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Vertical lines and points of a surface

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Introduction

The idea of a vertex of a plane curve (which for the ovals has led to the "four-vertex theorem") has been generalized also for skew curves, although we have not the universally accepted definition of such a point for those curves. As far as we know, the notion of a vertex has not so far been generalized to surfaces.

This paper is an attempt at such a generalization.

Basing ourselves on some ideas introduced by P. Szymański [1], we define first the notion of vertical direction at a given point on the surface and next we determine the vertex as a point for which every direction is vertical. In order to make the calculations simpler we define the vertical direction by the vanishing of the so called geodesic derivative of the normal curvature, although—as it seems to us—it would be better to define the vertical direction as a direction along which the so called longitudinal curvature reaches its extremal values.

The theorems concern vertical lines and points given in the present paper do not allow us to assert that the notions introduced will turn out important. In spite of it there are some interesting facts, as for example the connections of vertical directions with the so called Codazzi tensor.

Chapter I has an introductory character: it contains (with notation not quite the same as adopted by the author) the most important ideas from paper [1], which are fundamental for further considerations.

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I. Preliminary remarks

1. We shall be concerned with quantities of the first, second and third class (1) connected with a two-dimensional surface embedded in the three-dimensional Euclidean space. The quantities of the first class

(1) The notions introduced by F. Minding.

are those quantities which are only functions of a point P of a surface ($\boldsymbol{r}$ — radius-vector of that point)

$$A = f(\boldsymbol{r}) . \tag{1.1}$$

A quantity which is a function of a point P and a direction (with a determined sense) characterized by the unit vector $\boldsymbol{t}$ fixed at P and lying in the tangent plane of a surface at P is called a quantity of the second class

$$B = f(\boldsymbol{r}, \boldsymbol{t}) . \tag{1.2}$$

Finally a quantity of the third class is a function of a point P, direction $\boldsymbol{t}$ and a plane (π) passing through this direction and characterized by its unit normal vector $\boldsymbol{p}$:

$$C = f(\boldsymbol{r}, \boldsymbol{t}, \boldsymbol{p}) . \tag{1.3}$$

Let the surface considered be represented by the vector equation

$$\boldsymbol{r} = \boldsymbol{r}(u^1, u^2) \tag{1.4}$$

where u^1, u^2 are the parameters giving the regular Gaussian system of curvilinear coordinates, the field of $\boldsymbol{r}$ vectors is of class C^3, and

$$[\boldsymbol{r}_1 \boldsymbol{r}_2] \neq 0 . \tag{1.5}$$

Partial differentiation with respect to u^k is denoted by the suffix k, so that

$$\boldsymbol{r}_k = \frac{\partial \boldsymbol{r}}{\partial u_k}, \quad \boldsymbol{r}_{ij} = \frac{\partial^2 \boldsymbol{r}}{\partial u^i \partial u^j}, \quad \boldsymbol{r}_{ijk} = \frac{\partial^3 \boldsymbol{r}}{\partial u^i \partial u^j \partial u^k}, \ldots$$

and the square bracket means the vector product. All suffixes may assume the values 1 and 2.

Since the total differential of the radius-vector of a surface (1.4) is equal to

$$d\boldsymbol{r} = \boldsymbol{r}_k du^k , \tag{1.6}$$

the unit vector $\boldsymbol{t}$ of the direction on a surface may be represented in the form

$$\boldsymbol{t} = \frac{d\boldsymbol{r}}{dr} = \boldsymbol{r}_k \mu^k , \tag{1.7}$$

where

$$dr = |d\boldsymbol{r}| \tag{1.8}$$

and (²)

$$\mu^k = \frac{du^k}{dr} . \tag{1.9}$$

(²) The symbol dr is not a differential. It becomes the differential of the lenght of a certain curve lying on the surface when $\boldsymbol{t}$ is a vector tangent to that curve.

Calculating the square of the length of the vector $\boldsymbol{t}$, we get

$$(1.10) \qquad (\boldsymbol{r}_i \boldsymbol{r}_j)\mu^i\mu^j = 1 .$$

Since the scalar product

$$(1.11) \qquad (\boldsymbol{r}_i \boldsymbol{r}_j) = g_{ij}$$

represents the components of the metric tensor of our surface, (1.10) may be written in the form

$$(1.12) \qquad g_{ij}\mu^i\mu^j = 1 .$$

A quadratic differential form (1.12) is a modification of the classic 1-st Gaussian quadratic form. Its discriminant

$$(1.13) \qquad g = \det|g_{ij}|$$

is by (1.5) always positive.

2. At each point of a surface there exists a unit normal vector defined by

$$(1.14) \qquad \boldsymbol{m} = \frac{1}{\sqrt{g}}[\boldsymbol{r}_1 \boldsymbol{r}_2] .$$

If we take in the tangent plane the unit vector $\boldsymbol{l}$ orthogonal to $\boldsymbol{t}$ and equal to

$$(1.15) \qquad \boldsymbol{l} = [\boldsymbol{t}\boldsymbol{m}] ,$$

then we obtain on a surface at the given point M and for the given direction $\boldsymbol{t}$ the orthonormal right-handed trihedron $(M; \boldsymbol{t}, \boldsymbol{m}, \boldsymbol{l},)$.

The vectors $\boldsymbol{l}$, $\boldsymbol{r}_k$ are evidently linearly dependent; thus we may write

$$(1.16) \qquad \boldsymbol{l} = \boldsymbol{r}_k \omega^k .$$

We now find the relation between the coefficients ω^k and μ^k. It follows from (1.15), (1.7) and (1.16) that

$$\boldsymbol{r}_k \omega^k = [\boldsymbol{r}_i \boldsymbol{m}]\mu^i .$$

Scalar multiplication by $\boldsymbol{r}_l$ of both sides of the above relation gives

$$(1.17) \qquad (\boldsymbol{r}_l \boldsymbol{r}_k)\omega^k = (\boldsymbol{r}_l \boldsymbol{r}_i \boldsymbol{m})\mu^i ,$$

where the brackets on the right side of (1.17) denote the mixed product of three vectors.

We now introduce the so called Ricci symbol ε_{il}, in our case with two suffixes. It is defined, however — in another way than it is commonly done [3] — as the mixed product of the vectors $\boldsymbol{r}_k$ and $\boldsymbol{m}$, so that

$$(1.18) \qquad \varepsilon_{il} = \frac{1}{\sqrt{g}}(\boldsymbol{r}_i \boldsymbol{r}_l \boldsymbol{m}) .$$

[3] Compare [3], p. 84.

Evidently ε_{il} is skew-symmetric ($\varepsilon_{li} = -\varepsilon_{il}$). We shall also use the symbol ε^{lj}. It is determined by the relation

(1.19) $$\varepsilon_{il}\varepsilon^{lj} = \delta_i^j$$

where δ_i^j is Kronecker's symbol and the summation index l is placed on the different places. It is easy to prove that

(1.20) $$\varepsilon^{ik} = -\varepsilon_{ik} \, .$$

Using (1.11) and (1.18) the relation (1.17) may be written in the form

$$g_{lk}\omega^k = \sqrt{g}\,\varepsilon_{li}\mu^i \, .$$

Multiplying the above equality by ε^{jl} we obtain

$$\varepsilon^{jl}g_{lk}\omega^k = \sqrt{g}\,\varepsilon_{il}\varepsilon^{lj}\mu^i = \sqrt{g}\,\mu^j \, ,$$

and finally the required relation between ω^k and μ^j is

(1.21) $$\mu^j = \frac{1}{\sqrt{g}}\,\varepsilon^{jl}g_{lk}\omega^k \, .$$

3. In the above-mentioned paper [1] the notions of directional curvatures (vector and scalar) of a surface are introduced. They are: the vector curvature of a surface in the direction of $\boldsymbol{t}$

(1.22) $$\boldsymbol{\varkappa} = \left[\boldsymbol{m}\,\frac{d\boldsymbol{m}}{dr}\right]$$

and its two components along the vectors $\boldsymbol{l}$ and $\boldsymbol{t}$ — the longitudinal curvature $\boldsymbol{\lambda}$ and the transversal curvature $\boldsymbol{\tau}$, so that

(1.23) $$\boldsymbol{\varkappa} = \boldsymbol{\lambda} + \boldsymbol{\tau} = \lambda\boldsymbol{l} + \tau\boldsymbol{t} \, .$$

We shall be concerned next with the scalar quantities λ and τ, which may be represented, as follows from (1.22) and (1.23), by the formulae

(1.24) $$\lambda = -\left(\boldsymbol{t}\,\frac{d\boldsymbol{m}}{dr}\right),$$

(1.25) $$\tau = \left(\boldsymbol{l}\,\frac{d\boldsymbol{m}}{dr}\right).$$

Since

(1.26) $$\frac{d\boldsymbol{m}}{dr} = \boldsymbol{m}_k\,\mu^k$$

and using (1.7) and (1.26), we may express the scalar curvature λ defined by formula (1.24) as the quadratic form — analogous to the 2-nd fundamental form of Gauss —

$$\lambda = -(\boldsymbol{r}_j\mu^j)(\boldsymbol{m}_k\mu^k) = -(\boldsymbol{r}_j\boldsymbol{m}_k)\mu^j\mu^k$$

or

$$\lambda = h_{jk}\mu^j\mu^k\,, \tag{1.27}$$

where the coefficients h_{jk} are defined by the formulae

$$h_{jk} = -(\boldsymbol{r}_j\boldsymbol{m}_k) = (\boldsymbol{m}\boldsymbol{r}_{jk}); \tag{1.28}$$

this means that they are the components of the second tensor of a surface.

Similarly for τ, on the basis of (1.16), (1.26), (1.28) and the relation inverse to (1.21), we get the following quadratic form

$$\tau = (\boldsymbol{r}_j\omega^j)(\boldsymbol{m}_i\mu^i) = (\boldsymbol{r}_j\boldsymbol{m}_i)\omega^j\mu^i = \frac{1}{\sqrt{g}}\varepsilon^{jl}g_{lk}h_{ji}\mu^k\mu^i$$

or

$$\tau = \varrho_{ik}\mu^i\mu^k \tag{1.29}$$

where we introduce the notation

$$\varrho_{ik} = \frac{1}{\sqrt{g}}\varepsilon^{jl}g_{lk}h_{ji}\,. \tag{1.30}$$

It should be emphasized that both quantities of the second class: the longitudinal curvature λ and the transversal curvature τ of the surface in the direction $\boldsymbol{t}$, coincide at the given point M with the normal curvature and the geodesic torsion respectively of the curve passing through M and tangent to the direction $\boldsymbol{t}$.

