

Spectral properties of doubly commuting hyponormal operators

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Abstract. The paper describes some spectral properties of doubly commuting pairs of hyponormal operators. A generalization of certain results of Putnam [2], II, III, is given using the notion of joint spectrum introduced by J. Taylor [4].

I. Let T_1, T_2 be a doubly commuting pair of hyponormal operators in a complex Hilbert space H (below we give a few examples of such pairs). In what follows we extend several results of Putnam, proved for a single hyponormal operator in [2], II, III, to the case of a commuting pair. Let us first recall a known result concerning the relation between the joint spectrum $\sigma(T_1, T_2)$ of Taylor and the joint approximate point spectrum $\sigma_\pi(T_1^*, T_2^*)$. Let S_1, S_2 be a doubly commuting pair of operators (not necessarily hyponormal). By the result of Vasilescu [5] (of Curto [1]), it is known that $(0, 0) \notin \sigma(S_1, S_2)$ iff the following operators are invertible:

$$S_1^* S_1 + S_2^* S_2, \quad S_1 S_1^* + S_2^* S_2, \quad S_1^* S_1 + S_2 S_2^* \quad \text{and} \quad S_1 S_1^* + S_2 S_2^*.$$

Hence for T_1, T_2 we have

$$(0) \quad \sigma(T_1, T_2) = \overline{\sigma_\pi(T_1^*, T_2^*)}$$

(note that $T_s T_s^* \leq T_s^* T_s$, $s = 1, 2$).

Though we shall deal mainly with the pair T_1, T_2 , we start with two results concerning commutators of commuting self-adjoint pairs.

Let $J_s = J_s^*$, $s = 1, 2$ and $J_1 J_2 = J_2 J_1$ (J_s is bounded for $s = 1, 2$). Suppose that U_1, U_2 are unitary and commute. Assume also that $U_1 J_2 = J_2 U_1$ and $U_2 J_1 = J_1 U_2$. Denote by $\max J_1 J_2$ ($\min J_1 J_2$) the maximum (minimum) point of $\sigma(J_1 \cdot J_2)$. We have the following theorem (this is an analogue of Theorem 2.2.2 of [2]).

THEOREM 1. *Let J_s, U_s , $s = 1, 2$, satisfy the above conditions. Suppose also that*

$$U_s^* J_s U_s - J_s = D_s, \quad \text{where either } D_s \geq 0 \text{ or } D_s \leq 0, \quad s = 1, 2.$$

Then

$$\|D_1 D_2\| \leq (1/2 \pi^2) m(\sigma(U_1, U_2))d, \quad \text{where } d = \max J_1 J_2 - \min J_1 J_2.$$

Proof. Suppose that $D_s \geq 0$, $s = 1, 2$, and

$$U_s = \int_{[0, 2\pi]^2} e^{i\lambda_s} dE, \quad s = 1, 2.$$

Let

$$S = \{(\lambda_1, \lambda_2) \in [0, 2\pi]^2, (e^{i\lambda_1}, e^{i\lambda_2}) \in \sigma(U_1, U_2)\}.$$

Denote by $S_c = [0, 2\pi]^2 \setminus S$ the complement of S . Let $f(\lambda) \neq 0$ be a function defined on $[0, 2\pi]^2$ of class C^k ($k > 1$) such that $f(\lambda) = 0$ on S . It follows that

$$F(\lambda_1, \lambda_2) = \sum_{-\infty}^{\infty} c_m e^{i(m_1 \lambda_1 + m_2 \lambda_2)} \quad \text{with } \sum_{-\infty}^{\infty} |c_m| < \infty \quad \text{and } \int f(\lambda) dE_\lambda = 0.$$

Now we have

$$\sum_{-m_1, -n_1}^{m_1, n_1} c_{k_1 k_2} U_1^{k_1} U_2^{k_2} = \int \sum_{-m_1, -n_1}^{m_1, n_1} c_{k_1 k_2} e^{i(k_1 \lambda_1 + k_2 \lambda_2)} dE,$$

and so the above sum converges to zero in uniform topology. Hence

$$(D_1 D_2)^{1/2} (c_{00} + \sum'_{k_1, k_2} c_{k_1 k_2} U_1^{k_1} U_2^{k_2}) = 0,$$

where \sum' denotes summation over all k_1, k_2 except $k_1 = 0, k_2 = 0$.

