar{I}}$, e.g. that F is a $D_{\bar{I}}$-function.

We consider the set $X_{\bar{I}}$ and take an arbitrary point $\underset{0}{\bar{x}} \in X_{\bar{I}}$ of the form

$$\begin{aligned} \underset{0}{\bar{x}}_i &= \underset{0}{x}_i && \text{for } i \in I_\Omega, \\ \underset{0}{\bar{x}}_{\hat{i}} &= v_{\hat{i}} && \text{for } \hat{i} \in \hat{I}_\Omega,\ \hat{I}_\Omega = \bar{I} \setminus I_\Omega, \\ \underset{0}{\bar{x}}_{\bar{j}} &= \underset{0}{a}^i_{\bar{j}} \underset{0}{x}_i && \text{for } \bar{j} \in \bar{J}\ (i \in I_\Omega), \end{aligned} \tag{20}$$

where $\underset{0}{x}$ is the point previously fixed in X_{I_Ω} and $v_{\hat{i}}$ and $\underset{0}{a}_{\bar{j}}$ are arbitrarily chosen in V and R, respectively, provided that $(\underset{0}{x}_i, v_{\hat{i}})$ form a linearly independent system of vectors. At the same time let us consider the family of points $x \in X_{I_\Omega}$ given by the formulas

$$x_i = \underset{0}{x}_i \qquad \text{for } i \in I_\Omega,$$

$$(21) \qquad x_{\hat{i}} = a^i_{\hat{i}} \underset{0}{x}_i \qquad \text{for } \hat{i} \in \hat{I}_\Omega \ (i \in I_\Omega),$$

$$x_{\bar{j}} = \underset{0}{a}^i_{\bar{j}} \underset{0}{x}_i \qquad \text{for } \bar{j} \in \bar{J} \ (i \in I_\Omega),$$

where $a^i_{\hat{i}}$ are the parameters of the family.

It may be seen that for each system of $a^i_{\hat{i}}$ the point x given by (21) together with the point $\underset{0}{\bar{x}}$ satisfies condition (11), where I is replaced by I_Ω. Furthermore, for our function f all the assumptions in the second part of Lemma 4 are fulfilled, and by (12) we get the equality $f(\underset{0}{\bar{x}}) = f(x)$. This equality means that for the function F defined in (18) we have the equalities

$$(22) \qquad F(a_{\hat{i}}, \underset{0}{a}_{\bar{j}}) = F(\underset{0}{a}_{\hat{i}}, \underset{0}{a}_{\bar{j}}) \quad \text{for each } a_{\hat{i}} = \|a^i_{\hat{i}}\|.$$

Thus we have proved that F is constant on the element from $D_{\bar{I}}$ passing through the point $(\underset{0}{a}_{\hat{i}}, \underset{0}{a}_{\bar{j}}) \in U$. But $\underset{0}{a}_{\bar{j}} = \|\underset{0}{a}^i_{\bar{j}}\|$ appearing in formulas (20) may change arbitrarily, so the function F is a $D_{\bar{I}}$-function.

If $\bar{I} \neq I_\Omega$ varies in $K_\Omega(p)$ arbitrarily, then we obtain a family $\{D_{\bar{I}}\}$ of $s = |\Omega|-1$ decompositions of U, being a quasi-catesian s-decomposition of U. Using Lemma 1, where X is replaced by U and f by F, we obtain that F is a constant function, which was to be proved.

Theorem 1 is therefore proved.

We illustrate this theorem in

EXAMPLE 2. We consider the situation given in Example 1. For the sequence $K = (1, \ldots, 7)$ we take the following family of its subsequences: $\{(2, 3), (1, 4), (3, 5, 7), (6)\}$. This family is a covering for K, but it is not closed. However, a covering of the form $\{(2, 3), (1, 4), (3, 5, 7), (6), \emptyset\}$ is a closed covering for K.