When we seek the directions (so called principal directions) for which the longitudinal curvature λ take its extreme values, we are led to the conclusion that the transversal curvature τ must then disappear, this means that we have an equation

$$\varrho_{ik}\mu^i\mu^k = 0\,. \tag{1.31}$$

The values of the principal curvatures λ_1 and λ_2 may be obtained from the well-known equation

$$\lambda^2 - 2H\lambda + K = 0\,, \tag{1.32}$$

where

$$2H = \lambda_1 + \lambda_2 \tag{1.33}$$

and

$$K = \lambda_1\lambda_2 \tag{1.34}$$

denotes respectively the mean and the total (Gaussian) curvatures of a surface.

4. Among several intrinsic differential operations on geometric quantities connected with a surface and introduced in the above mentioned paper ([1], p. 44-55) of special interest for us is the notion of the so called derivative at a given point $\boldsymbol{M}$, in a given direction $\boldsymbol{t}$ and a given plane (π) characterized by its normal vector $\boldsymbol{p}$. This derivative applied to the quantities of the second class $B(\boldsymbol{r}, \boldsymbol{t})$ is defined by the following formula ([1], p. 49, formulae (36) and (37)):

$$\frac{dB}{dr_{(p)}} = \frac{\partial B}{\partial u^k}\mu^k + \frac{\partial B}{\partial \mu^k}\cdot\frac{d\mu^k}{dr_{(p)}}; \tag{1.35}$$

moreover the following conditions must be satisfied:

(1.36) $(\boldsymbol{t}\,d\boldsymbol{t}) = 0$ (because of the unity of $\boldsymbol{t}$)

(1.37) $(\boldsymbol{p}\,d\boldsymbol{t}) = 0$ (because of the parallelism of the vector $d\boldsymbol{t}$ of infinitesimal translation of $\boldsymbol{t}$ to the plane (π)).

In order to evaluate $\dfrac{d\mu^k}{dr_{(p)}}$ we apply formula (1.35) in the case where $B = \boldsymbol{t}$ and then we obtain

$$\frac{d\boldsymbol{t}}{dr_{(p)}} = \boldsymbol{t}_k\mu^k + \frac{\partial \boldsymbol{t}}{\partial \mu^k}\cdot\frac{d\mu^k}{dr_{(p)}}. \tag{1.38}$$

But

$$\boldsymbol{t}_k = \partial_k(\boldsymbol{r}_i\mu^i) = \boldsymbol{r}_{ik}\mu^i$$

and

$$\frac{\partial \boldsymbol{t}}{\partial \mu^k} = \frac{\partial}{\partial \mu^k}(\boldsymbol{r}_i\mu^i) = \boldsymbol{r}_i\frac{\partial \mu^i}{\partial \mu^k} = \boldsymbol{r}_i\delta^i_k = \boldsymbol{r}_k.$$

Taking it into account we obtain from (1.38)

$$\frac{d\boldsymbol{t}}{dr_{(p)}} = \boldsymbol{r}_{ik}\mu^i\mu^k + \boldsymbol{r}_k\frac{d\mu^k}{dr_{(p)}}. \tag{1.39}$$

Denoting by $\boldsymbol{w}$ the vector

$$\boldsymbol{w} = \boldsymbol{r}_{ik}\mu^i\mu^k, \tag{1.40}$$

relation (1.39) may be written in the form

$$\frac{d\boldsymbol{t}}{dr_{(p)}} = \boldsymbol{w} + \boldsymbol{r}_k\frac{d\mu^k}{dr_{(p)}}.$$

Multiplying scalarly by $\boldsymbol{t}$ and by $\boldsymbol{p}$ the above equality and taking into account conditions (1.36) and (1.37), we get the following system of equations:

$$\begin{aligned}(\boldsymbol{t}\boldsymbol{w}) + (\boldsymbol{t}\boldsymbol{r}_k)\frac{d\mu^k}{dr_{(p)}} &= 0,\\ (\boldsymbol{p}\boldsymbol{w}) + (\boldsymbol{p}\boldsymbol{r}_k)\frac{d\mu^k}{dr_{(p)}} &= 0.\end{aligned} \tag{1.41}$$

In order to obtain the explicit form of the expressions $\frac{d\mu^k}{dr_{(p)}}$ we denote by D the determinant

$$D = \begin{vmatrix} (\boldsymbol{t}\boldsymbol{r}_1) & (\boldsymbol{t}\boldsymbol{r}_2) \\ (\boldsymbol{p}\boldsymbol{r}_1) & (\boldsymbol{p}\boldsymbol{r}_2) \end{vmatrix} = ([\boldsymbol{t}\boldsymbol{p}][\boldsymbol{r}_1\boldsymbol{r}_2]) = \sqrt{g}(\boldsymbol{t}\boldsymbol{p}\boldsymbol{m})\,;$$

thus

$$D = -\sqrt{g}(\boldsymbol{p}\boldsymbol{l}) \tag{1.42}$$

and by D^k the determinants

$$D^k = \varepsilon^{ik}\begin{vmatrix} (\boldsymbol{t}\boldsymbol{r}_i) & (\boldsymbol{t}\boldsymbol{w}) \\ (\boldsymbol{p}\boldsymbol{r}_i) & (\boldsymbol{p}\boldsymbol{w}) \end{vmatrix} = \varepsilon^{ik}([\boldsymbol{t}\boldsymbol{p}][\boldsymbol{r}_i\boldsymbol{w}])$$

or

$$D^k = -\varepsilon^{ik}(\boldsymbol{q}\boldsymbol{w}\boldsymbol{r}_i)\,, \tag{1.43}$$

where we introduce the vector

$$\boldsymbol{q} \overset{\text{def}}{=} [\boldsymbol{t}\boldsymbol{p}]\,. \tag{1.44}$$

Since

$$\frac{d\mu^k}{dr_{(p)}} = \frac{D^k}{D}\,, \tag{1.45}$$

we finally have

$$\frac{d\mu^k}{dr_{(p)}} = \frac{\varepsilon^{ik}(\boldsymbol{q}\boldsymbol{w}\boldsymbol{r}_i)}{\sqrt{g}(\boldsymbol{p}\boldsymbol{l})} \tag{1.46}$$

([1], p. 51, formulae (44)).

The particular case of the notion considered is the so called geodesic derivative ([1], p. 77-79) of the quantity $B(\boldsymbol{r}, \boldsymbol{t})$. It is the derivative (1.35) taken in the plane normal to the surface. Then

$$\boldsymbol{p} = \boldsymbol{l} \quad \text{and} \quad \boldsymbol{q} = -\boldsymbol{m} \tag{1.47}$$

and expression (1.46) becomes

$$\frac{d\mu^k}{dr_{(l)}} = -\frac{1}{\sqrt{g}}\,\varepsilon^{ik}(\boldsymbol{m}\boldsymbol{w}\boldsymbol{r}_i) \tag{1.48}$$

([1], p. 78, formulae (164)).

We shall be further concerned essentially with the geodesic derivative of the longitudinal curvature λ. It is easy to prove that the quantity obtained is an invariant of the transformations of the curvilinear system of coordinates on the surface and that the following relation holds:

$$\frac{d\lambda}{dr_{(l)}} = -\left(\boldsymbol{t}\,\frac{d^2\boldsymbol{m}}{dr^2}\right). \tag{1.49}$$

II. Vertical directions

1. Let us consider a regular piece of surface (S) of class C^3 and a plane curve (K) lying on it and passing through a given point M of this surface. Let the curve (K) be a line of intersection of the surface (S) and a plane (P) normal to it at the point M and passing in the given direction $\boldsymbol{t}$ (Fig. 1). The plane (P) evidently contains the unit normal vector $\boldsymbol{m}$ and is perpendicular to the vector $\boldsymbol{l} = [\boldsymbol{tm}]$. Suppose that the point M is a vertex of the plane curve (K), i.e. that at this point

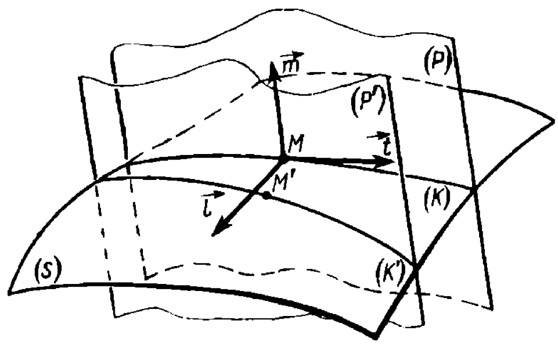

Fig. 1.

$$\frac{d\varkappa}{ds} = 0$$

where $\varkappa$ is the curvature of (K) and s denotes its natural parameter (the length of the arc). As we know, the curvature $\varkappa$ of the normal intersection (K) is equal to the normal curvature $\varkappa_n$ of this curve and $\varkappa_n$ coincides at the given point with the longitudinal curvature λ of the surface in the direction $\boldsymbol{t}$ tangent to (K) at this point.

Now we want to find at the point M such a direction $\boldsymbol{l}$ (orthogonal to $\boldsymbol{t}$) that if we pass from M to the neighbouring point M' of surface (S) in this direction and if we cut the surface by the plane (P') normal to it at M and parallel to plane (P), the vertex of the line of intersection (K') obtained in this way will be at M'. This leads directly (see Chap. I, p. 144) to the condition for such a direction: the derivative of λ at point M in the direction $\boldsymbol{t}$ and in the plane (P), or shortly the geodesic derivative of λ, must be zero. Thus we introduce the following definition.

DEFINITION 1. The *vertical direction* at a given point M is a direction of the vector $\boldsymbol{l}$ for which the geodesic derivative of longitudinal curvature λ vanishes.