Now, applying the Schwarz inequality we have

$$(1) \quad \|c_{00}(D_1 D_2)^{1/2} x\|^2 \leq \sum_{k_1, k_2} |c_{k_1 k_2}|^2 \sum_{k_1, k_2} \|(D_1 D_2)^{1/2} U_1^{k_1} U_2^{k_2} x\|^2.$$

We also have the equality

$$\sum_{-n}^n U_s^{*k} D_s U_s^k = U_s^{n+1} J_s U_s^{*n+1} - U_s^{*n} J_s U_s^n \quad (s = 1, 2)$$

(see [2], p. 17). Therefore

$$\begin{aligned} \sum_{-m, -n}^{m, n} U_1^{*k_1} U_2^{*k_2} D_1 D_2 U_1^{k_1} U_2^{k_2} &= \left(\sum_{-m}^m U_1^{*k_1} D_1 U_1^{k_1} \right) \left(\sum_{-n}^n U_2^{*k_2} D_2 U_2^{k_2} \right) \\ &= (U_1^{n+1} J_1 U_1^{*n+1} - U_1^{*n} J_1 U_1^n) (U_2^{m+1} J_2 U_2^{*m+1} - U_2^{*m} J_2 U_2^m). \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{-m, -n}^{m, n} \|(D_1 D_2)^{1/2} U_1^{k_1} U_2^{k_2} x\|^2 \\ &= (J_1 J_2 U_1^{*m+1} U_2^{*n+1} x, U_1^{*m+1} U_2^{*n+1} x) - (J_1 J_2 U_1^m U_2^{*n+1} x, U_1^m U_2^{*n+1} x) + \\ &\quad + (J_1 J_2 U_1^m U_2^n x, U_1^m U_2^n x) - (J_1 J_2 U_1^{*m+1} U_2^n x, U_1^{*m+1} U_2^n x). \end{aligned}$$

Consequently,

$$\sum_{-m, -n}^{m, n} \|(D_1 D_2)^{1/2} U_1^{k_1} U_2^{k_2} x\|^2 \leq 2d \cdot \|x\|^2$$

and so

$$\sum_{-m, -n}^{m, n} \|(D_1 D_2)^{1/2} U_1^{k_1} U_2^{k_2} x\|^2 \leq 2d \|x\|^2 - (D_1 D_2 x, x).$$

Take a sequence x_p of unit vectors such that

$$D_1 D_2 x_p - \|D_1 D_2\| x_p \rightarrow 0 \quad \text{as } p \rightarrow \infty$$

By the Parseval equality we have

$$\sum_{k_1, k_2} |c_{k_1, k_2}|^2 = \alpha \int |f(\lambda_1, \lambda_2)|^2 d\lambda_1 d\lambda_2, \quad \text{where } \alpha = 1/4\pi^2.$$

Putting $x = x_p$ in (1) and applying the above relations we get

$$\begin{aligned} & |\alpha \int f(\lambda) d\lambda_1 d\lambda_2|^2 \cdot (D_1 D_2 x_p, x_p) \\ & \leq [\alpha \int |f(\lambda)|^2 d\lambda - |\alpha \int f(\lambda) d\lambda|^2] [2d - (D_1 D_2 x, x)]. \end{aligned}$$

Passing with p to infinity we obtain

$$\begin{aligned} (2) \quad & |\alpha \int f(\lambda) d\lambda_1 d\lambda_2|^2 \|D_1 D_2\| \\ & \leq [\alpha \int |f(\lambda)|^2 d\lambda_1 d\lambda_2 - |\alpha \int f(\lambda) d\lambda|^2] [2d - \|D_1 D_2\|]. \end{aligned}$$

Choose a sequence $f_p \in C^{(k)} [0, 2\pi]^2$ ($k > 1$) such that

- (a) $f_p|_S = 0$,
- (b) $f_p \rightarrow \psi_{S_c}$ almost everywhere with respect to the Lebesgue measure (denoted by m),
- (c) $\|f_p\|_x < M$ for every p .

