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On the continuity of the Faber mapping

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Dedicated to the memory of Franciszek Leja

Abstract. Let K be a nondegenerate continuum which does not separate the complex plane. In geometric terms a sufficient condition is expressed guaranteeing that K is a Faber set.

1. Introduction. As usual, we shall denote by $\boldsymbol{N}$ the set $\{0, 1, 2, \ldots\}$, by $\boldsymbol{Z}$ the set of all integers, by $\boldsymbol{R}$ the set of all real numbers and by $\boldsymbol{C}$ the set of all complex numbers. For $M \subset \boldsymbol{C}$ we shall denote by ∂M and $\operatorname{diam} M$ the boundary and the diameter of M, respectively. Given $\varepsilon > 0$, we put

$$\mathscr{H}^1_\varepsilon(M) = \inf \sum_{n=1}^{\infty} \operatorname{diam} M_n,$$

where the infimum is taken over all sequences of sets $M_n \subset \boldsymbol{C}$ such that $\operatorname{diam} M_n \leqslant \varepsilon$ and $M \subset \bigcup_{n=1}^{\infty} M_n$. The linear measure ($=$ length) of M is defined by

$$\mathscr{H}^1(M) = \lim_{\varepsilon \to 0+} \mathscr{H}^1_\varepsilon(M).$$

Given a function f defined on the interval $\langle a, b\rangle$, we shall denote by $\operatorname{var}[f\colon \langle a, b\rangle]$ its total variation on $\langle a, b\rangle$.

In the year 1903 G. Faber [5] published in Mathematische Annalen the paper *Über polynomische Entwickelungen.* In this work, stimulated by reflections about the theorem of Runge, Faber introduced a sequence of polynomials corresponding to any domain bounded by an analytic Jordan curve. These polynomials, which turned out to be extremely useful in complex function theory (see for example [7], where extensive references may be found), are now called *Faber polynomials.* They can be defined in various ways. For our purposes it will be convenient to give a definition by means of the generating function.

Let $K \subset \boldsymbol{C}$ be a nondegenerate continuum (i.e., a compact connected set containing at least two different points) not dividing $\boldsymbol{C}$ (i.e., $\boldsymbol{C} \setminus K$ is connected). Denote $\psi(w)$ the conformal mapping of $C\setminus\Delta$, where Δ is the unit disc $\Delta = \{w \in C;\ |w| \leqslant 1\}$, onto $C\setminus K$, normalized by $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. In the following we shall always suppose $\psi'(\infty) = 1$.

This assumption means that the logarithmic capacity of K is equal to one. Hence in this case ψ has in $C\setminus\Delta$ the following expansion:

$$\psi(w) = w + a_0 + \frac{a_1}{w} + \ldots \tag{1}$$

The Faber polynomials $F_n(t) = z^n + \ldots$, $n \in N$, are defined by

$$\frac{w\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{z^n}. \tag{2}$$

Let us denote by $A(K)$ the Banach space of all complex functions f continuous on K and analytic in all interior points of K equipped with the norm $\|f\|_K = \max_{w\in K} |f(w)|$. From (2) we can readily define a mapping T: $w^n \to F_n(z)$, which may at once be extended by linearity to a mapping of the set of all polynomials, namely

$$(Tp)(z) = \sum_{n=0}^{N} a_n F_n(z) \quad \text{for } p(w) = \sum_{n=0}^{N} a_n w^n.$$

T is called the *Faber mapping*. If there exists a constant $M < \infty$ such that $\|TP\|_K \leqslant M\|P\|_\Delta$, then T may be uniquely extended to a continuous mapping of the whole Banach space $A(\Delta)$ into $A(K)$ because of the density of the set of all polynomials in $A(\Delta)$. Following the terminology introduced in 1974 by Dynkin [4], we shall call K in this case a Faber set.

Examples of Faber sets are: Jordan domains bounded by a Ljapunov curve [4]; convex domains; domains with bounded rotation (beschränkte Randdrehung) in the sense of Radon–Paatero [9] (cf. also [7], Satz 2, p. 51). J. E. Andersson [1] remarks that in fact in [9] the following stronger result is contained: K is a Faber set if for every $z \in \partial K$ the condition

$$\liminf_{R\to 1+} \int_0^{2\pi} |d_\tau \arg(\psi(Re^{i\tau}) - z)| \leqslant C < \infty \tag{3}$$

is fulfilled.

In the present paper we shall be concerned with a geometric condition on K guaranteeing that K will be a Faber set.

