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The punctured plane: alternating projections and L^2-angles

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Abstract. The paper exploits further the possibilities contained in the method of alternating projections [9], [12]. We apply Fourier analysis to study L^2-angles in multiply connected domains.

The method of alternating projections in various forms appears in the works by Schwarz, von Neumann, Wiener, Kaczmarz and others. Its fundamental role in the theory of holomorphic functions was discovered more recently (1983) in [9]. It permits us to describe the Bergman projection [1] in a domain $D = A \cup B$ in terms of (more elementary) Bergman projections in A and B. An angle between two corresponding subspaces of $L^2(D)$ is called the *L^2-angle between A and B* and is denoted by $\gamma(A, B)$. By construction, $\gamma(A, B)$ is invariant under biholomorphic mappings. It is a nice surprise that in many cases the alternating projections procedure can be carried out by explicit analytic culculations, and the value of the corresponding L^2-angle can be determined. Such investigations are greatly facilitated by fundamental relations between Fourier analysis, Laplace transforms and Bergman theory described by the author in [11], [12] and based on the L^2-type formulation of the Laplace transform due to Dzhrbashyan [2], [3], and Genchev [5], [6]. These developments point out to a powerful extension of Fourier analysis as envisaged by Mackey in [8] p. 309: *Going into the complex domain permits to extend Fourier analysis beyond its normal range.* Striking a philosophical note, one is tempted to say that we are witnessing here a manifestation of both the vitality and unity of mathematics, and to suspect that these qualities are essential for the recognized ability of mathematics to explain the empirical phenomena. In more ordinary terms, we may say that the present article represents a conscious step in a longer research program. We are concerned with the punctured plane $D = \boldsymbol{C}\setminus\{0\}$ considered as a union $D = A \cup B$, where

$$(1)\qquad A = \{z \in \boldsymbol{C};\ z \notin [0, \infty)\}, \quad B = \{z \in \boldsymbol{C};\ z \notin (-\infty, 0]\}.$$

The domain D is multiply connected, and we shall use this example to demonstrate that the Genchev transform is useful to study the L^2-angles not only in simply connected, but also in multiply connected domains. We shall describe in detail the relevant alternating projections procedure, and we shall prove that the L^2-angle $\gamma(A, B)$ for (1) is equal to zero.

1. Preliminary remarks. As usual, the space of all functions holomorphic and square integrable in a domain $W \subset C$ is denoted by $L^2H(W)$. Given a biholomorphic mapping φ: $U \to W$, the problem for $L^2H(W)$ can usually be restated as a problem for $L^2H(U)$, with reference to the canonical isometry φ^*: $L^2H(W) \to L^2H(U)$ given by $(\varphi^* f)(z) = f(\varphi(z))\varphi'(z)$. It is easy to verify the commutativity of Diagram 2, provided that Diagram 1 is commutative.

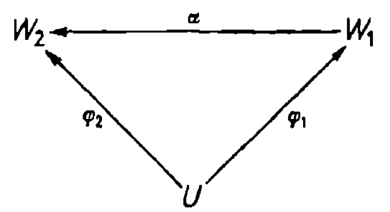

Diagram 1

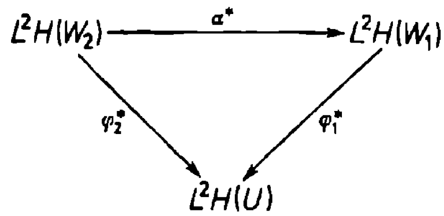

Diagram 2

Assuming that U equals A or B, we shall use the mapping $\varphi(z) = i\ln z$ to map U onto a tube in the complex plane. Different choices of the branch of logarithm will yield different mappings φ_1 and φ_2. In this particular case, α is a translation and (since $\alpha' \equiv 1$) the mapping α^* is given by the formula

$$(2) \qquad (\alpha^* f)(w) = f(\alpha(w)).$$

Now, let us consider a tube $T = \{z \in C;\ \operatorname{Re} z \in (a, b)\}$ and a function $f \in L^2H(T)$. The Genchev transform of f is independent of $x = \operatorname{Re} z$ and defined by

$$(3) \qquad G_f(t) = e^{2\pi tx} \int_{-\infty}^{\infty} e^{2\pi ity} f(x+iy)\,dy$$

which describes an element

$$G_f \in L^2(R, (e^{-4\pi at} - e^{-4\pi bt})/(4\pi t)).$$

By a fundamental result of Genchev [5] (simplified in [11]) the correspondence $f \mapsto G_f$ defines a unitary isomorphism of $L^2H(T)$ onto the space

$$L^2(R, (e^{-4\pi at} - e^{-4\pi bt})/(4\pi t)).$$

Assume that $g_1(w) = g_2(w+c)$ where $g_2 \in L^2H(T+c)$. Then it is easy to verify [11] that the respective Genchev transforms G_{g_1} and G_{g_2} are related by the formula

$$(4) \qquad G_{g_1}(t) = e^{-2\pi tc} G_{g_2}(t).$$

When $c = 2\pi$ it will be convenient to say that the transform G_{g_1} is obtained by *pulling G_{g_2} to the left*. As shown by (4) this amounts to multiplying G_{g_2} by