This is possible in view of the results of [3], VI. Putting $f = f_p$ in (2) and letting $p \rightarrow \infty$ we see that (2) also holds for $f = \psi_{S_c}$.

Since

$$\int \psi_{S_c}^2 d\lambda = \int \psi_{S_c} d\lambda = m(S_c) = 4\pi^2 - m(S),$$

we can write

$$a\alpha \|D_1 D_2\| \leq [a\alpha - (\alpha a)^2] [2d - \|D_1 D_2\|], \quad \text{where } a = m(S_c).$$

Hence

$$\|D_1 D_2\| \leq (1/2 \pi^2) m(S) d.$$

The proof is complete.

As an application of Theorem 1 we obtain the following result.

THEOREM 2. Let A_1, A_2 be a commuting pair of selfadjoint operators and let B_1, B_2 be a doubly commuting pair of operators. Suppose that the following conditions are satisfied:

- (a) $A_s B_s - B_s A_s = C_s$, where $C_s \geq 0$ or $C_s \leq 0$,
 (b) $A_1 B_2 = B_2 A_1$ and $A_2 B_1 = B_1 A_2$.

Then

$$\|C_1 C_2\| \leq (d/2\pi^2) m(\sigma(A_1, A_2)),$$

where $d = \max(\operatorname{Im} B_1, \operatorname{Im} B_2) - \min(\operatorname{Im} B_1, \operatorname{Im} B_2)$.

Proof (following the ideas of Putnam [2]). We may assume that $C_s \geq 0$, $s = 1, 2$. Denoting $\operatorname{Im} B_s = J_s$ we have (by (a))

$$(3) \quad A_s J_s - J_s A_s = -iC_s, \quad s = 1, 2.$$

Note that $U_s = (A_s - iI)(A_s + iI)^{-1}$ is a unitary operator and by (3) we can write $U_s J_s U_s^* - J_s = 2Q_s^* C_s Q_s$, where $Q_s = (A_s - iI)^{-1}$.

Let $D_s = 2Q_s^* C_s Q_s \geq 0$. Since $U_k J_s = J_s U_k$ for $s \neq k$ (this can be easily derived from our assumptions), we can apply Theorem 1.

Hence

$$(4) \quad m(\sigma(U_1, U_2)) \geq (2\pi^2/d) \cdot \|D_1 D_2\|.$$

But $D_1 D_2 = D_2 D_1 \geq 0$ (by (b)), and so

$$\begin{aligned} \frac{1}{4} \|D_1 D_2\| &= \|Q_1^* C_1 Q_1 Q_2^* C_2 Q_2\| \geq \sup_{\|x\|=1} (Q_1^* Q_2^* C_1 C_2 Q_1 Q_2 x, x) \\ &\geq \|C_1 C_2\| \inf_{\|x\|=1} \|Q_1 Q_2 x\|^2. \end{aligned}$$

By the definition of Q_s we can write $(A_s = \int \lambda_s dE) Q_s = \int (\lambda_s - i)^{-1} dE$, and so

$$\|Q_1 Q_2 x\|^2 \geq (1 + \|A_1\|^2)^{-1} \|Q_2 x\|^2 \geq (1 + \|A_1\|^2)^{-1} (1 + \|A_2\|^2)^{-1} \|x\|^2.$$

The above estimations and (4) prove that

$$(5) \quad m(\sigma(U_1, U_2)) \geq (8\pi^2/d) \|C_1 C_2\| [(1 + \|A_1\|^2)(1 + \|A_2\|^2)]^{-1}.$$

If

$$(e^{i\varphi_1}, e^{i\varphi_2}) = \left(\frac{\lambda_1 - i}{\lambda_1 + i}, \frac{\lambda_2 - i}{\lambda_2 + i} \right), \quad 0 < \varphi_s < 2\pi, \lambda_s \in \mathbf{R},$$

then

$$d\varphi_1 d\varphi_2 = 4 [(1 + \lambda_1^2)(1 + \lambda_2^2)]^{-1} d\lambda_1 d\lambda_2.$$