Let $\Pi = \{\zeta \in C;\ |\zeta| = 1\}$ be the unit circle and $D\{\zeta \in C;\ |\zeta| < 1\}$ the open unit disc. For $z \in C$ denote by π_z: $\zeta \to (\zeta - z)/|\zeta - z|$ the mapping of the set $C\setminus\{z\}$ onto Π. For compact $Q \subset C$ we define, for $\theta \in \langle 0, 2\pi\rangle$,

$$N_z^Q(\theta) = \sum \chi_Q(u), \qquad u \in Q\setminus\{z\}, \qquad \pi_z(u) = e^{i\theta},$$

where the sum is taken over all $u \in \pi_z^{-1}(\theta)$ and χ_Q denotes the characteristic function of Q. Thus $N_z^Q(\theta)$ is the number (finite or infinite) of all points lying at the intersection of Q with the half-line $\{z+te^{i\theta};\ t>0\}$. N_z^Q is Borel measurable ([14], p. 217) and therefore we may adopt the following

DEFINITION. Let $K \subset C$ be a compact set. For each $z \in C$ we define

$$v^K(z) = \int_{\Pi} N_z^K(\theta)\, d\mathscr{H}^1(\theta).$$

This quantity is sometimes called the *cyclic variation* of K at the point z. Further we define

$$V(K) = \sup_{\zeta \in K} v^K(\zeta).$$

Our result may now be formulated as follows:

THEOREM. *Let $K \subset C$ be a nondegenerate continuum which does not divide C. Let*

$$V(\partial K) < \infty. \tag{4}$$

Then K is a Faber set.

Let us make some remarks before going to prove the theorem. Condition (4) can be fulfilled also for curves which are not smooth and contain many angular points. Especially all Jordan curves with bounded rotation fulfil (4) (cf. [15]). It is not hard to construct a Jordan curve having countably many angular points of magnitude $\frac{1}{2}\pi$. Such a curve cannot be of bounded rotation, because a curve of bounded rotation can have only a finite number of angular points of magnitude not less than a given angle (cf. [13], p. 72). Therefore our theorem generalizes the result of Kővari and Pommerenke. The essential result of the present paper is in fact Theorem 3.2, which shows that the geometric condition (4) implies the uniform estimate (3) of the cyclic variation of level curves of the conformal mapping (1) at every point of ∂K. On the other hand, condition (4) is not fulfilled for many arcs $Q(f)$ given by the equation $y = f(x)$, $0 \leqslant x \leqslant 1$, where $f\colon \langle 0, 1\rangle \to \mathbf{R}$ is continuously differentiable. Let $C^1(\langle 0, 1\rangle)$ be the Banach space of all continuously differentiable functions f on the interval $\langle 0, 1\rangle$, $f(0) = 0$, equipped with the norm $\|f\| = \max_{x \in \langle 0,1\rangle} |f'(x)|$. Then the set

$$\{f \in C^1(\langle 0, 1\rangle);\ v^{Q(f)}(\zeta) = \infty \ \ \forall \zeta \in Q(f)\}$$

is residual in $C^1(\langle 0, 1\rangle)$ (cf. [10]).

The paper is based on two deep results. The first one is Ważewski's characterization of rectifiable continua [17]. The second one is Young's theory of length [18], which is based on a deep study of prime ends of a simply connected domain realized in [16] and which makes it possible to pass to the boundary values of the derivative of the conformal mapping (1). The author owes thanks to Dr. J. Král for drawing his attention to paper [18].

2. In this section we summarize some fundamental results of Young's theory [18] of length in the generality needed for our purposes and some known auxiliary facts needed for proving the theorems of Section 3. The aim of Young's theory [18] is to extend the definition of length (and area, of course, but this is of no interest in the present paper) to continuous images of more general plane sets than the segment, for example to the continuous image of the boundary of any simply connected domain, where the classical definitions no longer apply.

Let $A \subset \boldsymbol{R}^k$. The function $M(x)$: $A \to N \cup \{\infty\}$ is said to define a multiple system of which it is the multiplicity function. Any set of points is identified with the multiple system whose multiplicity function is the characteristic function of that set. If $\mu(E)$ is any measure defined for sets E of an addition class, we define its extension $\mu(M)$ to multiple systems by the formula

$$\mu(M) = \sum_{n=1}^{\infty} \mu(M_n), \tag{5}$$

where $M_n = \{x \in A;\ M(x) \geqslant n\}$. In order to make this definition correct, the notion of an additive class must be extended in an appropriate manner to multiple systems M so as to contain all sets M_n. It is clear how to do this and therefore we may refer the reader to [18], p. 277, for a formal definition. In this paper we will take for $\mu(M)$ only the length $\mathscr{H}^1(M)$.

Let $K \subset \boldsymbol{C}$ be closed, $A \subset \boldsymbol{R}^k$, f: $K \to A$ being continuous. We denote by $M(x; f; \mathscr{U})$ the number (finite or $+\infty$) of points $u \in \mathscr{U} \subset K$ fulfilling the equation $f(u) = x$. We write briefly $M(x)$ if $\mathscr{U} = K$. $M(x)$ is Borel measurable and, if K is a segment, then $\mathscr{H}^1(M)$ is the classical length of the curve $f(u)$ ([18], (3.3), p. 278 and (4.1), p. 280).

