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On the method of upper and lower solutions in abstract cones[1]

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Dedicated to the memory of Jacek Szarski

Abstract. Consider the IVP

$$(*) \qquad u' = f(t, u), \quad u(0) = u_0$$

in a real Banach space E. When f does not satisfy any monotone property, the definition of upper lower solutions can be formulated in terms of functionals from K^*, the dual cone of K, as follows: $\varphi(v_0' - f(t, \sigma)) \leqslant 0$ for all σ such that $u_0(t) < \sigma < w_0(t)$ and $\varphi(v_0(t) - \sigma) = 0$, $\varphi(w_0' - f(t, \sigma)) \geqslant 0$ for all σ such that $v_0(t) \leqslant \sigma < w_0(t)$ and $\varphi(w_0(t) - \sigma) = 0$. It is known that even when $E = R^n$ and K is an arbitrary cone in E^n, the statement that there exists a solution of $(*)$ such that $v_0 < u < w_0$ on $I = [0, T]$ is not valid. In this paper by strengthening the definition of upper lower solutions, it is shown the above result is true.

I. Introduction and preliminaries. Let E be a real Banach space with $\|\cdot\|$ and let E^* denote the dual of E. Let $K \subset E$ be a cone, that is, a closed convex subset such that $\lambda K \subset K$ for every $\lambda \geqslant 0$ and $K \cap \{-K\} = \{0\}$. By means of K a partial order $\leqslant$ is defined as $v \leqslant u$ iff $u - v \in K$. We let $K^* = [\varphi \in E^*\colon \varphi(u) \geqslant 0$ for all $u \in K]$.

A cone K is said to be *normal* if there exists a real number $N > 0$ such that $0 \leqslant v \leqslant u$ implies $\|v\| \leqslant N\|u\|$, where N is independent of u, v. We shall always assume in this paper that K is a normal cone.

Let α denote the Kuratowski's measure of noncompactness, the properties of which may be found in [2], [4].

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For any $v_0, w_0 \in C[I, E]$ such that $v_0(t) \leqslant w_0(t)$ on I, where $I = [0, T]$, we define the conical segment

$$[v_0, w_0] = [u \in E\colon\ v_0(t) \leqslant u \leqslant w_0(t),\ t \in I].$$

Let us consider the IVP

$$u' = f(t, u), \quad u(0) = u_0, \tag{1.1}$$

where $f \in C[I \times E, E]$. Suppose that $v_0, w_0 \in C^1[I, E]$ and

$$v_0' \leqslant f(t, v_0), \quad w_0' \geqslant f(t, w_0) \quad \text{on } I. \tag{1.2}$$

Then v_0, w_0 are called *lower* and *upper solutions* of (1.1) defined in a natural way.

A function f is said to be *quasimonotone relative to* K if

$$v \leqslant u \quad \text{and} \quad \varphi(v-u) = 0, \quad \varphi \in K^* \quad \text{implies} \quad \varphi\big(f(t, v)\big) \leqslant \varphi\big(f(t, u)\big).$$

If $E = R^n$ and $K = R^n_+$, the standard cone, the inequalities induced by K are componentwise and the quasimonotonicity of f is reduced to

$$v \leqslant u \quad \text{and} \quad v_i = u_i, \quad 1 \leqslant i \leqslant n \quad \text{implies} \quad f_i(t, v) \leqslant f_i(t, u).$$

In this special case one can prove the following result.

THEOREM A. *Let $E = R^n$ and $K = R^n_+$. Suppose that v_0, w_0 satisfy* (1.2) *with $v_0(t) \leqslant w_0(t)$ on I and that f is quasimonotone. Then there exists a solution $u(t)$ of* (1.1) *on I such that $v_0(t) \leqslant u(t) \leqslant w_0(t)$ on I provided $v_0(0) \leqslant u_0 \leqslant w_0(0)$.*

If f is not known to be quasimonotone, we need to strengthen lower and upper solutions as follows: for each i, $1 \leqslant i \leqslant n$,

$$\begin{aligned} &v_{0i}' \leqslant f_i(t, \sigma) \quad \text{for all } \sigma \text{ such that } v_0(t) \leqslant \sigma \leqslant w_0(t) \text{ and } v_{0i}(t) = \sigma_i, \\ &w_{0i}' \geqslant f_i(t, \sigma) \quad \text{for all } \sigma \text{ such that } v_0(t) \leqslant \sigma \leqslant w_0(t) \text{ and } w_{0i}(t) = \sigma_i. \end{aligned} \tag{1.3}$$

We then have the following classical result of Müller.

THEOREM B. *Let $E = R^n$ and $K = R^n_+$. Suppose that v_0, w_0 satisfy* (1.3). *Then the conclusion of Theorem* A *holds.*

See for the details of proofs [1], [5].

We observe that the proofs of Theorems A and B depend crucially on the modification of f, that is $\tilde{f}$, where $\tilde{f}(t, u) = f\big(t, p(t, u)\big)$ and

$$p_i(t, u) = \max[v_{0i}(t), \min\{u_i, w_{0i}(t)\}] \quad \text{for each } i.$$

Clearly this modification makes sense only when $K = R^n_+$.