According to this we may find the vertical directions from the equation

$$\frac{d\lambda}{dr_{(l)}} = 0\,. \tag{2.1}$$

Since [form. (1.35)]

$$\frac{d\lambda}{dr_{(l)}} = \frac{\partial\lambda}{\partial u^k}\mu^k + \frac{\partial\lambda}{\partial\mu^k}\cdot\frac{d\mu^k}{dr_{(l)}}, \tag{2.2}$$

condition (2.1) takes the form

$$\frac{\partial\lambda}{\partial u^k}\mu^k + \frac{\partial\lambda}{\partial\mu^k}\cdot\frac{d\mu^k}{dr_{(l)}} = 0\,, \tag{2.3}$$

where [form. (1.48)]

$$\frac{d\mu^k}{dr_{(l)}} = -\frac{1}{\sqrt{g}}\varepsilon^{lk}(\boldsymbol{m}\boldsymbol{w}\boldsymbol{r}_l)\,. \tag{2.4}$$

By means of (1.40) these expressions may be written in another way namely as the quadratic forms

$$\frac{d\mu^k}{dr_{(l)}} = -\frac{1}{\sqrt{g}}\varepsilon^{lk}(\boldsymbol{m}\boldsymbol{r}_{ij}\boldsymbol{r}_l)\mu^i\mu^j\,. \tag{2.5}$$

Their coefficients may easily be expressed by means of the Christoffel symbols of the second kind. It is well known ([2], p. 123, form. 2) that

$$\boldsymbol{r}_{ij} = \Gamma^p_{ij}\boldsymbol{r}_p + h_{ij}\boldsymbol{m}\,.$$

Multiplying vectorially the above equality by $\boldsymbol{r}_l$ and next scalarly by $\boldsymbol{m}$, we obtain

$$(\boldsymbol{r}_l\boldsymbol{r}_{ij}\boldsymbol{m}) = \Gamma^p_{ij}(\boldsymbol{r}_l\boldsymbol{r}_p\boldsymbol{m}) = \sqrt{g}\,\varepsilon_{lp}\Gamma^p_{ij} \qquad \text{or} \qquad \varepsilon_{lp}\Gamma^p_{ij} = \frac{1}{\sqrt{g}}(\boldsymbol{r}_l\boldsymbol{r}_{ij}\boldsymbol{m})\,.$$

Finally multiplying both sides of this relation by ε^{lk} and using (1.19), we get

$$\Gamma^k_{ij} = -\frac{1}{\sqrt{g}}\varepsilon^{lk}(\boldsymbol{r}_l\boldsymbol{r}_{ij}\boldsymbol{m})\,. \tag{2.6}$$

Thus finally (2.5) takes the form

$$\frac{d\mu^k}{dr_{(l)}} = -\Gamma^k_{ij}\mu^i\mu^j\,. \tag{2.7}$$

Since the derivatives $\dfrac{\partial\lambda}{\partial u^k}$ and $\dfrac{\partial\lambda}{\partial\mu^k}$ have the following expressions:

$$\frac{\partial\lambda}{\partial u^k} = \partial_k h_{ij}\mu^i\mu^j \tag{2.8}$$

and

$$\frac{\partial\lambda}{\partial\mu^k} = \frac{\partial}{\partial\mu^k}(h_{ij}\mu^i\mu^j) = h_{ij}\left(\frac{\partial\mu^i}{\partial\mu^k}\mu^j + \frac{\partial\mu^j}{\partial\mu^k}\mu^i\right) = 2h_{ik}\mu^i\,, \tag{2.9}$$

equation (2.3) may be written in the form

$$(\partial_k h_{ij} - 2h_{kl}\Gamma^l_{ij})\mu^i\mu^j\mu^k = 0\,, \tag{2.10}$$

or shortly

$$P_{ijk}\mu^i\mu^j\mu^k = 0\,, \tag{2.11}$$

where the coefficients P_{ijk} are

$$P_{ijk} \overset{\text{def}}{=} \partial_k h_{ij} - 2h_{kl}\Gamma^l_{ij}\,. \tag{2.12}$$

The symmetry of P_{ijk} with respect to the first two suffixes implies only six different coefficients (2.12) and equation (2.11) becomes

$$P_{111}(\mu^1)^3+(P_{112}+2P_{121})(\mu^1)^2\mu^2+(P_{221}+2P_{122})\mu^1(\mu^2)^2+P_{222}(\mu^2)^3=0\,.$$

Denoting by P, Q, R and S the four coefficients of the cubic form in the left side of the above equation, we have

$$P(\mu^1)^3+Q(\mu^1)^2\mu^2+R\mu^1(\mu^2)^2+S(\mu^2)^3=0 \tag{2.13}$$

where

$$\begin{aligned}
P &= P_{111} = \partial_1 h_{11}-2h_{11}\Gamma^1_{11}-2h_{12}\Gamma^2_{11}\,,\\
Q &= P_{112}+2P_{121} = \partial_2 h_{11}+2\partial_1 h_{12}-4h_{11}\Gamma^1_{12}-2h_{12}\Gamma^1_{11}-4h_{12}\Gamma^2_{12}-2h_{22}\Gamma^2_{11}\\
R &= P_{221}+2P_{122} = \partial_1 h_{22}+2\partial_2 h_{12}-4h_{22}\Gamma^2_{12}-2h_{12}\Gamma^2_{22}-4h_{12}\Gamma^1_{12}-2h_{11}\Gamma^1_{22}\\
S &= P_{222} = \partial_2 h_{22}-2h_{12}\Gamma^1_{22}-2h_{22}\Gamma^2_{22}\,.
\end{aligned} \tag{2.14}$$

It is easy to prove that the coefficients P_{ijk} do not represent any tensor although the whole equation (2.11) is an invariant under a parameter transformation (see Chap. I, p. 144). We may thus infer that there exists a form of this equation for which the above-mentioned coefficients are already the components of some tensor. Indeed, if we write according to (2.9)

$$\frac{\partial\lambda}{\partial\mu^k}=h_{kj}\mu^j+h_{ik}\mu^i\,,$$

then, substituting this together with (2.7) and (2.8) in (2.2), we obtain

$$\begin{aligned}
\frac{d\lambda}{dr_{(l)}} &= (\partial_k h_{ij})\mu^i\mu^j\mu^k-h_{jl}\Gamma^l_{ik}\mu^i\mu^j\mu^k-h_{il}\Gamma^l_{jk}\mu^i\mu^j\mu^k\\
&= (\partial_k h_{ij}-h_{jl}\Gamma^l_{ik}-h_{il}\Gamma^l_{jk})\mu^i\mu^j\mu^k\,.
\end{aligned}$$

The expression in brackets is a covariant derivative of the second fundamental tensor h_{ij} of the surface, and therefore we may write

$$\frac{d\lambda}{dr_{(l)}}=\nabla_k h_{ij}\mu^i\mu^j\mu^k\,, \tag{2.15}$$

where

$$\nabla_k h_{ij}=\partial_k h_{ij}-\Gamma^l_{ik}h_{jl}-\Gamma^l_{jk}h_{il}\,. \tag{2.16}$$

The covariant derivative $\nabla_k h_{ij}$ is a tensor already (the so called Codazzi tensor (see [5], p. 45)) (4). But in further calculations it will be more convenient to use the coefficients P_{ijk}.

(4) Equation (2.11) is equivalent to the relation $C_{ijk}\mu^i\mu^j\mu^k=0$, where $C_{ijk}\stackrel{\text{def}}{=}\nabla_k h_{ij}$ is a symmetric part of the Codazzi tensor.

The variables μ^k according to (1.7) characterize the direction $\boldsymbol{t}$. Now we want to introduce the variables ω^k characterizing the direction $\boldsymbol{l}$. Inserting (1.21) in (2.11) we get

$$P_{ijk}\frac{1}{\sqrt{g}}g_{\alpha l}\varepsilon^{i\alpha}\omega^l\frac{1}{\sqrt{g}}g_{\beta m}\varepsilon^{j\beta}\omega^m\frac{1}{\sqrt{g}}g_{\gamma n}\varepsilon^{k\gamma}\omega^n = 0$$

or shortly

$$p_{lmn}\omega^l\omega^m\omega^n = 0\,, \tag{2.17}$$

where the new coefficients p_{lmn} are

$$p_{lmn} = g_{\alpha l}\varepsilon^{i\alpha}g_{\beta m}\varepsilon^{j\beta}g_{\gamma n}\varepsilon^{k\gamma}P_{ijk}\,. \tag{2.18}$$

Also these coefficients are symmetric with respect to the first two suffixes and, just as (2.11), equation (2.17) reduces to

$$p(\omega^1)^3+q(\omega^1)^2\omega^2+r\omega^1(\omega^2)^2+s(\omega^2)^3 = 0\,, \tag{2.19}$$

where we write

$$\begin{aligned}
p &= p_{111} = P(g_{12})^3-Q(g_{12})^2g_{11}+Rg_{12}(g_{11})^2-S(g_{11})^3\,,\\
q &= p_{112}+2p_{121} = 3P(g_{12})^2g_{22}-Qg_{12}[(g_{12})^2+2g_{11}g_{22}]+\\
&\qquad +Rg_{11}[g_{11}g_{22}+2(g_{12})^2]-3S(g_{11})^2g_{12}\,,\\
r &= p_{221}+2p_{122} = 3Pg_{12}(g_{22})^2-Qg_{22}[g_{11}g_{22}+2(g_{12})^2]+\\
&\qquad +Rg_{12}[(g_{12})^2+2g_{11}g_{22}]-3Sg_{11}(g_{12})^2\,,\\
s &= p_{222} = P(g_{22})^3-Qg_{12}(g_{22})^2+R(g_{12})^2g_{22}-S(g_{12})^3\,.
\end{aligned} \tag{2.20}$$

The formulae obtained may be simplified if we introduce on our surface the orthogonal curvilinear coordinates. Then we have $g_{12} = 0$, and (2.20) reduces to

$$\begin{aligned}
p &= -S(g_{11})^3\,,\\
q &= R(g_{11})^2g_{22}\,,\\
r &= -Qg_{11}(g_{22})^2\,,\\
s &= P(g_{22})^3\,.
\end{aligned} \tag{2.21}$$