Let $K \subset \boldsymbol{C}$ be a compact set, f: $K \to \boldsymbol{R}^k$ being continuous. Two points of K will be termed equivalent if they lie on the same connected subset of K throughout which $f(u)$ is constant. Let E_x^* be any finite set consisting of non-equivalent points $u \in K$ for which $f(u) = x$. For $\mathscr{U} \subset K$ we denote by $M^*(x; f; \mathscr{U})$ the upper bound of the number of points of the various $E_x^* \subset \mathscr{U}$. We write $M^*(x)$ for $M^*(x; f; K)$.

Evidently $M^*(x) \leqslant M(x)$ and $M^*(x) = M(x)$ if $M(x) < \infty$.

Lemma 2.1 (i) *$M^*(x)$ is Borel measurable;*

(ii) $\mu(M) = \mu(M^*)$ *if* $\mu(M) < \infty$ (cf. [18], (5.4), p. 287, and (5.8), p. 289).

To this end let $K \subset \boldsymbol{C}$ be a continuum, f: $K \to A \subset \boldsymbol{R}^k$ continuous. We say that the intrinsic length of f is the number $\mathscr{H}^1(M^*)$, where M^* is the multiplicity system defined by the multiplicity function $M^*(x; f; K)$. Let G be any domain contained in $\boldsymbol{C} \backslash K$. The boundary length $L_{\mathrm{fr}}(f; G)$ is defined as the number $\mathscr{H}^1(N(G))$, where $N(G)$ is the multiplicity system defined by the multiplicity function $N(x; f; G)$, the number of prime ends (in the sense of

Carathéodory, cf. [2], [16] for the definition) of G throughout which $f(u)$ assumes the constant value x.

EXAMPLE 2.1. Let K be the sum of Π with its radius $\langle 0, 1\rangle$, G the bounded component of K, $f(u) = u$. Then the intrinsic length of K is $2\pi+1$, but $L_{\mathrm{fr}}(f, G) = 2\pi+2$, because we must add the length of the radius twice.

The following fundamental assertion holds:

PROPOSITION 2.1 ([18], Theorem (7.2), p. 294, and (7.3), p. 295). *Let $K \subset C$ be a continuum, $f\colon K \to A \subset R^k$ being continuous. Let $\mathscr{U} \subset K$ be such that the intrinsic length $L^*(f|\mathscr{U})$ of $f|\mathscr{U}$ is finite. Let G be complementary domains of K whose boundaries are contained in $\mathscr{U}$. Then*

(i) $\sum\limits_{G} L_{\mathrm{fr}}(f; G) \leqslant 2L^*(f|\mathscr{U})$;

(ii) *if $L_{\mathrm{fr}}(f; G)$ is finite, it is the classical length of the closed curve $g(e^{it}) = f(\varphi(e^{it}))$, where φ is the conformal mapping of D onto G.*

PROPOSITION 2.2. *Let $Q \subset C$ be a compact set fulfilling* (4). *Then $\mathscr{H}^1(Q) < \infty$* ([6], Proposition 1.1, p. 448).

PROPOSITION 2.3. *Let $Q \subset C$ be a compact set consisting of a finite number of components. Then for every $z \in C$ the estimate $v^Q(z) \leqslant V(Q)+\pi$ holds* ([6], Proposition 2.1, p. 454).

Let us remark that the proof of the last proposition is based upon Ważewski's deep characterization of rectifiable continua [17]; the formulation of Ważewski's result is contained also in [6], p. 452–453.

3. Our aim in this section is to prove Theorem 3.2. We start with an auxiliary assertion.

LEMMA 3.1. *Let $G \subset C$ be a simply connected domain whose boundary contains at least two different points, $\mathscr{H}^1(G) < \infty$. Then the set of prime ends of G consists only of prime ends of the first kind, that is, the prime ends of G reduce to points.*

This lemma is proved in [16] as a consequence of (11.3), p. 15.

PROPOSITION 3.1. *Let K be a non-degenerated continuum which does not divide C and fulfils* (4), *ψ being the conformal mapping* (1). *Then the function*

$$\psi(e^{i\tau}) = \lim_{\substack{w \to e^{i\tau} \\ w \in C\setminus \Delta}} \psi(w), \qquad 0 \leqslant \tau \leqslant 2\pi,$$

is absolutely continuous and the following holds:

(i) $$\frac{d}{d\tau}\psi(e^{i\tau}) = ie^{i\tau} \lim_{R \to 1+} \psi'(Re^{i\tau})$$

almost everywhere (a.e.) on $\langle 0, 2\pi\rangle$.