Hence for

$$S = \{(\varphi_1, \varphi_2), (e^{i\varphi_1}, e^{i\varphi_2}) \in \sigma(U_1, U_2)\}$$

we have

$$\begin{aligned} m(\sigma(U_1, U_2)) &= m(S) = 4 \int_{\sigma(A_1, A_2)} [(1 + \lambda_1^2)(1 + \lambda_2^2)]^{-1} d\lambda_1 d\lambda_2 \\ &\leq 4m(\sigma(A_1, A_2)). \end{aligned}$$

For any $t > 0$ let $\tilde{A}_s = tA_s$. Since $tC_s = \tilde{A}_s B_s - B_s \tilde{A}_s$, we obtain (by (5))

$$\begin{aligned} 4m(\sigma(\tilde{A}_1, \tilde{A}_2)) &= 4t^2 m(\sigma(A_1, A_2)) \\ &\geq \frac{8\pi^2}{d} \|C_1 C_2\| t^2 [(1 + t\|A_1\|^2)(1 + t\|A_2\|^2)]^{-1}. \end{aligned}$$

Dividing the above inequality by t^2 and letting $t \rightarrow 0$ we get

$$m(\sigma(A_1, A_2)) \geq \frac{2\pi^2}{d} \|C_1 C_2\|.$$

The proof is complete.

Now we give an application of Theorem 2 to a doubly commuting pair of hyponormal (cohyponormal) operators. Namely, we have the following result (cf. [2], Theorem 2.2.1).

THEOREM 3. *Let T_1, T_2 be a doubly commuting pair of hyponormal (cohyponormal) operators. If $T_s = H_s + iJ_s$ and $T_s^* T_s - T_s T_s^* = D_s$, then*

$$\|D_1 D_2\| \leq (4/\pi^2) \|J_1 J_2\| m(\sigma(H_1, H_2)).$$

Proof. By the above assumptions and notation we have

$$H_s iJ_s - iJ_s H_s = -C_s, \quad \text{where } D_s = 2C_s.$$

According to Theorem 2,

$$(1/4) \|D_1 D_2\| \leq (d/2\pi^2) m(\sigma(H_1, H_2)),$$

where $d = \max(J_1 J_2) - \min(J_1 J_2)$. But $d \leq 2\|J_1 J_2\|$ and so

$$(6) \quad \|D_1 D_2\| \leq (4/\pi^2) \|J_1 J_2\| m(\sigma(H_1, H_2)).$$

The proof is complete.

Remark. By symmetry we also have

$$\|D_1 D_2\| \leq (4/\pi^2) \|H_1 H_2\| m(\sigma(J_1, J_2)).$$

Now we give an application of Theorem 3. However, first we need to generalize the well-known result of Putnam: if $T = H + iJ$ is hyponormal, then $\sigma(H) = \text{pr}_x \sigma(T)$, where pr_x denotes the projection onto the real axis. See [2], Theorem 3.4.1.

THEOREM 4. *Let T_1, T_2 be a doubly commuting pair of hyponormal*

(cohyponormal) operators. Denote by $\text{pr}_x: C^2 \rightarrow R^2$ the projection $\text{pr}_x(z_1, z_2) = (x_1, x_2)$, where $z_s = x_s + iy_s$. Then

$$\text{pr}_x(\sigma(T_1, T_2)) = \sigma(H_1, H_2).$$

Proof. (a) First we prove the inclusion $\text{pr}_x(\sigma(T_1, T_2)) \supseteq \sigma(H_1, H_2)$. Let $(\lambda_1, \lambda_2) \in \sigma(H_1, H_2)$. Choose a sequence $\{f_n\}$ of unit vectors such that

$$(\lambda_s - H_s)f_n \rightarrow 0, \quad s = 1, 2.$$

Then

$$J_s(H_s - \lambda_s)f_n \rightarrow 0, \quad s = 1, 2.$$

Since

$$(H_s - \lambda_s)J_s - J_s(H_s - \lambda_s) = -iC_s, \quad \text{where } 2C_s = D_s = [T_s^*, T_s] \geq 0,$$

we have

$$(f_n, (H_s - \lambda_s)J_s f_n) - (f_n, J_s(H_s - \lambda_s)f_n) = -i\|C_s^{1/2} f_n\|^2 \rightarrow 0, \quad s = 1, 2.$$