If K is an arbitrary cone, inequalities (1.2) need no change. On the other hand, inequalities (1.2) can be formulated in terms of functionals from K^*, namely, for $\varphi \in K^*$

$$\begin{aligned} &\varphi\big(v_0' - f(t, \sigma)\big) \leqslant 0 \quad \text{for all } \sigma \text{ such that } v_0(t) \leqslant \sigma \leqslant w_0(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \varphi\big(v_0(t) - \sigma\big) = 0, \\ &\varphi\big(w_0' - f(t, \sigma)\big) \geqslant 0 \quad \text{for all } \sigma \text{ such that } v_0(t) \leqslant \sigma \leqslant w_0(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \varphi\big(w_0(t) - \sigma\big) = 0. \end{aligned} \tag{1.4}$$

This version of condition (1.3) allows us to consider cones K other than the standard cone. The question is whether Theorems A and B hold even when K is an arbitrary cone. Theorem B may not be valid even in R^n as was shown by Volkmann [6]. Consider the example in R^3. Let $K = [u \in R^3: (u_1^2 + u_2^2)^{1/2} \leqslant u_3]$. Take $v_0 = (0, 0, 0)$, $w_0 = (2, 0, 2)$, $f_1 = f_3 = 0$ and

$$f_2 = \begin{cases} u_1 & \text{if } u_1 \in [0, 1], \\ 2 - u_1 & \text{if } u_1 \in [1, 2], \\ 0 & \text{otherwise}. \end{cases}$$

But the solution through $u_0 = (1, 0, 1)$ is $u(t) = (1, t, 1)$ which does not remain in the sector $[v_0, w_0]$. Note also that f is Lipschitzian. Thus it is evident that f being Lipschitzian is not sufficient to prove Theorem B in the set up corresponding to (1.4).

In this paper we consider this open problem and show that if the lower and upper solutions given by (1.4) are further strengthened, Theorem B is valid. We mention that a result corresponding to Theorem A has been recently proved [3] in an arbitrary cone by extending the monotone iterative technique.

II. Existence via method of upper and lower solutions. Let us begin by listing the following conditions for convenience.

(A_1) For any bounded set B in $[v_0, w_0]$,

$$a\big(f(I \times B)\big) \leqslant La(B);$$

(A_2) $\|f(t, u_1) - f(t, u_2)\| \leqslant L\|u_1 - u_2\|$, $t \in I$, $u_1, u_2 \in [v_0, w_0]$;

(A_3) $v_0, w_0 \in C^1[I, E]$ with $v_0(t) \leqslant w_0(t)$ on I such that there is an $M > 0$ satisfying

$$\varphi[v_0' - f(t, \sigma) + M(v_0 - \sigma)] \leqslant 0, \qquad \varphi[w_0' - f(t, \sigma) + M(w_0 - \sigma)] \geqslant 0,$$

for all $\sigma \in [v_0, w_0]$ and $\varphi \in K^*$.

Remark 1. If f is assumed to be uniformly continuous then condition (A_1) is superfluous. For, in that case,

$$a\big(f(I \times B)\big) = \max_I a\big(f(t, B)\big)$$

and consequently (A_2) implies (A_1). See [2], [4].

Remark 2. If, in addition, σ in (A_3) is such that for every $\varphi \in K^*$, $\varphi(v_0(t)-\sigma) = 0$ and $\varphi(w_0(t)-\sigma) = 0$, then condition (A_3) reduces to condition (1.4). Suppose now that f satisfies

(A_4) $f(t, u_1)-f(t, u_2) \geqslant -M(u_1-u_2)$ whenever $u_2 \leqslant u_1$ and $u_1, u_2 \in [v_0, w_0]$ for some $M > 0$.

Then it is easy to show that condition (1.2) implies (A_3). We note also that (A_4) implies that f is quasimonotone relative to K.

As was mentioned earlier, when (A_1), (1.2) and (A_4) are satisfied, a result corresponding to Theorem A is true. See [3] for details.

Let us consider the linear IVP

$$u' = F(t, u), \quad u(0) = u_0, \tag{2.1}$$

where $F(t, u) = f(t, \eta(t))-M(u-\eta(t))$ and $\eta \in C[I, E]$ is such that $v_0(t) \leqslant \eta(t) \leqslant w_0(t)$ on I. We need the following lemma proved in [3].

LEMMA 2.1. *Let assumption* (A_1) *hold. Then the IVP* (2.1) *has a unique solution* $u(t)$ *on* I.