(ii) *There exists a finite constant M such that for every $R > 1$*

$$\int\limits_0^{2\pi} |\psi'(Re^{i\tau})|\, d\tau \leqslant M.$$

Proof. According to Proposition 2.2 we have $H^1(\partial K) < \infty$ and by Lemma 3.1 the prime ends of $C \setminus K$ are only of the first kind. Consequently $\psi(w)$ is continuous ([11], Theorem 2, p. 407) in $C \setminus D$. Denote its extension to Π also by $\psi(w)$. Now denote by id the continuous function defined on ∂K by the formula $\mathrm{id}\, u = u$. All connected subsets of ∂K on which id is constant consist of single points. Hence $M^*(x; \mathrm{id}; \partial K) = 1$ for every $x \in \partial K$ and consequently the intrinsic length of id equals $\mathscr{H}^1(\partial K)$, which is finite. According to Proposition 2.1 (i) we have $L_{\mathrm{fr}}(\mathrm{id}; C \setminus K) \leqslant 2\mathscr{H}^1(\partial K) < \infty$. The conformal mapping of D onto $C \setminus K$ is given by the function $\varphi(\zeta) = \psi(w)$, $\zeta = 1/w$. So φ is continuous on Δ and $\varphi(e^{it}) = \psi(e^{i\tau})$ if we write $\zeta = re^{it}$, $w = Re^{i\tau}$. The length of the curve $\varphi(e^{it})$, $0 \leqslant t \leqslant 2\pi$, equals $L_{\mathrm{fr}}(\mathrm{id}; C \setminus K)$ according to Proposition 2.1 (ii) and therefore it is finite. Hence $\mathrm{var}\,[\varphi(e^{it}); \langle 0, 2\pi\rangle] < \infty$. Now let us consider the function

$$g(\zeta) = \varphi(\zeta) - 1/\zeta, \qquad g(0) = \lim_{\zeta \to 0} (\varphi(\zeta) - 1/\zeta),$$

which is holomorfic in D. Because of the finiteness of $\mathrm{var}\,[\varphi(e^{it}); \langle 0, 2\pi\rangle]$ we also have $\mathrm{var}\,[g(e^{it}); \langle 0, 2\pi\rangle] < \infty$ and therefore the following assertions hold (cf. [3], Theorems 3.10 and 3.11, p. 142): $g(e^{it})$ is absolutely continuous, $g' \in H^1$, and consequently

$$\int_0^{2\pi} |g'(re^{it})|\, dt \leqslant M < \infty, \qquad g'(e^{it}) = \lim_{r \to 1-} g'(re^{it})$$

exists a.e. on $\langle 0, 2\pi\rangle$ and $\frac{d}{dt} g(e^{it}) = ie^{it} g'(e^{it})$ a.e. on $\langle 0, 2\pi\rangle$. According to the definition of g we obtain $\frac{d}{dt}\varphi(e^{it}) = ie^{it}\varphi'(e^{it})$ a.e. on $\langle 0, 2\pi\rangle$ and

$$\int_0^{2\pi} |\varphi'(re^{it})|\, dt \leqslant \int_0^{2\pi} |g'(re^{it})| + 2\pi/r^2 \leqslant M + 2\pi/r^2.$$

Because of $\dfrac{d\varphi(\zeta)}{d\zeta} = \dfrac{d\psi(w)}{dw} \cdot (-w^2)$ we obtain (i) and

$$\int_0^{2\pi} |\psi'(Re^{i\tau})|\, d\tau \leqslant M/R^2 + 2\pi < M + 2\pi < \infty,$$

and so (ii) is proved.

Proposition 3.2. *Let $\psi(e^{i\tau})$ be the function of the preceding proposition and $z \in C \setminus K$ a given point such that $v^{\partial K}(z) < \infty$. Then the function $\chi_z(\tau) = \arg(\psi(e^{i\tau}) - z) = \mathrm{Im} \log(\psi(e^{i\tau}) - z)$ is absolutely continuous and*

$$\mathrm{var}\,[\chi_z(\tau);\ \langle 0, 2\pi\rangle] \leqslant 2v^{\partial K}(z). \tag{6}$$