Thus $\|C_s f_n\| \rightarrow 0$, $s = 1, 2$, and so $(H_s - \lambda_s)J_s f_n \rightarrow 0$, $s = 1, 2$. But $J_t H_s = H_s J_t$ for $t \neq s$ and $J_1 J_2 = J_2 J_1$; identifying $J_s f_n$ ($s = 1, 2$) with the previous f_n , we see that $(H_s - \lambda_s)J_s J_t f_n \rightarrow 0$ for every s, t . Similarly, for any polynomial $p(J_1, J_2)$, $(H_s - \lambda_s)p(J_1, J_2)f_n \rightarrow 0$, $s = 1, 2$. Now if $\Phi(\lambda_1, \lambda_2)$ is any continuous function on R^2 , then, approximating $\Phi(J_1, J_2)$ by polynomials in J_1, J_2 we have

$$(H_s - \lambda_s)\Phi(J_1, J_2)f_n \rightarrow 0, \quad s = 1, 2.$$

Let E be the spectral measure of J_1, J_2 , i.e. $J_s = \int \lambda_s dE$. Assume that $\sigma(J_1, J_2) \subseteq [c, d] \times [c, d] = \Delta_1$. We have $\|E(\Delta_1)f_n\| = 1$, $n = 1, 2, \dots$. Dividing Δ_1 into the four equal squares we see that for at least one of these squares, say Δ_2 , the inequality $\|E(\Delta_2)f_n^{(2)}\| \geq 1/4$ holds for all n , where $\{f_n^{(2)}\}$ is a subsequence of $\{f_n^{(1)}\} = \{f_n\}$. Continuing, we produce a sequence of squares $\Delta_1, \Delta_2, \dots$ and a double-indexed sequence of vectors $\{f_n^{(k)}\}$, where $\Delta_{k+1} \subset \Delta_k$, $m(\Delta_k) = ((d-c)/2^k)^2$, $\{f_n^{(k+1)}\}$ is a subsequence of $\{f_n^{(k)}\}$, and $\|E(\Delta_k)f_n^{(k)}\| \geq 1/4^{k-1}$, $k, n = 1, 2, 3, \dots$. Let (λ'_1, λ'_2) be the limit point of (c_{k_1}, c_{k_2}) , where

$$\Delta_k = [(c_{k_1}, c_{k_2}), (d_{k_1}, c_{k_2}), (d_{k_1}, e_{k_2}), (c_{k_1}, e_{k_2})].$$

Take $\gamma_k \rightarrow 0$ ($\gamma_k > 0$).

Denote by $\tilde{\Delta}_k$ the γ_k -neighbourhood of Δ_k . Choose a sequence Φ_k of continuous functions on R^2 such that

$$\Phi_k(x) = \begin{cases} 1, & x \in \Delta_k, \\ \text{between 0 and 1,} & x \in \tilde{\Delta}_k \setminus \Delta_k, \\ 0, & x \in R^2 \setminus \tilde{\Delta}_k. \end{cases}$$

Next, define the sequence

$$g_{kn} = \frac{\Phi_k(J_1, J_2) f_n^{(k)}}{\|\Phi_k(J_1, J_2) f_n^{(k)}\|}.$$

Since $E(\omega)g_{kn} = 0$, for any Borel set ω such that $\omega \cap \tilde{\Delta}_k = \emptyset$ we can write

$$\begin{aligned} \|(J_1 - \lambda'_1)g_{kn}\|^2 &= \iint (\lambda_1 - \lambda'_1)^2 d(Eg_{kn}, g_{kn}) \\ &\leq (d_{k1} - c_{k1} + \gamma_k)^2, \quad n = 1, 2, \dots, \\ \|(J_2 - \lambda'_2)g_{kn}\|^2 &= \iint (\lambda_2 - \lambda'_2)^2 d(Eg_{kn}, g_{kn}) \\ &\leq (e_{k2} - c_{k2} + \gamma_k)^2, \quad n = 1, 2, \dots \end{aligned}$$

Hence there exists a subsequence n_k such that for $x_k = g_{kn_k}$ we have

$$\|(H_s - \lambda'_s)x_k\| \rightarrow 0, \quad \|(J_s - \lambda'_s)x_k\| \rightarrow 0, \quad s = 1, 2.$$

This completes the proof of inclusion (a).