Although then the form of the coefficients P, Q, R and S does not change, but the Christoffel symbols Γ^k_{ij} in formulae (2.14) take the following form:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2g_{11}}\partial_1 g_{11}\,, &\quad \Gamma^2_{11} &= -\frac{1}{2g_{22}}\partial_2 g_{11}\,,\\
\Gamma^1_{12} &= \frac{1}{2g_{11}}\partial_2 g_{11}\,, &\quad \Gamma^2_{12} &= \frac{1}{2g_{22}}\partial_1 g_{22}\,,\\
\Gamma^1_{22} &= -\frac{1}{2g_{11}}\partial_1 g_{22}\,, &\quad \Gamma^2_{22} &= \frac{1}{2g_{22}}\partial_2 g_{22}\,.
\end{aligned} \tag{2.22}$$

For further investigations we want to simplify relations (2.14) still more. We can do this assuming that the parametric lines $u^1 = \text{const}$ and $u^2 = \text{const}$ coincide with the curvature lines of the surface ([5]). Then, as we know, besides the condition of orthogonality $g_{12} = 0$, the condition of conjugatness, i.e. $h_{12} = 0$ must be satisfied. Thus coefficients (2.14) by the use of (2.22) become

$$
\begin{aligned}
P &= \partial_1 h_{11} - \frac{h_{11}}{g_{11}} \partial_1 g_{11} = g_{11} \partial_1 \left(\frac{h_{11}}{g_{11}}\right), \\
Q &= \partial_2 h_{11} - \partial_2 g_{11} \left(2 \frac{h_{11}}{g_{11}} - \frac{h_{22}}{g_{22}}\right), \\
R &= \partial_1 h_{22} - \partial_1 g_{22} \left(2 \frac{h_{22}}{g_{22}} - \frac{h_{11}}{g_{11}}\right), \\
S &= \partial_2 h_{22} - \frac{h_{22}}{g_{22}} \partial_2 g_{22} = g_{22} \partial_2 \left(\frac{h_{22}}{g_{22}}\right).
\end{aligned}
\tag{2.23}
$$

Since in the coordinate system formed of curvature lines the principal curvatures λ_1 and λ_2 are equal

$$
\lambda_1 = h_{11}/g_{11}, \qquad \lambda_2 = h_{22}/g_{22}, \tag{2.24}
$$

we may write (2.23) in the following way:

$$
\begin{aligned}
P &= g_{11} \partial_1 \lambda_1, \\
Q &= \partial_2 h_{11} - \partial_2 g_{11} (2\lambda_1 - \lambda_2), \\
R &= \partial_1 h_{22} - \partial_1 g_{22} (2\lambda_2 - \lambda_1), \\
S &= g_{22} \partial_2 \lambda_2.
\end{aligned}
\tag{2.25}
$$

The two middle coefficients Q and R with respect to the Mainardi-Codazzi equations, which in our curvature system are

$$
\begin{aligned}
\partial_2 h_{11} &= \tfrac{1}{2}(\lambda_1 + \lambda_2) \partial_2 g_{11}, \\
\partial_1 h_{22} &= \tfrac{1}{2}(\lambda_1 + \lambda_2) \partial_1 g_{22}
\end{aligned}
\tag{2.26}
$$

and with respect to the relations

$$
\begin{aligned}
\partial_2 \lambda_1 &= \frac{1}{2g_{11}} \partial_2 g_{11} (\lambda_2 - \lambda_1), \\
\partial_1 \lambda_2 &= -\frac{1}{2g_{22}} \partial_1 g_{22} (\lambda_2 - \lambda_1)
\end{aligned}
\tag{2.27}
$$

obtained by the differentiation of expressions (2.24), take the final form

$$
\begin{aligned}
Q &= \tfrac{3}{2} \partial_2 g_{11} (\lambda_2 - \lambda_1) = 3 g_{11} \partial_2 \lambda_1, \\
R &= -\tfrac{3}{2} \partial_1 g_{22} (\lambda_2 - \lambda_1) = 3 g_{22} \partial_1 \lambda_2.
\end{aligned}
$$

([5]) Such a system of coordinates is regular everywhere except at the umbilic points, which we exclude from general considerations.

So in the curvature lines system the coefficients of equation (2.13) are

$$
(2.28) \qquad \begin{aligned} P &= g_{11}\partial_1\lambda_1, & R &= 3g_{22}\partial_1\lambda_2, \\ Q &= 3g_{11}\partial_2\lambda_1, & S &= g_{22}\partial_2\lambda_2 \end{aligned}
$$

and the coefficients of equation (2.19) may be expressed by the formulae

$$
(2.29) \qquad \begin{aligned} p &= -(g_{11})^3 g_{22}\partial_2\lambda_2, & r &= -3(g_{11})^2(g_{22})^2\partial_2\lambda_1, \\ q &= 3(g_{11})^2(g_{22})^2\partial_1\lambda_2, & s &= g_{11}(g_{22})^3\partial_1\lambda_1 . \end{aligned}
$$

Finally the equation for vertical directions (2.19) may be written in the following form:

$$
(2.30) \qquad (g_{11})^2\partial_2\lambda_2(\omega^1)^3 - 3g_{11}g_{22}\partial_1\lambda_2(\omega^1)^2\omega^2 + \\ + 3g_{11}g_{22}\partial_2\lambda_1\omega^1(\omega^2)^2 - (g_{22})^2\partial_1\lambda_1(\omega^2)^3 = 0
$$

or

$$
(2.31) \qquad g_{11}(\omega^1)^2[g_{11}\partial_2\lambda_2\omega^1 - 3g_{22}\partial_1\lambda_2\omega^2] - g_{22}(\omega^2)^2[g_{22}\partial_1\lambda_1\omega^2 - 3g_{11}\partial_2\lambda_1\omega^1] = 0 .
$$

We see from (2.31) that if one of the principal curvatures of the surface is constant, which is valid for so called pipe surfaces ([4], p. 75), then this equation has a double root $(\omega^1)^2 = 0$ or $(\omega^2)^2 = 0$; the second root satisfies the equation

$$
(2.32) \qquad g_{11}\partial_2\lambda_2\omega^1 - 3g_{22}\partial_1\lambda_2\omega^2 = 0
$$

or the equation

$$
(2.33) \qquad g_{22}\partial_1\lambda_1\omega^2 - 3g_{11}\partial_2\lambda_1\omega^1 = 0 .
$$

This case in particular occurs for all developable surfaces, for which, as is well known, one of the principal curvatures vanishes. The only exceptions are: the plane — then also the second principal curvature is equal to zero — and the cylinder of revolution, for which the second principal curvature is constant. In these cases the equation of vertical directions is identically satisfied ([6]). Moreover, the identical vanishing of equation (2.30) holds for the sphere. These are of course all the possible cases. And thus we have the following theorem:

THEOREM 1. *The only surfaces for which every direction is vertical are: the plane, the sphere and the cylinder of revolution.*

2. For further applications of the notion of a vertical direction it will be convenient to express the coefficients P_{ijk} and P, Q, R, S, by the derivatives of the radius-vector $\boldsymbol{r}$ of a surface. To begin with, we shall prove that for the second member on the right side of formula (2.2) we have the relation

$$
(2.34) \qquad \frac{\partial\lambda}{\partial\mu^k}\cdot\frac{d\mu^k}{dr_{(1)}} = 2(\boldsymbol{w}\boldsymbol{m}_i)\mu^i .
$$

([6]) Compare with the theorem about the vanishing of the Codazzi tensor ([5], p. 46).

Indeed, since we may write

$$[\boldsymbol{r}_j \boldsymbol{m}] = \frac{1}{\sqrt{g}} \varepsilon^{ps}(\boldsymbol{r}_j \boldsymbol{r}_p)\boldsymbol{r}_s = \frac{1}{\sqrt{g}} \varepsilon^{ps} g_{jp} \boldsymbol{r}_s ,$$

we obtain from (2.4)

$$\frac{d\mu^k}{dr_{(l)}} = -\frac{1}{\sqrt{g}} \varepsilon^{jk}(\boldsymbol{r}_j \boldsymbol{m}\boldsymbol{w}) = -\frac{1}{\sqrt{g}} \varepsilon^{jk} \frac{1}{\sqrt{g}} \varepsilon^{ps} g_{jp}(\boldsymbol{r}_s \boldsymbol{w}) = -g^{ks}(\boldsymbol{r}_s \boldsymbol{w}) \tag{2.35}$$

and in consequence

$$\frac{\partial\lambda}{\partial\mu^k} \cdot \frac{d\mu^k}{dr_{(l)}} = -2h_{ik}\mu^i g^{ks}(\boldsymbol{r}_s \boldsymbol{w}) = -2h_i^s(\boldsymbol{r}_s \boldsymbol{w})\mu^i .$$

By the Weingartenf ormulae ([2], p. 194, form. 4) $\boldsymbol{m}_i = -h_i^s \boldsymbol{r}_s$, we finally have

$$\frac{\partial\lambda}{\partial\mu^k} \cdot \frac{d\mu^k}{dr_{(l)}} = 2(\boldsymbol{m}_i \boldsymbol{w})\mu^i ,$$

which was to be proved.