Proof. One may join the point z with the point at infinity by a Jordan arc γ lying in $C \setminus K$. Hence in the simply connected domain $C \setminus \{\gamma\}$ there exists a holomorphic function $\log(\zeta - z)$, and so on $\langle 0, 2\pi\rangle$ the function $\chi_z(\tau)$ is well defined. Its absolute continuity results easily from the following facts: $\psi(e^{i\tau})$ is absolutely continuous (Proposition 3.1); $|\psi(e^{i\tau}) - z| \geqslant \alpha > 0$ on $\langle 0, 2\pi\rangle$; $\log(1+u) = \mathcal{O}(u)$ for $u \to 0$. So it remains to prove (6). Denote by $A_z(\theta) = \{\zeta \in \partial K;\ \arg(\zeta - z) = \theta\}$, $M_z(\theta)$ the number of points in $A_z(\theta)$ (possibly infinite). On account of the choice of the function $\log(\zeta - z)$ the sets $A_z(\theta + 2j\pi)$, $A_z(\theta + 2k\pi)$ are disjoint for $j \neq k$. Otherwise there would exist a point $\zeta \in \partial K$ such that $\arg(\zeta - z) = \theta + 2j\pi$ and at the same time $\arg(\zeta - t) = \theta + 2k\pi$; hence the function $\log(\zeta - z)$ cannot be univalent in $C\setminus\{\gamma\}$. Hence for every $\theta \in \langle 0, 2\pi)$ we get $N_z^{\partial K}(\theta) = \sum_{k \in Z} M_z(\theta + 2k\pi)$, and with respect to the compactness of ∂K and continuity of $\arg(\zeta - z)$ the sum includes only a finite number of terms, so that we can write

$$\int_{-\infty}^{\infty} M_z(\theta)\, d\mathscr{H}^1(\theta) = \sum_{k \in Z} \int_0^{2\pi} M_z(\theta + 2k\pi)\, d\mathscr{H}^1(\theta) = \int_0^{2\pi} \sum_{k \in Z} M_z(\theta + 2k\pi)\, d\mathscr{H}^1(\theta)$$

$$= \int_0^{2\pi} N_z^{\partial K}(\theta)\, d\mathscr{H}^1(\theta) = v^{\partial K}(z).$$

Consequently, by hypothesis, we get

$$\int_{-\infty}^{\infty} M_z(\theta)\, d\mathscr{H}^1(\theta) = v^{\partial K}(z) < \infty. \tag{7}$$

But (cf. par example [8], p. 21)

$$\int_{-\infty}^{\infty} M_z(\theta)\, d\mathscr{H}^1(\theta) = \int_0^{\infty} m_z(\lambda)\, d\mathscr{H}^1(\lambda),$$

where $m_z(\lambda) = \mathscr{H}^1(\{\theta;\ M_z(\theta) > \lambda\})$. Since $M_z(\theta) \in N \cup \{\infty\}$, one obtains

$$\int_0^{\infty} m_z(\lambda)\, d\lambda = \sum_{n=0}^{\infty} \mathscr{H}^1(\{\theta;\ M_z(\theta) > n\}) = \sum_{n=1}^{\infty} \mathscr{H}^1(\{\theta;\ M_z(\theta) \geqslant n\}),$$

which, by Definition — see (5) — is equal to $\mathscr{H}^1(M_z)$, where M_z is the multiplicity system defined by the multiplicity function $\theta \to M_z(\theta)$. Hence on account of (7) we get $\mathscr{H}^1(M_z) = v^{\partial K}(z) < \infty$. According to Lemma 2.1

$$\mathscr{H}^1(M_z^*) = \mathscr{H}^1(M_z) = v^{\partial K}(z) < \infty,$$

which means that the inner length of the continuous function $\arg(\zeta - z)|\partial K$, which is by definition equal to $\mathscr{H}^1(M_z^*)$, is finite. Hence by Proposition 2.1 (i)

$$L_{fr}(\arg(\zeta - z);\ C\setminus K) \leqslant 2\mathscr{H}^1(M_z^*) = 2v^{\partial K}(z) < \infty,$$

so that by (ii) of the same proposition $L_{fr}(\arg(\zeta - z);\ C\setminus K)$ equals the classical length of the curve $\arg(\varphi(e^{it}) - z)$, $t \in \langle 0, 2\pi\rangle$, where φ is the conformal mapping of D onto $C\setminus K$, which is therefore finite. Hence also the length of the continuous curve

$$\chi_z(\tau) = \arg(\psi(e^{i\tau}) - z) = \arg(\varphi(e^{-it}) - z)$$

is finite. But this length is equal to the total variation of χ_z on $\langle 0, 2\pi\rangle$, whence

$$\operatorname{var}[\chi_z(\tau);\ \langle 0, 2\pi\rangle] \leqslant 2v^{\partial K}(z)$$

and (6) is proved.

THEOREM 3.1. *Let K be a nondegenerate continuum not dividing C and fulfilling* (4). *Let ψ be the conformal mapping* (1). *Then for each point $z \in C\backslash K$ the following estimate holds*:

$$\int_0^{2\pi} \left|\operatorname{Im} \frac{ie^{i\tau}\psi'(e^{i\tau})}{\psi(e^{i\tau}) - z}\right| d\tau \leqslant 2(V(\partial K) + \pi). \tag{8}$$

Proof. By Proposition 2.3 one has $v^{\partial K}(z) \leqslant V(\partial K) + \pi$. But $V(\partial K) + \pi < \infty$ from (4). By (6) of Proposition 3.2 one gets

$$\operatorname{var}[\chi_z(\tau);\ \langle 0, 2\pi\rangle] \leqslant 2(V(\partial K) + \pi). \tag{9}$$