Now we shall give a generalization of Theorem 1. For this purpose we formulate the following two lemmas:

LEMMA 5. *Let there be given a covering $K_\Omega(p) \subset K(p)$ of K and it's closed subcoverings $K_{\Omega_\varrho}(p) \subset K_\Omega(p)$, $\varrho = 1, \ldots, r$, such that*

$$K_\Omega(p) = \bigcup_{\varrho=1,\ldots,r} K_{\Omega_\varrho}(p).$$

Denoting by I_{Ω_ϱ} the lower bound of $K_{\Omega_\varrho}(p)$, and by X_{Ω_ϱ} and $X_{I_{\Omega_\varrho}}$ the sets corresponding, by (16) *and* (17) *respectively, to the covering $K_{\Omega_\varrho}(p)$, we may state that*

$$\Big(\bigcup_\varrho X_{\Omega_\varrho}\Big)^* = \bigcup_\varrho X^*_{\Omega_\varrho} \Leftrightarrow \forall_{I \in K_\Omega(p)} \forall_\varrho \colon I \neq I_{\Omega_\varrho} \Rightarrow I \cap I_{\Omega_\varrho} \neq I.$$

Proof. From Lemma 3 and Remark 2 we obtain that $X^*_{\Omega_\varrho} = X_{I_{\Omega_\varrho}}$ for each ϱ. From Lemma 2 and Remark 4 it follows that the GL(V)-orbit in $X_{I_{\Omega_\varrho}}$ reaches the GL(V)-orbit passing through a given point $\underset{0}{x} \in X_\Omega$ iff $I_0 \neq I_{\Omega_\varrho}$ and $I_0 \cap I_{\Omega_\varrho} = I_0$, where $I_0 = \iota(\underset{0}{x})$. Furthermore, from Remark 1 we get

$$\big(\bigcup_\varrho X_{\Omega_\varrho}\big)^* \subset \bigcup_\varrho X^*_{\Omega_\varrho}.$$

All these facts, combined with the definition of the star operation, complete the proof.

By the same assumptions and denotations as in Lemma 5 we formulate

LEMMA 6. *For each* $\varrho_0 \in \{2, \ldots, r\}$ *we have*

$$X_{\Omega_{\varrho_0}} \cap \big(\bigcup_{\varrho<\varrho_0} X_{\Omega_\varrho}\big) \neq \emptyset \Leftrightarrow K_{\Omega_{\varrho_0}} \cap \big(\bigcup_{\varrho<\varrho_0} K_{\Omega_\varrho}\big) \neq \emptyset.$$

Proof. The given equivalence follows directly from the definition of sets X_{Ω_ϱ} (see (16)).

By the same denotations as in Lemma 5 we now formulate

THEOREM 2. *Let a covering $K_\Omega(p)$ of K be given such that there exists a sequence of closed subcoverings $K_{\Omega_\varrho}(p)$, $\varrho = 1, \ldots, r$, satisfying the following conditions*:

1° $K_\Omega(p) = \bigcup_\varrho K_{\Omega_\varrho}(p)$,

2° $\forall_{I \in K_\Omega(p)} \forall_\varrho$: $I \neq I_{\Omega_\varrho} \Rightarrow I \cap I_{\Omega_\varrho} \neq I$,

3° $K_{\Omega_{\varrho_0}} \cap \big(\bigcup_{\varrho<\varrho_0} K_{\Omega_\varrho}\big) \neq \emptyset$ *for each* $\varrho_0 \in \{2, \ldots, r\}$.

Then the set X_Ω given for our covering $K_\Omega(p)$ by (16) *is a* GL(V)*-connected set.*

Proof. By wirtue of Theorem 1 we obtain for each ϱ that the set X_{Ω_ϱ} is a GL(V)-connected set. By assumptions 2° and 3° and Lemmas 5 and 6, respectively, we may state that Proposition 1 can be applied to the family $\{X_{\Omega_\varrho}\}$ of sets in $X = V^q$. In consequence, we obtain that X_Ω really is a GL(V)-connected set, which was to be proved.

EXAMPLE 3. In the case of Example 1 the covering

$$\{(1,2,3),\ (2,3),\ (3,4,5),\ (2,3,6),\ (3,5,7),\ (2,5,7),\ (3,6),\ (3),\ (2)\}$$

of the sequence $K = (1, \ldots, 7)$ satisfies all the assumptions of Theorem 2. Indeed, if we put $K_{\Omega_1}(3) = \{(1,2,3), (3,4,5), (3,5,7), (3,6), (3)\}$ and $K_{\Omega_2}(3) = \{(1,2,3), (2,3), (2,3,6), (2,5,7), (2)\}$, then we obtain a two-element sequence of subcoverings which are closed and satisfy conditions 1°–3°.

Reçu par la Rédaction le 14. 6. 1973