Since

$$\lambda = h_{ij}\mu^i\mu^j = (\boldsymbol{m}\boldsymbol{r}_{ij})\mu^i\mu^j = (\boldsymbol{m}\boldsymbol{w}) ,$$

we have

$$\partial_k \lambda \mu^k = (\boldsymbol{m}_k \boldsymbol{w})\mu^k + (\boldsymbol{m}\boldsymbol{w}_k)\mu^k$$

and the geodesic derivative of the longitudinal curvature λ may be represented by the relation

$$\frac{d\lambda}{dr_{(l)}} = (\boldsymbol{m}_k \boldsymbol{w})\mu^k + (\boldsymbol{m}\boldsymbol{w}_k)\mu^k + 2(\boldsymbol{m}_k \boldsymbol{w})\mu^k = (\boldsymbol{m}\boldsymbol{w}_k)\mu^k + 3(\boldsymbol{m}_k \boldsymbol{w})\mu^k .$$

Next we have

$$\boldsymbol{w}_k = \partial_k(\boldsymbol{r}_{ij}\mu^i\mu^j) = \boldsymbol{r}_{ijk}\mu^i\mu^j$$

and finally

$$\frac{d\lambda}{dr_{(l)}} = (\boldsymbol{m}\boldsymbol{r}_{ijk})\mu^i\mu^j\mu^k + 3(\boldsymbol{m}_k \boldsymbol{r}_{ij})\mu^i\mu^j\mu^k . \tag{2.36}$$

It follows from the above that the coefficients P_{ijk} may be expressed in the form

$$P_{ijk} = (\boldsymbol{m}\boldsymbol{r}_{ijk}) + 3(\boldsymbol{m}_k \boldsymbol{r}_{ij}) \tag{2.37}$$

and the coefficients P, Q, R and S in the form

$$\begin{aligned} P &= (\boldsymbol{m}\boldsymbol{r}_{111}) + 3(\boldsymbol{m}_1 \boldsymbol{r}_{11}) , \\ Q &= 3[(\boldsymbol{m}\boldsymbol{r}_{112}) + (\boldsymbol{m}_2 \boldsymbol{r}_{11}) + 2(\boldsymbol{m}_1 \boldsymbol{r}_{12})] , \\ R &= 3[(\boldsymbol{m}\boldsymbol{r}_{221}) + (\boldsymbol{m}_1 \boldsymbol{r}_{22}) + 2(\boldsymbol{m}_2 \boldsymbol{r}_{12})] , \\ S &= (\boldsymbol{m}\boldsymbol{r}_{222}) + 3(\boldsymbol{m}_2 \boldsymbol{r}_{22}) . \end{aligned} \tag{2.38}$$

3. By means of the classification of cubic forms of two variables we now want to determine the number of solutions of equation (2.13). As is well known, ([6], p. 171, tab. V), the complete system of algebraic concomitants for the form

$$f = a_{ijk}\mu^i\mu^j\mu^k \tag{2.39}$$

contains, besides the form f, its double hessian, i.e. the quadratic form

$$\mathcal{H} = \frac{1}{18}\det\left|\frac{\partial^2 f}{\partial\mu^\alpha\partial\mu^\beta}\right|, \tag{2.40}$$

its discriminant Δ and the jacobian of the cubic form f and quadratic $\mathcal{H}$

$$\mathcal{J} = \frac{\partial(f, \mathcal{H})}{\partial(\mu^1, \mu^2)}. \tag{2.41}$$

The classification of form (2.39) is based on these four concomitants. To begin with, we assume that f does not vanish identically. The case where the cubic form is identically equal to zero will be considered further in Chap. IV. If $\Delta \neq 0$, then the given form is nonsingular; if $\Delta = 0$, then it is singular.

The nonsingular cubic form splits into three independent linear forms; moreover, if $\Delta < 0$, one of them is real and the other two conjugated complex, and if $\Delta > 0$, all three are real ones.

If the cubic form is singular, then it can split into the product of a linear form and the square of another linear form. Then the covariants $\mathcal{H}$ and $\mathcal{J}$ are not identically zero. If form (2.39) can be written as a cube of a linear form, then $\mathcal{H} \equiv 0$ and $\mathcal{J} \equiv 0$.

As follows from the above, it is sufficient to consider the hessian $\mathcal{H}$ of the cubic form and its discriminant Δ. Now we want to calculate these quantities for the form on the left side of equation (2.13)

$$\mathcal{H} = \frac{2}{9}\begin{vmatrix} 3P\mu^1 + Q\mu^2 & Q\mu^1 + R\mu^2 \\ Q\mu^1 + R\mu^2 & R\mu^1 + 3S\mu^2 \end{vmatrix}$$

$$= \frac{2}{9}[(3P\mu^1 + Q\mu^2)(R\mu^1 + 3S\mu^2) - (Q\mu^1 + R\mu^2)^2],$$

and finally the hessian $\mathcal{H}$ has the form

$$\mathcal{H} = \tfrac{2}{9}\{[3PR - Q^2](\mu^1)^2 + [9PS - QR]\mu^1\mu^2 + [3QS - R^2](\mu^2)^2\}. \tag{2.42}$$

Its discriminant

$$\Delta = \frac{1}{81}\begin{vmatrix} 2(3PR - Q^2) & 9PS - QR \\ 9PS - QR & 2(3QS - R^2) \end{vmatrix}$$

$$= \frac{1}{81}[4(3PR - Q^2)(3QS - R^2) - (9PS - QR)^2]$$

may be written finally as follows:

$$\Delta = \tfrac{1}{27}(12PQRS - 27P^2S^2 - 4PR^3 - 2SQ^3 + Q^2R^2)\,. \tag{2.43}$$

If at the given point M expression (2.43) is equal to zero, then at this point there exist on the surface (S) one or two real vertical directions. Moreover, in this case the following equalities

$$3PR - Q^2 = 9PS - QR = 3QS - R^2 = 0 \tag{2.44}$$

must be satisfied.

One real vertical direction may be obtained also when Δ is different from zero and negative. The existence of three real different vertical directions occurs only if expression (2.43) is positive.

III. Vertical lines

1. The notion of vertical direction introduced in Chap. II is for us rather an auxiliary notion. The fundamental one is the idea of a vertical line defined as follows:

DEFINITION 2. A curve on a surface (S) whose tangent at each point is along a vertical direction is called a *vertical line.*

To obtain a differential equation of the vertical lines on a surface (S), we notice that, similarly to $\mu^k = du^k/dr$, (1.9), the coefficients ω^k may be represented in the form

$$\omega^k = \frac{dv^k}{dr}\,, \tag{3.1}$$

where the system of lines $v^1 = \text{const}$ and $v^2 = \text{const}$ is orthogonal to the system of the parameter lines: $u^1 = \text{const}$, $u^2 = \text{const}$. Inserting (3.1) in (2.19) and multiplying by $(dr)^2$, we get the following equation of the family of vertical lines

$$p\,(dv^1)^3 + q\,(dv^1)^2dv^2 + r\,dv^1(dv^2)^2 + s\,(dv^2)^3 = 0\,, \tag{3.2}$$

where the coefficients p, q, r, s are given for an arbitrary system by formulae (2.20) and for the system formed by the lines of curvature — by formulae (2.29). To simplify the calculations we shall further use the curvature system on the surface in question (then of course lines $v^k = \text{const}$ and $u^k = \text{const}$ coincide). So the equation of vertical lines (see (2.30)) may be written as follows:

$$\begin{aligned}(g_{11})^2\partial_2\lambda_2(du^1)^3 - 3g_{11}g_{22}\partial_1\lambda_2(du^1)^2du^2 &+ \\ + 3g_{11}g_{22}\partial_2\lambda_1du^1(du^2)^2 - (g_{22})^2\partial_1\lambda_1(du^2)^3 &= 0\,.\end{aligned} \tag{3.3}$$

The solutions of this equation represent in general three one-parameter families of curves on the surface (S).

In some cases (compare with example 1b below) it is more suitable to use the differential equation of the orthogonal trajectories of the family of vertical lines, viz.

$$P(du^1)^3 + Q(du^1)^2du^2 + Rdu^1(du^2)^2 + S(du^2)^3 = 0\,, \tag{3.4}$$

obtained in the same manner as equation (3.3).

2. Let us now consider the problem of vertical lines on several particular classes of surfaces.

1° At first we investigate developable surfaces. Excluding the cases of the plane and of the cylinder of revolution (see Chap. II, p. 148) we assume that, for example, $\lambda_2 = 0$ while the second principal curvature $\lambda_1 = 2H \neq \text{const}$. So the equation of vertical lines on developable surfaces is

$$g_{22}(du^2)^2[g_{22}\partial_1 H\, du^2 - 3g_{11}\partial_2 H\, du^1] = 0\,. \tag{3.5}$$

One (double) family of vertical lines coincides with the lines of curvature $u^2 = \text{const}$, the second one is a solution of the equation

$$g_{22}\partial_1 H\, du^2 - 3g_{11}\partial_2 H\, du^1 = 0\,. \tag{3.6}$$

Now we want to consider the vertical lines on particular kinds of developable surfaces, i.e. on cylinders, cones and torsoids.

a) Cylindric surfaces. An arbitrary cylindric surface, whose generators $u^1 = \text{const}$ are along the direction of the unit vector $\boldsymbol{e}$, and whose base curve $\boldsymbol{\rho} = \boldsymbol{\rho}(u^1)$ (u^1 — natural parameter) lies in the plane perpendicular to $\boldsymbol{e}$, may be represented by the vector equation

$$\boldsymbol{r}(u^1, u^2) = \boldsymbol{\rho}(u^1) + u^2\boldsymbol{e}\,. \tag{3.7}$$

Hence we find that

$$\boldsymbol{r}_1 = \boldsymbol{\rho}'\,, \qquad \boldsymbol{r}_2 = \boldsymbol{e}\,,$$

$$g_{11} = 1\,, \qquad g_{12} = 0\,, \qquad g_{22} = 1\,, \qquad g = 1$$

and

$$\boldsymbol{r}_{11} = \boldsymbol{\rho}''\,, \qquad \boldsymbol{r}_{12} = 0\,, \qquad \boldsymbol{r}_{22} = 0\,,$$

$$h_{11} = -(\boldsymbol{e}\boldsymbol{\rho}'\boldsymbol{\rho}'')\,, \qquad h_{12} = 0\,, \qquad h_{22} = 0\,,$$

so that

$$\lambda_1 = -(\boldsymbol{e}\boldsymbol{\rho}'\boldsymbol{\rho}'')\,, \qquad \lambda_2 = 0$$

and finally

$$H = -\tfrac{1}{2}(\boldsymbol{e}\boldsymbol{\rho}'\boldsymbol{\rho}'')\,. \tag{3.8}$$