But by the same proposition χ_z is absolutely continuous on $\langle 0, 2\pi\rangle$ and hence

$$\operatorname{var}[\chi_z(t);\ \langle 0, 2\pi\rangle] = \int_0^{2\pi} |\chi_z'(\tau)|\, d\tau$$

(cf. for example [12], Theorem 8, p. 279). Because of

$$\chi_z'(\tau) = \frac{d}{d\tau}\chi_z(\tau) = \frac{d}{d\tau}\operatorname{Im}\log(\psi(e^{i\tau}) - z)$$

$$= \operatorname{Im}\frac{d}{d\tau}\log(\psi(e^{i\tau}) - z) = \operatorname{Im}\frac{(d/d\tau)\psi(e^{i\tau})}{\psi(e^{i\tau}) - z}$$

a.e. on $\langle 0, 2\pi\rangle$ one has by Proposition 3.1 (i)

$$\operatorname{var}[\chi_z(\tau);\ \langle 0, 2\pi\rangle] = \int_0^{2\pi} \left|\operatorname{Im}\frac{ie^{i\tau}\psi'(e^{i\tau})}{\psi(e^{i\tau}) - z}\right| d\tau,$$

where $\psi'(e^{i\tau}) = \lim\limits_{R\to 1+} \psi(Re^{i\tau})$ a.e. on $\langle 0, 2\pi\rangle$, and from (9) we get (8).

Now we will define in an obvious way Hardy's class $H^1(C\backslash\Delta)$ as the set of all functions ψ holomorphic in $C\backslash\Delta$ (which means that $\lim\limits_{w\to\infty}\psi(w)$ exists and lies in C) such that

$$\int_0^{2\pi} |\psi(Re^{i\tau})|\, d\tau \leqslant M < \infty \qquad \text{for all } R > 1.$$

Analogously we define $h^1(C\backslash\Delta)$ as the set of all functions h harmonic in $C\backslash\Delta$ (which means that $\lim_{w\to\infty} h(w)$ exists and is finite) such that

$$\int_0^{2\pi} |h(Re^{i\tau})|\,d\tau \leqslant M < \infty \quad \text{for all } R > 1.$$

Applying the conformal mapping $w = 1/\zeta$, we can easily see that $\psi \in H^1(C\backslash\Delta)$, $h \in h^1(C\backslash\Delta)$, respectively, if and only if the functions $\varphi(\zeta) = \psi(w)$, $g(\zeta) = h(w)$ lie in $H^1(D)$, $h^1(D)$, respectively. Hence from [3], Theorem 3.1, p. 34. and Theorem 1.1, p. 2, we get the following

LEMMA 3.2 (i) *$\psi \in H^1(C\backslash\Delta)$ if and only if it can be represented as a Poisson integral of its radial boundary values which exist a.e. on $\langle 0, 2\pi\rangle$, that is, if*

$$\psi(Re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \Pi_R(\tau-\theta)\psi(e^{i\tau})\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{w+e^{i\tau}}{w-e^{i\tau}}\psi(e^{i\tau})\,d\tau,$$

where

$$w = Re^{i\theta}, \qquad \Pi_R(\tau) = \frac{R^2-1}{R^2-2R\cos\tau+1}.$$

(ii) *$h \in h^1(C\backslash\Delta)$ if and only if there exists a measure on $\langle 0, 2\pi\rangle$ such that*

$$h(re^{i\theta}) = \int_0^{2\pi} \Pi_R(\tau-\theta)\,d\mu(\tau).$$

THEOREM 3.2. *Let K be a nondegenerate continuum not dividing C and fulfilling* (4), *and ψ the conformal mapping* (1). *Let $z \in \partial K$. Then*

$$\operatorname{Im}\frac{iw\psi'(w)}{\psi(w)-z} \in h^1(C\backslash\Delta), \tag{10}$$

$$\sup_{R>1}\int_0^{2\pi}\left|\operatorname{Im}\frac{iRe^{i\tau}\psi'(Re^{i\tau})}{\psi(Re^{i\tau})-z}\right|d\tau \leqslant 2(V(\partial K)+3\pi). \tag{11}$$