(b) The opposite inclusion is immediate.

Let $(z_1, z_2) \in \overline{\sigma_\pi(T_1^*, T_2^*)}$, where $z_s = \lambda_s + i\mu_s$. Writing $R_s = T_s - z_s I$, we have

$$R_s R_s^* = (H_s - \lambda_s)^2 + (J_s - \mu_s)^2 + C_s, \quad C_s \geq 0, \quad s = 1, 2.$$

By our assumption there exists a sequence f_n ($\|f_n\| = 1$) such that

$$((R_1 R_1^* + R_2 R_2^*) f_n, f_n) \rightarrow 0.$$

Hence

$$((H_1 - \lambda_1)^2 + (H_2 - \lambda_2)^2 f_n, f_n) = \|(H_1 - \lambda_1) f_n\|^2 + \|(H_2 - \lambda_2) f_n\|^2 \rightarrow 0,$$

and so $(\lambda_1, \lambda_2) \in \sigma(H_1, H_2)$.

This completes the proof of the opposite inclusion and of the theorem.

Now we are ready to formulate a theorem which generalizes Theorem 3.7.1 of [2]. This is our main application of the previous results.

THEOREM 5. *Let T_1, T_2 be a doubly commuting pair of hyponormal (cohyponormal) operators. If $T_s = H_s + iJ_s$ and $T_s^* T_s - T_s T_s^* = D_s$, then either*

(a) *each of the sets $\sigma(H_1), \sigma(H_2), \sigma(J_1), \sigma(J_2)$ contains an interval*

or

(b) *$m_2(\sigma(T_1, T_2)) \geq \pi^2 \|D_1 D_2\|$; here m_2 denotes the Lebesgue measure in \mathbb{R}^4 .*

Proof. We follow the main ideas of [2] and combine them with Theorems 3 and 4. Recall that

$$\sigma(T_1, T_2) = \overline{\sigma_\pi(T_1^*, T_2^*)}.$$

Let $H_s = \int \lambda_s dE$. We may assume that $\sigma(H_1)$ does not contain any interval. If $\sigma(H_1, H_2) \subset \text{Int}((\alpha_1, \beta_1] \times (\alpha_2, \beta_2]) = \tilde{\Delta}$, then

$$(Z) \quad \partial\tilde{\Delta} \cap \sigma(H_1, H_2) = \emptyset.$$

Let $\Delta = (a_1, b_1] \times (a_2, b_2]$ be an arbitrary rectangle for which (Z) is satisfied. Write $J_s^{\Delta} = E(\Delta)J_sE(\Delta)$ and $H_s^{\Delta} = E(\Delta)H_sE(\Delta)$. Let (λ_1, λ_2) be the "middle" point of Δ . Since $H_sJ_s - J_sH_s = iC_s$, where $2C_s = D_s$, we have

$$(H_s^{\Delta} - \mu_s)J_s^{\Delta} - J_s^{\Delta}(H_s^{\Delta} - \mu_s) = iC_s^{\Delta}.$$

Now it is easy to check that $T_s^{\Delta} = E(\Delta)T_sE(\Delta)$ satisfy the assumptions of Theorem 3. Thus

$$\|D_1^{\Delta}D_2^{\Delta}\| \leq (4/\pi^2)\|(H_1^{\Delta} - \mu_1)(H_2^{\Delta} - \mu_2)\| \cdot m(\sigma(J_1^{\Delta}, J_2^{\Delta})).$$

Hence we can write

$$\begin{aligned} \|(D_1D_2)^{1/2}E(\Delta)x\|^2 &= (D_1D_2E(\Delta)x, E(\Delta)x) = (E(\Delta)D_1D_2E(\Delta)x, E(\Delta)x) \\ &= (D_1^{\Delta}D_2^{\Delta}E(\Delta)x, E(\Delta)x) \leq \|D_1^{\Delta}D_2^{\Delta}E(\Delta)x\| \cdot \|E(\Delta)x\| \\ &\leq (4/\pi^2)\|(H_1^{\Delta} - \mu_1)(H_2^{\Delta} - \mu_2)\| \cdot m(\sigma(J_1^{\Delta}, J_2^{\Delta})) \cdot \|E(\Delta)x\|^2. \end{aligned}$$