Since the mean curvature H does not depend here of u^2, we have $\partial_2 H = 0$ and according to (3.6) we conclude that besides the lines $u^2 = \text{const}$, i.e. the family of curves parallel to the base curve of the cylinder, also some of the generators $u^1 = \text{const}$ are vertical lines, namely those, which pass through the points of the base curve for which

$$\frac{d}{du^1}(\boldsymbol{e}\boldsymbol{\rho}'\boldsymbol{\rho}'') = 0 . \tag{3.9}$$

But $(\boldsymbol{e}\boldsymbol{\rho}'\boldsymbol{\rho}'') = |\varkappa|$ where $\varkappa$ denotes the curvature of the base curve and (3.9) coincide with the condition for the existence of vertices. It means that to vertical lines of cylindric surface belong those generators which pass through the vertices of the base curve $\boldsymbol{\rho} = \boldsymbol{\rho}(u^1)$.

b) Cones. If we put the vertex of a cone at the centre 0 of the radius-vectors and as its base curve $\boldsymbol{\rho} = \boldsymbol{\rho}(u^1)$ (u^1 — natural parameter) we take a spherical curve (on the unit sphere with centre at 0), then we may write the vector equation of the cone in the form

$$\boldsymbol{r}(u^1, u^2) = u^2\boldsymbol{\rho}(u^1) . \tag{3.10}$$

Hence

$$\boldsymbol{r}_1 = u^2\boldsymbol{\rho}' , \qquad \boldsymbol{r}_2 = \boldsymbol{\rho} ,$$

$$g_{11} = (u^2)^2 , \qquad g_{12} = 0 , \qquad g_{22} = 1 , \qquad g = (u^2)^2$$

and

$$\boldsymbol{r}_1 = u^2\boldsymbol{\rho}'' , \qquad \boldsymbol{r}_{12} = 0 , \qquad \boldsymbol{r}_{22} = 0 ,$$

$$h_{11} = -u^2(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') , \qquad h_{12} = 0 , \qquad h_{22} = 0 ,$$

so that

$$\lambda_1 = -\frac{1}{u^2}(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') , \qquad \lambda_2 = 0 ,$$

and finally

$$H = -\frac{1}{2u^2}(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') . \tag{3.11}$$

Besides the curves $u^2 = \text{const}$, i.e. the family of orthogonal trajectories of the generators of the cone (lines similar in form to the base curve), we have as the vertical lines the curves which according to (3.6) satisfy the equation

$$-\frac{1}{u^2}(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'')du^2 + 3(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'')du^1 = 0 . \tag{3.12}$$

But $\boldsymbol{\rho} = \boldsymbol{m}$ (the normal to the unit sphere), whence: $\boldsymbol{\rho}' = \boldsymbol{t}$, $\boldsymbol{\rho}'' = \varkappa_1\boldsymbol{n}$ (from the first Frenet formula), and therefore we have

$$(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') = \varkappa_1(\boldsymbol{m}\boldsymbol{t}\boldsymbol{n}) = \varkappa_1 \sin\theta = \varkappa_g , \tag{3.13}$$

where $\varkappa_1$ denotes the curvature of the base curve, $\boldsymbol{n}$ — its unit vector along the principal normal, θ — the angle between vectors $\boldsymbol{n}$ and $\boldsymbol{m}$, and $\varkappa_g$ — the geodesic curvature of the base curve with respect to the sphere.

Now (3.12) may be written as

$$\frac{1}{u^2}\varkappa_g' du^2 + 3\varkappa_g du^1 = 0 . \tag{3.14}$$

From (3.14) it is seen that the vertical lines are the generators passing through those points of the base curve for which

$$\varkappa_g' = 0 . \tag{3.15}$$

These points may be called "the geodesic vertices" of the spherical curve. Evidently the great and small circles of the sphere have at each point the geodesic vertex.

Besides these "particular solutions" of equation (3.12) or (3.14) there exist vertical lines satisfying the equation ($\varkappa_g' \neq 0$)

$$\frac{du^2}{u^2} + 3\frac{\varkappa_g}{\varkappa_g'} du^1 = 0 \tag{3.16}$$

or, in another form,

$$\frac{du^2}{u^2} + \operatorname{ctg}\varphi(u^1)\, du^1 = 0 , \tag{3.17}$$

where $\varphi(u^1)$ is the angle between an arbitrary curve from the family of verticals and the respective generator $u^1 = \text{const}$ of the cone.

Equation (3.16) cannot in general be effectively integrated; on the contrary, this may easily be doen in our case with equation (3.4) of orthogonal trajectories of vertical lines.

On the basis of formulae (2.38) we calculate the coefficients P, Q, R and S for the cone. Since

$$\boldsymbol{r}_{111} = u^2\boldsymbol{\rho}''' , \quad \boldsymbol{r}_{112} = \boldsymbol{\rho}'' , \quad \boldsymbol{r}_{221} = 0 , \quad \boldsymbol{r}_{222} = 0$$

and

$$\boldsymbol{m} = [\boldsymbol{\rho}'\boldsymbol{\rho}] , \quad \boldsymbol{m}_1 = [\boldsymbol{\rho}''\boldsymbol{\rho}'] , \quad \boldsymbol{m}_2 = 0 ,$$

hence

$$P = -u^2(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') , \quad Q = -3(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') , \quad R = 0 , \quad S = 0 .$$

So the equation of orthogonal trajectories of vertical lines for the cone is

$$\frac{du^2}{u^2} + 3\frac{(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}''')}{(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'')} du^1 = 0 .$$

By integration we obtain

$$u^2 = C(\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'')^{-1/3} = C\varkappa_g^{-1/3} , \tag{3.18}$$

where C denotes the constant of integration.

Now we find this family of vertical lines for the elliptic cone. The radius-vector of its base curve has the form

$$\boldsymbol{\rho} = \boldsymbol{a}/a$$

where
$$\boldsymbol{a} = a\cos u^1\boldsymbol{i} + b\sin u^1\boldsymbol{j} + c\boldsymbol{k}$$
and
$$a = |\boldsymbol{a}| = \sqrt{a^2\cos^2 u^1 + b^2\sin^2 u^1 + c^2}\,.$$

The first and the second derivatives of $\boldsymbol{\rho}(u^1)$ are
$$\boldsymbol{\rho}' = \frac{1}{a}\boldsymbol{a}' - \frac{a'}{a^2}\boldsymbol{a}\,, \qquad \boldsymbol{\rho}'' = \frac{1}{a}\boldsymbol{a}'' - \frac{2a'}{a^2}\boldsymbol{a}' + \left(\frac{2a'^2}{a^3} - \frac{a''}{a^2}\right)\boldsymbol{a}$$
and hence

(3.19) $$\varkappa_g = (\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}'') = \frac{1}{a^3}(\boldsymbol{a}\boldsymbol{a}'\boldsymbol{a}'') = \frac{abc}{a^3}\,.$$

Differentiating this we obtain

(3.20) $$\varkappa_g' = (\boldsymbol{\rho}\boldsymbol{\rho}'\boldsymbol{\rho}''') = -3abc\,a'/a^4\,.$$

According to (3.18) the orthogonal trajectories of vertical lines are the curves
$$u^2 = Ca = C\sqrt{a^2\cos^2 u^1 + b^2\sin^2 u^2 + c^2}\,.$$

There are — as is easy to see — ellipses formed by the intersections of the cone by the planes perpendicular to its axis. Now its not difficult to imagine the vertical lines of our elliptic cone. In this case it is even possible to get their equation in an explicit form. Namely, integrating the equation
$$\frac{du^2}{u^2} - \frac{a'}{a}du^1 = 0$$
obtained from (3.16), if we insert (3.19) and (3.20), we have

(3.21) $$u^2 = C\frac{(\sin u^1)^{(a^2+c^2)/(b^2-a^2)}}{(\cos u^1)^{(b^2+c^2)/(b^2-a^2)}}\,.$$

Equation (3.21) gives, together with the above mentioned curves $u^2 = \text{const}$ and the chosen generators (four in this case), all the families of vertical lines on the elliptic cone.

c) Torsoides. The last kind of developable surfaces are the surfaces created by the straight lines tangent to the space curve, i.e. so called torsoids. Their vector equation is

(3.22) $$\boldsymbol{r}(u^1, u^2) = \boldsymbol{\rho}(u^1) + (u^2 - u^1)\boldsymbol{t}(u^1)\,,$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(u^1)$ is the equation of the edge of regression for the torsoid (u^1 — the natural parameter), $\boldsymbol{t} = d\boldsymbol{\rho}/du^1$ — the unit tangent vector of this curve. By the parameters chosen in this manner the straight lines $u^1 = \text{const}$ (the generators of the torsoid) and their orthogonal trajectories $u^2 = \text{const}$ form on this surface the curvature system of coordinates. From (3.22) we get

$$\boldsymbol{r}_1 = (u^2 - u^1)\varkappa_1(u^1)\boldsymbol{n}\,, \qquad \boldsymbol{r}_2 = \boldsymbol{t}(u^1)$$

where $\varkappa_1$ denotes the curvature of the edge of regression and $\boldsymbol{n}$ the unit vector along the principal normal. Further we have

$$g_{11} = (u^2 - u^1)^2\varkappa_1^2\,, \quad g_{12} = 0\,, \quad g_{22} = 1\,, \quad g = (u^2 - u^1)^2\varkappa_1^2\,,$$

and

$$\boldsymbol{r}_{11} = (u^2 - u^1)\varkappa_1\varkappa_2\boldsymbol{k} - (u^2 - u^1)\varkappa_1^2\boldsymbol{t} + \left[(u^2 - u^1)\frac{d\varkappa_1}{du^1} - \varkappa_1\right]\boldsymbol{n}\,,$$

$$\boldsymbol{r}_{12} = \varkappa_1\boldsymbol{n}\,, \qquad \boldsymbol{r}_{22} = 0\,,$$

where $\boldsymbol{k}$ is the unit vector along the binormal of the edge of regression and $\varkappa_2$ its torsion different from zero. Hence follows

$$h_{11} = |u^2 - u^1|\,\varkappa_1\varkappa_2\,, \qquad h_{12} = 0\,, \qquad h_{22} = 0$$

and finally

$$\lambda_1 = \varkappa_2/|u^2 - u^1|\,\varkappa_1\,, \qquad \lambda_2 = 0\,.$$

This implies that the mean curvature for our torsoid has the form

$$H(u^1, u^2) = \varkappa_2/2\,|u^2 - u^1|\,\varkappa_1\,. \tag{3.23}$$

Besides the family of the curves $u^2 = \text{const}$ (see form. (3.5)), to the vertical lines belong the curves satisfying the equation

$$\left[(u^2 - u^1)\left(\varkappa_1\frac{d\varkappa_2}{du^1} - \varkappa_2\frac{d\varkappa_1}{du^1}\right) + \varkappa_1\varkappa_2\right]du^2 - 3(u^2 - u^1)^2\varkappa_1^3\varkappa_2 du^1 = 0 \tag{3.24}$$

where for simplification we consider only one shell $(u^2 - u^1 > 0)$ of the torsoid. It is quite clear that none of its generators $u^1 = \text{const}$ is a vertical line.