Proof. First let us remark that (11) implies (3) with the constant $C \leqslant 2(V(\partial K)+3\pi)$. Conversely (3) implies (11) with the constant C. Namely, the function $\dfrac{iw\psi'(w)}{\psi(w)-z}$ is holomorphic in $C\backslash\Delta$, so that the function $\left|\operatorname{Im}\dfrac{iw\psi'(w)}{\psi(w)-z}\right|$ is subharmonic in $C\backslash\Delta$ and hence integrals (3) produce a nondecreasing function of R. The idea of the proof of Theorem 3.2 consists in constructing the measure μ_z of (ii) of Lemma 3.2 corresponding to the function $h(w) = \operatorname{Im}\dfrac{iw\psi'(w)}{\chi(w)-z}$ as a weak limit for $\varrho \to 1$ of measures μ_ϱ constructed by using (8). Let $z \in \partial K$. There exists an $\alpha \in \langle 0, 2\pi\rangle$ such that $z = \psi(e^{i\alpha})$ (if there are many such α, we choose any of them). Fix a $\varrho > 1$ and consider in $C\setminus\Delta$ the function

$$f_{\varrho,\alpha} = \frac{iw\psi'(w)}{\psi(w)-\psi(\varrho e^{i\alpha})} - \frac{iw}{w-\varrho e^{i\alpha}}.$$

Since ψ is one-to-one in $C\setminus\Delta$ and since

$$\operatorname*{res}_{w=\varrho e^{i\alpha}} \frac{iw\,\psi'(w)}{\psi(w)-\psi(\varrho e^{i\alpha})} = i\varrho e^{i\alpha} = \operatorname*{res}_{w=\varrho e^{i\alpha}} \frac{iw}{w-\varrho e^{i\alpha}},$$

$f_{\varrho,\alpha}$ is holomorphic in $C\setminus\Delta$. We show that in addition $f_{\varrho,\alpha} \in H^1(C\setminus\Delta)$. Obviously $\lim\limits_{w\to\infty} f_{\varrho,\alpha}(w) = 0$. Now choose $R_0 < \varrho$. The function $w/(\psi(w) - \psi(\varrho e^{i\alpha}))$ is continuous in the ring $1 \leqslant |w| \leqslant R_0$ and hence there exists a constant $M(R_0) < \infty$ so that for $R \leqslant R_0$

$$\int_0^{2\pi} |f_{\varrho,\alpha}(Re^{i\tau})\,d\tau| \leqslant M(R_0)\Big(\int_0^{2\pi} |\psi'(Re^{i\tau})|\,d\tau + 1\Big).$$

By Proposition 3.1 (ii) we then get

$$\int_0^{2\pi} |f_{\varrho,\alpha}(Re^{i\tau})|\,d\tau \leqslant M_{R_0} < \infty \qquad \text{for every } R > 1,$$

because the integrals $\int_0^{2\pi} |f_{\varrho,\alpha}(Re^{i\tau})|\,d\tau$ form a nonincreasing function of R. So $f_{\varrho,\alpha} \in H^1(C\setminus\Delta)$. Hence by Lemma 3.2 (i) ($w = Re^{i\theta}$)

$$f_{\varrho,\alpha}(w) = \frac{1}{2\pi}\int_0^{2\pi} \Pi_R(\tau-\theta)\, f_{\varrho,\alpha}(e^{i\tau})\,d\tau$$

and taking imaginary parts we obtain for $|w| > 1$

(12) $$\operatorname{Im}\frac{iw\psi'(w)}{\psi(w)-z} - \operatorname{Im}\frac{iw}{w-\varrho e^{i\alpha}}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \Pi_R(\tau-\theta)\left(\operatorname{Im}\frac{ie^{i\tau}\,\psi'(e^{i\tau})}{\psi(e^{i\tau})-\psi(\varrho e^{i\alpha})} - \frac{ie^{i\tau}}{e^{i\tau}-\varrho e^{i\alpha}}\right)d\tau.$$

Consider on $\langle 0, 2\pi\rangle$ the measure

$$d\nu_{\varrho,\alpha} = \frac{1}{2\pi}\operatorname{Im}\left(\frac{ie^{i\tau}\,\psi'(e^{i\tau})}{\psi(e^{i\tau})-\psi(\varrho e^{i\alpha})} - \frac{ie^{i\tau}}{e^{i\tau}-\varrho e^{i\alpha}}\right)d\tau.$$

We have

$$\|\nu_{\varrho,\alpha}\| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{Im}\frac{ie^{i\tau}\psi'(e^{i\tau})}{\psi(e^{i\tau})-\psi(\varrho e^{i\alpha})}\right|d\tau + \frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{Im}\frac{ie^{i\tau}}{e^{i\tau}-\varrho e^{i\alpha}}\right|d\tau.$$

The first integral is, by (8), not greater than $V/\pi+1$. The second one is estimated by 1. Namely, as is easily seen,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{Im}\frac{ie^{i\tau}}{e^{i\tau}-\varrho e^{i\alpha}}\right|d\tau = \frac{1}{2\pi}v^{\Pi}(\varrho e^{i\alpha}).$$

But $v^{\Pi}(\varrho e^{i\alpha}) \leqslant 2\pi$, as is apparent from the definition of v^{Π}. Hence