But

$$\|(H_1^{\Delta} - \mu_1)(H_2^{\Delta} - \mu_2)\| \leq m(\Delta)/4,$$

and so

$$(7) \quad 2\pi\|(C_1C_2)^{1/2}E(\Delta)x\| \leq [m(\Delta)m(\sigma(J_1^{\Delta}, J_2^{\Delta}))]^{1/2} \cdot \|E(\Delta)x\|.$$

We want to estimate $m(\Delta)m(\sigma(J_1^{\Delta}, J_2^{\Delta}))$, but first we observe what follows. If $(\dot{\mu}_1, \dot{\mu}_2) \in \sigma(J_1^{\Delta}, J_2^{\Delta})$, then by Theorem 4 there exist $(\dot{\lambda}_1, \dot{\lambda}_2)$ and $x_n = E(\Delta)x_n$ ($\|x_n\| = 1$) such that

$$(8) \quad (H_s - \dot{\lambda}_s)x_n \rightarrow 0, \quad s = 1, 2,$$

$$(9) \quad E(\Delta)(J_s - \dot{\mu}_s)x_n \rightarrow 0, \quad s = 1, 2.$$

We shall prove that

$$(10) \quad (J_s - \dot{\mu}_s)x_n \rightarrow 0, \quad s = 1, 2.$$

Since $(\dot{\lambda}_1, \dot{\lambda}_2) \in \sigma(H_1^{\Delta}, H_2^{\Delta})$, then $(\dot{\lambda}_1, \dot{\lambda}_2) \in \bar{\Delta}$ (the closure). On the other hand, $\partial\Delta \cap \sigma(H_1^{\Delta}, H_2^{\Delta}) = \emptyset$ (because $\partial\Delta \cap \sigma(H_1, H_2) = \emptyset$ and $\sigma(H_1^{\Delta}, H_2^{\Delta}) \subseteq \sigma(H_1, H_2)$) and so $(\dot{\lambda}_1, \dot{\lambda}_2) \in \text{Int}\Delta$.

Hence, putting $y_{ns} = (J_s - \dot{\mu}_s)x_n$, we get (by a reasoning similar to that in the proof of Theorem 4)

$$\|(H_1 - \dot{\lambda}_1)y_{ns}\|^2 + \|(H_2 - \dot{\lambda}_2)y_{ns}\|^2 \rightarrow 0, \quad s = 1, 2.$$

But $(\lambda_1, \lambda_2) \in \text{Int } \Delta$, so that $\|E(\mathbb{R}^2 \setminus \Delta) y_{ms}\| \rightarrow 0$. By (9) also $\|E(\Delta) y_{ms}\| \rightarrow 0$, $s = 1, 2$, and (10) must hold. Now we can estimate $m(\Delta)m(\sigma(J_1^A, J_2^A))$. By Theorem 4 we know that

$$\sigma(J_1, J_2) \subseteq \text{pr}_y(\sigma(T_1, T_2)),$$

and thus also

$$(11) \quad \sigma(J_1^A, J_2^A) \subseteq \text{pr}_y(\sigma(T_1, T_2)).$$

Let

$$\Omega = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4, (x_1, x_2) \in \Delta\}.$$

Divide Ω into at most countably many parallelepipeds P_i . Denote by \tilde{P}_i any P_i which has a non-empty intersection with $\sigma(T_1, T_2)$. It is clear (by (11)) that

$$(12) \quad m(\Delta)m(\sigma(J_1^A, J_2^A)) \leq \sum_{\tilde{P}_i} m_2(\tilde{P}_i) \stackrel{\text{df}}{=} S_{\Delta};$$

here m_2 stands for the Lebesgue measure in \mathbb{R}^4 . By our assumption (concerning $\sigma(H_1)$) we can find a partition of $\tilde{\Delta}$ into rectangles Δ_k such that $\partial\Delta_k \cap \sigma(H_1, H_2) = \emptyset$ for all k . Thus by (12) and (7) we have

$$2\pi \|(C_1 C_2)^{1/2} E(\Delta_k) x\| \leq (S_{\Delta_k})^{1/2} \|E(\Delta_k) x\|.$$