Equation (3.24) may be simplified for a certain group of torsoids, viz. when their edge of regression is the generalized screw curve, i.e. the curve for which

$$\varkappa_2/\varkappa_1 = \text{const}\,.$$

Then $\varkappa_1\dfrac{d\varkappa_2}{du^1} - \varkappa_2\dfrac{d\varkappa_1}{du^1} = 0$ and (3.24) reduces to

$$du^2 - 3(u^2 - u^1)\varkappa_1^2(u^1)du^1 = 0\,, \tag{3.25}$$

which after a change of variables may be reduced to the Riccati equation.

2°. The second class of surfaces which we want to investigate with respect to their vertical lines are the surfaces of revolution, of course except the sphere and the cylinder of revolution (see Chap. II, p. 148). The radius-vector of a point of an arbitrary surface of revolution may be written in the form

$$(3.26)\qquad \boldsymbol{r}(u^1, u^2) = f(u^1)\sin u^2 \boldsymbol{i} + f(u^1)\cos u^2 \boldsymbol{j} + u^1 \boldsymbol{k}$$

where the function $f(u^1)$ characterizes the shape of the meridian.

So we have

$$g_{11} = 1 + f'^2(u^1)\,, \quad g_{12} = 0\,, \quad g_{22} = f^2(u^1)\,, \quad g = f^2(u')[1 + f'^2(u^1)]$$

and

$$h_{11} = -f''(u^1)/\sqrt{1+f'^2(u^1)}\,, \quad h_{12} = 0\,, \quad h_{22} = f(u'^1)/\sqrt{1+f'^2(u^1)}\,,$$

whence

$$(3.27)\qquad \lambda_1 = -f''(u^1)/\big(1+f'^2(u^1)\big)^{3/2}\,, \quad \lambda_2 = 1/f(u^1)\sqrt{1+f'^2(u^1)}\,.$$

Since the curvature of meridian $\varkappa_1$ may be represented by the formula

$$\varkappa_1 = f''(u^1)/\big(1+f'^2(u^1)\big)^{3/2}$$

it follows that $\varkappa_1 = -\lambda_1$, moreover $\partial_2\lambda_1 = \partial_1\lambda_2 = 0$ and the equation of vertical lines on the surface of revolution according to (3.3) may be written in the form:

$$f^2(u^1)\,du^2\left[f^2(u^1)\frac{d\varkappa_1}{du^1}(du^2)^2 + 3\big(1+f'^2(u')\big)\frac{d\lambda_2}{du^1}(du^1)^2\right] = 0\,.$$

Rejecting the points of intersection of the meridians with the axis of revolution ($f(u^1) = 0$), which are the singular points of the surface or of the curvilinear system (in our case of the curvature system: they are umbilics), we see that among the vertical lines we recognize the curves $u^2 = \text{const}$, i.e. the meridians, and two other families of vertical lines satisfy the equation

$$(3.28)\qquad f^2(u^1)\frac{d\varkappa_1}{du^1}(du^2)^2 + 3\big(1+f'^2(u^1)\big)\frac{d\lambda_2}{du^1}(du^1)^2 = 0\,.$$

It follows from (3.28) that to the vertical lines belong also those parallels $u^1 = \text{const}$ for which

$$\frac{d\varkappa_1}{du^1} = 0\,,$$

i.e. the parallels passing through the vertices of the meridians.

For some values of u^1 equation (3.28) may have no real solutions. As is easy to prove, the vertical lines satisfying this equation will exist only on those parts of the surface on which the following inequality occurs

$$(3.29)\qquad \frac{d\varkappa_1}{du^1}\cdot\frac{d\lambda_2}{du^1} < 0\,.$$

In particular, some kinds of surfaces of revolution can not have vertical lines which are the solutions of (3.28).

IV. Vertical points

1. Besides the notions of vertical direction and a vertical line, the third notion which we want to introduce is a vertical point or shortly a vertex of a surface. It is defined in the following manner:

DEFINITION 3. The *vertex* of a surface (S) is a point for which all directions are vertical ones.

It follows from this definition that at a vertex the cubic form in (2.19) must identically vanish, i.e. all coefficients p, q, r, s must be simultaneously zeros. In view of their form (2.29) and excluding the singular points of a surface, when $g_{11} = 0$ or $g_{22} = 0$, we see that the vertical points are the solutions of the system of equations

$$\partial_k \lambda_1 = 0 , \quad \partial_k \lambda_2 = 0 . \tag{4.1}$$

This system in which the number of equations is greater than the number of unknowns is, as we know, solvable in some exceptional cases. In particular, when both principal curvatures λ_1 and λ_2 are constant, i.e. when the surface is a sphere, plane or cylinder of revolution (comp. Chap. II, p. 153, theorem 1). Each point of these surfaces is a vertex.

Since the Gaussian curvature K is given by formula (1.34), by differentiating it we obtain

$$\partial_i K = \lambda_1 \partial_i \lambda_2 + \lambda_2 \partial_i \lambda_1 .$$

If conditions (4.1) are satisfied at the point M, this implies the fulfilment at this point of the system

$$\partial_i K = 0 \tag{4.2}$$

and we have the following theorem:

THEOREM 2. *The vertices of a surface are at the same time points at which the necessary conditions for the extreme values of Gaussian curvature are satisfied.*

For the developable surfaces ($K \equiv 0$) system (4.2) is an identity and system (4.1) reduces to the following one:

$$\partial_i H = 0 . \tag{4.3}$$

As follows from our previous considerations (see Chap. III, p. 157, example 1° a, b, c) developable surfaces have no vertices at all but only the vertical lines.

The vertices of pipe surfaces for which one of the principal curvatures is constant may also be obtained from system (4.3).

2. The vertices of the surfaces of revolution may be found only (see Chap. III, p. 161, example 2°) among the roots of equation $f(u^1) = 0$, i.e. among the singular points of the surface or of the curvilinear system:

in our case among the umbilics. Then, of course, through the umbilic which is simultaneously the vertex of a surface pass an infinite number of vertical lines (the meridians) which coincide with the lines of curvature. In other cases the number of vertical lines passing through the vertex can be one, two, three or even zero. A similar situation occurs for example in the case of umbilics and lines of curvature. Now we want to consider this problem for the ellipsoid with unequal axes.

3°. Ellipsoid with unequal axes. The parametric equations of such an ellipsoid ($a^2 > b^2 > c^2$), if the Gaussian system of coordinates is formed by the lines of curvature, are ([7])

$$x^2 = \frac{a^2(a^2-u^1)(a^2-u^2)}{(a^2-b^2)(a^2-c^2)}, \quad y^2 = \frac{b^2(b^2-u^1)(b^2-u^2)}{(b^2-a^2)(b^2-c^2)}, \quad z^2 = \frac{c^2(c^2-u^1)(c^2-u^2)}{(c^2-a^2)(c^2-b^2)}, \tag{4.4}$$

where $a^2 \geqslant u^1 \geqslant b^2$ and $b^2 \geqslant u^2 \geqslant c^2$.

Writing

$$f(u^k) = (a^2-u^k)(b^2-u^k)(c^2-u^k), \tag{4.5}$$

we have

$$g_{11} = \frac{(u^1-u^2)u^1}{4f(u^1)}, \quad g_{12} = 0, \quad g_{22} = -\frac{(u^1-u^2)u^2}{4f(u^2)}$$

and

$$h_{11} = \frac{1}{4}\,abc\frac{u^1-u^2}{\sqrt{u^1u^2}f(u^1)}, \quad h_{12} = 0, \quad h_{22} = -\frac{1}{4}\,abc\frac{u^1-u^2}{\sqrt{u^1u^2}\,f(u^2)},$$

whence

$$\lambda_1 = \frac{abc}{(u^1)^{3/2}(u^2)^{1/2}}, \quad \lambda_2 = \frac{abc}{(u^1)^{1/2}(u^2)^{3/2}}. \tag{4.6}$$

Let us calculate the derivatives of λ_1 and λ_2:

$$\partial_1\lambda_1 = -\frac{3abc}{2(u^1)^{5/2}(u^2)^{1/2}}, \quad \partial_1\lambda_2 = -\frac{abc}{2(u^1)^{3/2}(u^2)^{3/2}}, \quad \partial_2\lambda_1 = -\frac{abc}{2(u^1)^{3/2}(u^2)^{3/2}}, \quad \partial_2\lambda_2 = -\frac{3abc}{2(u^1)^{1/2}(u^2)^{5/2}}. \tag{4.7}$$

As we see, they are always different from zero, and this implies that system (4.1) has no solutions! If we take into account the vanishing of p, q, r, s we obtain in view of (2.29) only the umbilics: $u^1 = u^2 = b^2$. The six vertices of our ellipsoid cannot be found in this way. This follows from the special kind of singularities, appearing in the system of curvature lines on the ellipsoid with unequal axes, viz. on the curves

$$u^1 = a^2 \quad \text{and} \quad u^1 = b^2$$

the coefficient g_{11} takes an infinite value and on the curves

$$u^2 = b^2 \quad \text{and} \quad u^2 = c^2$$

the same applies to the coefficient g_{22}. Since these very curves and their points of intersection are the required vertical lines and vertices of the ellipsoid, to obtain them we must remove from the equation of vertical lines the above-mentioned singularities.