(13) $$\|\nu_{\varrho,\alpha}\| \leqslant V/\pi+2.$$

Further, define on $\langle 0, 2\pi\rangle$ the measure $\sigma_{\varrho,\alpha}$ in such a way that

$$d\sigma_{\varrho,\alpha} = \frac{1}{2\pi}\,\mathrm{Im}\frac{ie^{i\tau}}{e^{i\tau}-1/\overline{\varrho e^{i\alpha}}}\,d\tau.$$

One has

$$\|\sigma_{\varrho,\alpha}\| = \frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{Im}\frac{ie^{i\tau}}{e^{i\tau}-e^{i\alpha}/\varrho}\right|d\tau = \frac{1}{2\pi}v^{\Pi}(e^{i\alpha}/\varrho) = 1.$$

Obviously $\dfrac{iw}{w-e^{i\alpha}/\varrho}$ is continuous in $C\backslash\Delta$, and so we get ($w = Re^{i\theta}$)

$$\frac{iw}{w-e^{i\alpha}/\varrho} = \frac{1}{2\pi}\int_0^{2\pi}\Pi_R(\tau-\theta)\frac{ie^{i\tau}}{e^{i\tau}-e^{i\alpha}/\varrho}\,d\tau.$$

Taking imaginary parts, we get

(14) $$\int_0^{2\pi}\Pi_R(\tau-\theta)\,d\sigma_{\varrho,\alpha}(\tau) = \mathrm{Im}\frac{iw}{w-e^{i\alpha}/\varrho}.$$

Denote $\mu_{\varrho,\alpha} = \nu_{\varrho,\alpha}+\sigma_{\varrho,\alpha}$. We have

$$\|\mu_{\varrho,\alpha}\| \leqslant \|\nu_{\varrho,\alpha}\|+\|\sigma_{\varrho,\alpha}\| \leqslant V/\pi+3.$$

Consequently, there exists a measure μ_z,

(15) $$\|\mu_z\| = V/\pi+3,$$

which is the weak limit of the measures $\mu_{\varrho,\alpha}$ for $\varrho\to 1+$. From (12) and (14) we obtain

$$(16)\qquad \operatorname{Im}\frac{iw\psi'(w)}{\psi(w)-\psi(\varrho e^{i\alpha})} = \int_0^{2\pi} \Pi_R(\tau-\theta)\,d\mu_{\varrho,\alpha}(\tau) + \operatorname{Im}\frac{iw}{w-\varrho e^{i\alpha}} - \operatorname{Im}\frac{iw}{w-e^{i\alpha}/\varrho}.$$

Since $\tau \to \Pi_R(\theta-\tau)$ is continuous on $\langle 0, 2\pi\rangle$, we get from (16) for $\varrho \to 1+$

$$(17)\qquad \operatorname{Im}\frac{iw\psi'(w)}{\psi(w)-z} = \int_0^{2\pi} \Pi_R(\tau-\theta)\,d\mu_z(\tau),$$

and so by Lemma 3.2 (ii)

$$\operatorname{Im}\frac{iw\psi'(w)}{\psi(w)-z} \in h^1(C\setminus\Delta)$$

and (10) is proved.

(11) follows from (10) by a standard procedure by using (15).

4. Now we are in a position to present the proof of the theorem of Section 1.

Choose $z \in \partial K$. (2) implies

$$F_n(z) = \frac{1}{2\pi i}\int_{|w|=R} w^{n-1}\frac{w\psi'(w)}{\psi(w)-z}\,dw$$

for every $R > 1$. (17) implies for $|w| \leqslant R$, by analytic completion,

$$\frac{w\psi'(w)}{\psi(w)-z} = \int_0^{2\pi} \frac{w+e^{i\tau}}{w-e^{i\tau}}\,d\mu_z(\tau),$$

so that Fubini's theorem gives

$$F_n(z) = \int_0^{2\pi}\left(\frac{1}{2\pi i}\int_{|w|=R} w^{n-1}\frac{w+e^{i\tau}}{w-e^{i\tau}}\,dw\right)d\mu_z(\tau) = 2\int_0^{2\pi} e^{in\tau}\,d\mu_z(\tau)$$

for $n \geqslant 1$ and $F_0(z) = \int_0^{2\pi} d\mu_z(\tau)$. Hence, if $p(z) = \sum_{n=1}^{N} a_n z^n$, we get

$$\sum_{n=0}^{N} a_n F_n(z) = 2\int_0^{2\pi} p(e^{i\tau})\,d\mu_z(\tau) - \int_0^{2\pi} p(0)\,d\mu_z(\tau),$$

so that $\|Tp\|_K = \|T_p\|_{\partial K} \leqslant 3\|p\|_\Delta \|\mu_z\|$ and with respect to (15)

$$\|Tp\|_K \leqslant 3(V/\pi + 3)\|p\|_\Delta.$$

The theorem is proved and in addition it is shown that the norm of the Faber mapping is less than $3(V/\pi+3)$.

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