Hence, by the Schwarz inequality,

$$2\pi \|(C_1 C_2)^{1/2} x\| \leq 2\pi \sum_k \|(C_1 C_2)^{1/2} E(\Delta_k) x\| \leq \left(\sum_k S_{\Delta_k}\right)^{1/2} \left(\sum_k \|E(\Delta_k) x\|^2\right)^{1/2}.$$

Thus

$$(13) \quad (2\pi)^2 \|C_1 C_2\| \leq \sum_k S_{\Delta_k}.$$

Since our division of $\tilde{\Delta}$ into Δ_k and of the corresponding Ω_k (see the definition of Ω) into P_{ik} is quite arbitrary (because of the assumption imposed on $\sigma(H_1)$), one can check that

$$m(\sigma(T_1, T_2)) = \inf_{P_1 \in S_{\Delta_k}} \sum_k S_{\Delta_k},$$

where only those partitions Δ_k of $\tilde{\Delta}$ are allowed for which

$$\partial\Delta_k \cap \sigma(H_1, H_2) = \emptyset.$$

Inequality (13) and the above remarks finish the proof.

We conclude our paper with a few examples of doubly commuting hyponormal operators.

II. Concluding remarks and examples. Although we have stated the results of this paper only for commuting (doubly commuting) pairs of

operators, it is easy to extend them to tuples of several operators. This is immediate for Theorems 1, 2 and 3 and less obvious for the rest.

Before we proceed to examples note that one can derive from Theorem 5 Putnam's result for a single hyponormal operator as follows. Let T be a hyponormal operator in H . Define

$$T_1 = T \otimes I, \quad T_2 = I \otimes T \quad \text{on } H \otimes H.$$

If

$$D = [T^*, T] = T^* T - T T^*,$$

then

$$D_1 = [T_1^*, T_1] = D \otimes I, \quad D_2 = [T_2^*, T_2] = I \otimes D.$$

But $\sigma(T_1, T_2) = \sigma(T) \times \sigma(T)$, by the result of [6] (or as can be checked directly).

Hence by Theorem 5 (b) we have

$$m_2(\sigma(T_1, T_2)) = (m(\sigma(T)))^2 \geq \pi^2 \|D_1 D_2\| = \pi^2 \|D \otimes D\| = \pi^2 \|D\|^2,$$

i.e.

$$m(\sigma(T)) \geq \pi \|D\|.$$

A similar reasoning (applying (a)) gives the second part of Putnam's result.

Now we shall give a few examples of doubly commuting hyponormal operators.

EXAMPLE 1. Let $A \in L(H)$, $B \in L(K)$ be two arbitrary hyponormal operators. Suppose we are given $N_2 \in L(K)$, $N_1 \in L(H)$ two normal operators such that $N_1 A = A N_1$, $N_2 B = B N_2$. Define the operators

$$T_1 = A \otimes N_2, \quad T_2 = N_1 \otimes B.$$

It is clear that T_1, T_2 are hyponormal operators in $H \otimes K$, and by Fuglede's theorem they doubly commute.

EXAMPLE 2. Let A, B, C, D be operators in H . Assume that they satisfy the following conditions:

- (a) $\|Ax\| \geq \|B^*x\|$, $\|Bx\| \geq \|A^*x\|$, $\|Cx\| \geq \|D^*x\|$, $\|Dx\| \geq \|C^*x\|$,
 (b) $CB = AD$, $DA = BC$, $B^*D = CA^*$, $A^*C = DB^*$.

Then the operators

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \quad \text{on } H \oplus H$$

are both hyponormal and doubly commute.

EXAMPLE 3. Let (Ω, μ) be a measure space with a finite measure μ . If $F_i: \Omega \rightarrow L(H(\lambda))$, $i = 1, 2$, are essentially bounded functions, $F_1(\lambda)$, $F_2(\lambda)$ are hyponormal and doubly commute (μ -almost everywhere), then $\int \oplus F_1(\lambda) d\mu(\lambda)$ and $\int \oplus F_2(\lambda) d\mu(\lambda)$ are also hyponormal in $\int \oplus H(\lambda) d\mu(\lambda)$ and doubly commute.

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