According to (3.3) we have

$$-\frac{3}{32}abc(u^1-u^2)^3\left[\frac{(u^1)^2}{f^2(u^1)(u^1)^{1/2}(u^2)^{5/2}}(du^1)^3 - \right.$$
$$-\frac{u^1u^2}{f(u^1)f(u^2)(u^1)^{3/2}(u^2)^{3/2}}(du^1)^2du^2 - \frac{u^1u^2}{f(u^1)f(u^2)(u^1)^{3/2}(u^2)^{3/2}}du^1(du^2)^2 +$$
$$\left. + \frac{(u^2)^2}{f^2(u^2)(u^1)^{5/2}(u^2)^{1/2}}(du^2)^3\right] = 0 .$$

Multiplying by the product $(u^1)^{5/2}(u^2)^{5/2}f^2(u^1)f^2(u^2)$ we can reduce this equation to the form

$$(u^1)^4f^2(u^2)(du^1)^3-(u^1)^2(u^2)^2f(u^1)f(u^2)(du^1)^2du^2 -$$
$$-(u^1)^2(u^2)^2f(u^1)f(u^2)du^1(du^2)^2+(u^2)^4f^2(u^1)(du^2)^3 = 0$$

or

$$[(u^1)^2f(u^2)du^1-(u^2)^2f(u^1)du^2][(u^1)^2f(u^2)(du^1)^2-(u^2)^2f(u^1)(du^2)^2] = 0 . \tag{4.8}$$

Decomposing the second bracket into linear factors, we have the following equations of the three families of vertical lines for the ellipsoid with unequal axes:

$$\begin{aligned} &(u^1)^2f(u^2)du^1-(u^2)^2f(u^1)du^2 = 0 ,\\ &u^1\sqrt{f(u^2)}\,du^1-u^2\sqrt{f(u^1)}\,du^2 = 0 ,\\ &u^1\sqrt{f(u^2)}\,du^1+u^2\sqrt{f(u^1)}\,du^2 = 0 . \end{aligned} \tag{4.9}$$

We see that now the identical vanishing of the cubic form on the left side of (4.9) occurs for those values of u^1 and u^2 for which at the same time following equations are satisfied

$$f(u^1) = (a^2-u^1)(b^2-u^1)(c^2-u^1) = 0 ,$$
$$f(u^2) = (a^2-u^2)(b^2-u^2)(c^2-u^2) = 0 .$$

These equations give the vertices of our surface which evidently coincide with those known in analytic geometry of quadrics. The particular solutions of equations (4.9) are curves

$$u^1 = a^2 , \quad u^2 = c^2 , \quad u^1 = b^2 \quad \text{or} \quad u^2 = b^2 .$$

At each vertex of the ellipsoid two of these vertical lines intersect (Fig. 2) ([7]). We obtain the remaining verticals by solving equations (4.9). First of them may be expressed by means of elementary functions, namely in the form

$$(4.10)\qquad \left(\frac{a^2-u^2}{a^2-u^1}\right)^{\frac{a^4}{(b^2-a^2)(c^2-a^2)}}\left(\frac{b^2-u^2}{b^2-u^1}\right)^{\frac{b^4}{(a^2-b^2)(c^2-b^2)}}\left(\frac{c^2-u^2}{c^2-u^1}\right)^{\frac{c^4}{(a^2-c^2)(b^2-c^2)}}=A$$

where A denotes the constant of integration. Other two solutions contain the elliptic integrals.

3. The considerations and calculations made in Chap. III point to the existence of a close connexion between the vertical lines of a surface and its lines of curvature. For developable surfaces and the surfaces of revolution (and also for pipe surfaces) one of the families of vertical lines is always the family of lines of curvature.

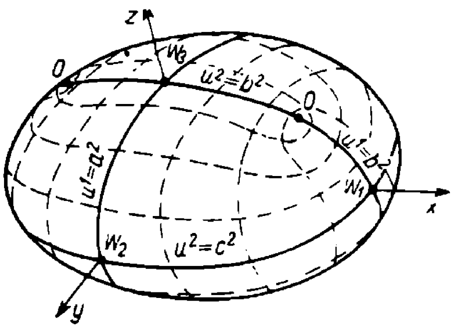

Fig. 2

Since we have taken for calculations the curvature system on the surface, it is easy now to determine the reciprocal situation of vertical lines and lines of curvature. It follows from (3.3) that, if we put in this equation $u^1 = \text{const}$ or $u^2 = \text{const}$, then either $\partial_1\lambda_1$ or $\partial_2\lambda_2$ must vanish, when one of the families of vertical lines coincide with one family of lines of curvature. If two families of vertical lines are simultaneously families of lines of curvature, then the derivatives $\partial_1\lambda_1$ and $\partial_2\lambda_2$ are both equal to zero, i.e. $\lambda_1 = \lambda_1(u^2)$ and $\lambda_2 = \lambda_2(u^1)$.

4. The vertical and the principal directions are connected with one another in some other way.

To find this relation let us consider the derivative of longitudinal curvature λ of the surface (S) not in the normal plane (the geodesic derivative) but in an arbitrary one passing through the direction $\boldsymbol{t}$. In other words, we want to consider now the oblique intersections of our surface. Let the normal vector $\boldsymbol{p}$ of such a plane form with vector $\boldsymbol{m}$ an angle θ different from zero. According to the definition of such a derivative (see Chap. I, p. 144, formulae (1.35) and (1.46)) the only difference between

([7]) A nearer investigation of equations (4.9) shows that the vertices are the singular points of these equations. For example the vertex W_1 is the saddle point of the first equation.

it and the geodesic derivative lies in the quantities $d\mu^k/dr$. Expressing the vector $\boldsymbol{p}$ by means of vectors $\boldsymbol{l}$ and $\boldsymbol{m}$, we have

$$\boldsymbol{p} = \cos\theta \boldsymbol{m} + \sin\theta \boldsymbol{l}\,, \tag{4.11}$$

whence

$$\boldsymbol{q} = [\boldsymbol{pt}] = \sin\theta \boldsymbol{m} - \cos\theta \boldsymbol{l} \tag{4.12}$$

and also

$$(\boldsymbol{pl}) = \sin\theta\,. \tag{4.13}$$

Inserting (4.12) and (4.13) in (1.46) we get, using (1.48),

$$\frac{d\mu^k}{dr_{(p)}} = \frac{d\mu^k}{dr_{(l)}} - \frac{1}{\sqrt{g}}\varepsilon^{ik}\operatorname{ctg}\theta(\boldsymbol{lwr}_i)\,,$$

thus the derivative of curvature λ in an arbitrary plane of the direction $\boldsymbol{p}$ may be written in the following way:

$$\frac{d\lambda}{dr_{(p)}} = \frac{d\lambda}{dr_{(l)}} - \frac{\operatorname{ctg}\theta}{\sqrt{g}}\varepsilon^{ik}(\boldsymbol{lwr}_i)\frac{\partial\lambda}{\partial\mu^k}\,. \tag{4.14}$$

Since we have

$$[\boldsymbol{r}_i\boldsymbol{l}] = \big[\boldsymbol{r}_i[\boldsymbol{tm}]\big] = -\boldsymbol{m}(\boldsymbol{r}_i\boldsymbol{t})$$

and

$$(\boldsymbol{lwr}_i) = -(\boldsymbol{mw})(\boldsymbol{r}_i\boldsymbol{t}) = -\lambda g_{ij}\mu^j\,,$$

inserting this together with relation (2.9) in (4.14) we obtain

$$\frac{d\lambda}{dr_{(p)}} = \frac{d\lambda}{dr_{(l)}} + 2\operatorname{ctg}\theta\cdot\frac{1}{\sqrt{g}}\lambda\varepsilon^{ik}g_{ij}h_{kl}\mu^j\mu^l\,.$$

But according to (1.30) and by (1.29) we finally have

$$\frac{d\lambda}{dr_{(p)}} = \frac{d\lambda}{dr_{(l)}} + 2\operatorname{ctg}\theta\lambda\tau\,. \tag{4.15}$$

This is the required formula ([8]). As we see, the derivative of λ in an arbitrary plane may be expressed in the form of a sum of the cubic form $d\lambda/dr_{(l)}$ and of the form of the fourth degree which is the product of two quadratic forms λ and τ.

Omitting the trivial case of asymptotic directions when $\lambda = 0$, we may formulate the following theorem:

([8]) If we denote by $\varkappa_g$ the geodesic curvature of the oblique intersection at the point M, then we have $\varkappa_g = \lambda\operatorname{ctg}\theta$ and equation (4.15) takes the form:

$$\frac{d\lambda}{dr_{(p)}} = \frac{d\lambda}{dr_{(l)}} + 2\tau\varkappa_g\,.$$

This is the first formula of Forsyth ([5], p. 48, form. 12a).

THEOREM 3. *If the given direction l is simultaneously vertical, i.e. $d\lambda/dr_{(l)} = 0$, and principal, i.e. $\tau = 0$, then it is the direction for which the derivative of λ in an arbitrary plane passing through the direction t orthogonal to l vanishes.*

Of course the opposite theorem is also true. A similar situation occurs when expression (4.15) is satisfied identically, i.e. when the vertices and the umbilics of a surface coincide.

5. The examples of vertical lines considered in this paper for several kinds of surfaces do not satisfy in many cases our intuitive idea of such lines. There are often curves, which "optically" could not be distinguished as the verticals (for cylinders — the intersections by planes perpendicular to their generators, for cones — the intersections by concentric spheres and for surfaces of revolutions — the meridians). According to our geometrical imagination, as the "proper" verticals we would like to regard the chosen generators for cylindric and conic surfaces and the chosen parallels for the surfaces of revolution (see examples 1° and 2° from Chap. III). This divergence between the results obtained and our geometrical intuition arises partly from the analytic definition of the vertical direction. Condition (2.1) is a necessary and not sufficient condition for the extreme values of λ in the normal plane. But the same situation occurs in the case of vertices for the plane curve, where we also have an analytic definition instead of the geometrical one.

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