For any $\eta \in C[I, E]$ such that $v_0(t) \leqslant \eta(t) \leqslant w_0(t)$ on I, define the mapping A by $A\eta = u$, where $u = u(t)$ is the unique solution of (2.1) corresponding to η. Concerning the mapping A we have

LEMMA 2.2. *Suppose that assumptions* (A_1) *and* (A_3) *hold. Then* A *maps the sector* $[v_0, w_0]$ *into itself.*

Proof. Let $\eta \in C[I, E]$ be such that $\eta \in [v_0, w_0]$ and let $u = A\eta$. For any $\varphi \in K^*$, set $p(t) = \varphi[u(t)-v_0(t)]$ so that $p(0) \geqslant 0$. Then for all $\sigma \in [v_0, w_0]$,

$$p' \geqslant \varphi[f(t, \eta)-M(u-\eta)-f(t, \sigma)+M(v_0-\sigma)],$$

in view of (A_3). Choosing $\sigma = \eta$, we have $p' \geqslant -Mp$ which implies $p(t) \geqslant p(0)e^{-Mt} \geqslant 0$ on I. This proves $v_0(t) \leqslant u(t)$ on I. A similar argument shows that $u(t) \leqslant w_0(t)$ on I. Hence $u = A\eta \in [v_0, w_0]$. Since η is arbitrary the proof is complete.

In view of Lemma 3.2 we can define the sequence $u_n = Au_{n-1}$ with $u_0 = v_0$ or w_0 satisfying $u_n \in [v_0, w_0]$ on I. We now need the following lemma which is proved in [3].

LEMMA 2.3. *Let* K *be a normal cone and let the assumptions of Lemma 2.2 hold. Then the sequence* $\{u_n(t)\}$ *is uniformly bounded, equicontinuous and relatively compact on* I.

By Lemma 2.3, we can conclude by Ascoli's theorem that there exists uniformly convergent subsequences of $\{u_n\}$. Suppose that $u_n(t)-u_{n-1}(t) \to 0$ as $n \to \infty$, then it is clear from the definition of $\{u_n\}$ that the limit of any subsequence is the unique solution of (1.1) on I. It then follows that a selection of a subsequence is unnecessary and the full sequence

$\{u_n(t)\}$ converges uniformly to the unique solution $u(t)$ on I such that $u(t) \in [v_0, w_0]$ on I. Thus it is sufficient to prove that $m(t) = 0$ on I, where $m(t) = \limsup_{n\to\infty} \|u_n(t) - u_{n-1}(t)\|$. To this end, we have

LEMMA 2.4. *Let K be a normal cone and let assumptions* (A_1), (A_2) *and* (A_3) *hold. Then $m(t) = 0$ on I.*

Proof. Since f maps bounded sets into bounded sets, we let $\|f(t, u)\| \leqslant N$ for $t \in I$ and $u \in [v_0, w_0]$. Then for $t_1, t_2 \in I$,

$$\begin{aligned}\|u_n(t_1) - u_{n-1}(t_1)\| &\leqslant \|u_n(t_2) - u_{n-1}(t_2)\| + 2N|t_1 - t_2| \\ &\leqslant m(t_2) + 2N|t_1 - t_2| + \varepsilon\end{aligned}$$

for large n, given $\varepsilon > 0$. Hence $m(t_1) \leqslant m(t_2) + 2N|t_1 - t_2| + \varepsilon$. Since t_1, t_2 can be interchanged and $\varepsilon > 0$ is arbitrary, we obtain

$$|m(t_1) - m(t_2)| \leqslant 2N|t_1 - t_2|$$

which proves that $m(t)$ is continuous on I. Now (A_2) yields

$$\begin{aligned}\|u_{n+1}(t) - u_n(t)\| &\leqslant \int_0^t \Big[\big\|f(s, u_n(s)) - f(s, u_{n-1}(s))\big\| + \\ &\qquad + M\|u_n(s) - u_{n-1}(s)\| + M\|u_{n+1}(s) - u_n(s)\|\Big] ds \\ &\leqslant \int_0^t [(L+M)\|u_n(s) - u_{n-1}(s)\| + M\|u_{n+1}(s) - u_n(s)\|] ds.\end{aligned}$$

For a fixed $t \in (0, T]$, there is a sequence of integers $n_1 < n_2 < \ldots$, such that $\|u_{n+1}(t) - u_n(t)\| \to m(t)$ as $n = n_k \to \infty$ and that $m^*(s) = \lim_{n=nk\to\infty} \|u_n(s) - u_{n-1}(s)\|$ exists uniformly on I. It therefore follows because of the fact $m^*(s) \leqslant m(s)$,

$$m(t) \leqslant (L+2M)\int_0^t m(s)\,ds \quad \text{on } I,$$

which implies that $m(t) \leqslant m(0)e^{(L+2M)t}$, $t \in I$. Since $m(0) = 0$, we have $m(t) = 0$ on I proving the lemma.

We have therefore proved the following main result of the paper.

THEOREM 2.1. *Assume that the cone K is normal and that conditions* (A_1), (A_2) *and* (A_3) *are satisfied. Then there exists a unique solution $u(t)$ of* (1.1) *on I such that $v_0(t) \leqslant u(t) \leqslant w_0(t)$ on I provided $v_0(0) \leqslant u_0 \leqslant w_0(0)$.*

COROLLARY 2.1. *Let $E = R^n$ and let* (A_2), (A_3) *hold. Then there exists a unique solution of* (1.1) *on I such that $u(t) \in [v_0, w_0]$ provided $v_0(0) \leqslant u_0 \leqslant w_0(0)$.*

This corollary is itself an extension of Müller's result [5] and answers affirmatively the open question in view of the counterexample of Volkmann [6].

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