$$q = e^{-4\pi^2 t}.$$

When $c = -2\pi$, we shall say that G_{g_1} is obtained by *pulling G_{g_2} to the right.* Again (4) shows that this is equivalent to multiplying G_{g_2} by

$$q^{-1} = e^{4\pi^2 t}.$$

2. The proof of $\gamma(A, B) = 0$. It is easy to see that the only function which is L^2-holomorphic both in A and B (and therefore L^2-holomorphic in the punctured plane) is the zero function. Therefore, the general expression for the L^2-angle in $D = A \cup B$ (see [12]),

$$\cos\gamma(A, B) = \sup_{\substack{f \in L^2H(A)\setminus\{0\} \\ f \perp L^2H(D)}} \|P_B f\|/\|f\|, \tag{4a}$$

takes in the considered case a simpler form

$$\cos\gamma(A, B) = \sup_{f \in L^2H(A)\setminus\{0\}} \|P_B f\|/\|f\|. \tag{4b}$$

Here P_B is the Bergman projection in B and $\gamma(A, B) \in [0, \pi/2]$. The domain A is mapped biholomorphically by $i\ln z$ $(\ln(-1) = i\pi)$ onto the tube $\mathrm{Re}\, w \in (-2\pi, 0)$. Denote by h the Genchev transform of that function in the tube which corresponds to the function $f \in L^2H(A)\setminus\{0\}$ under the canonical isometry. To compute $\|P_B f\|^2$ we use the decomposition $f = f_+ + f_-$, where

$$f_+ = f \cdot \chi_{\mathrm{Im}\, z > 0}, \quad f_- = f \cdot \chi_{\mathrm{Im}\, z < 0}.$$

The main idea is to rewrite the present problem as a problem for tubes. A halfplane contained in the plane with removed radius is then mapped onto a tube of width π contained in a tube of width 2π. Here we can use the known expression for endogeneous operators in terms of the Genchev transform.

The images of several consecutive sectors in the punctured plane under the mapping $w = i\ln z$ are shown in Fig. 1.

Note two images of B under "adjacent" branches of $i\log z$, which appear in Fig. 1c. The function $P_B f$, which we now want to describe, is the sum of $P_B f_+$ and $P_B f_-$. It is easy to describe the image of $P_B f_+$ in the tube II (over $(-\pi, \pi)$), and the image of $P_+ f_-$ in the tube III (over $(-3\pi, -\pi)$) in terms of their Genchev transforms h_{II} and h_{III}. In fact the first function results from an endogeneous operator in the tube II by applying it to the image of $f_+ = f|_{\mathrm{Im}\, z > 0}$, while the second function results from an endogeneous operator in the tube III

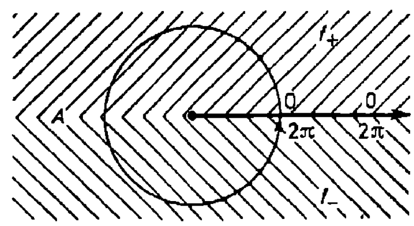

Fig. 1a

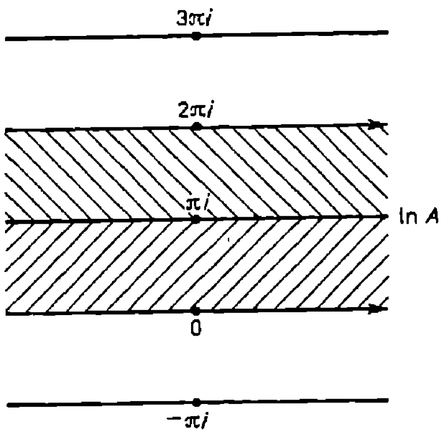

Fig. 1b

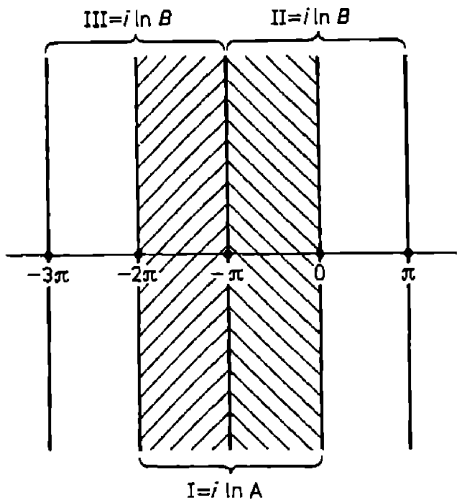

Fig. 1c

by applying it to the image of $f_- = f_{|\mathrm{Im}\, z<0}$. Therefore by Theorem 6 and Corollary 4 of [12]

$$h_{III} = h \cdot \frac{q}{1+q}, \quad h_{II} = h \cdot \frac{1}{1+q} \quad (q = e^{-4\pi^2 t}).$$

We shall find next the image of $P_B f_-$ in the tube II in terms of its Genchev transform h^{II}, using the knowledge of the corresponding image in the tube III. By the previous remarks h^{II} is obtained by pulling h_{III} to the right; more precisely,

$$h^{II} = q^{-1} h_{III} = h \cdot 1/(1+q).$$

The sum

$$h_{II} + h^{II} = h \cdot 2/(1+q)$$

is the Genchev transform of the image of $P_B f = P_B f_+ + P_B f_-$ in the tube II. Therefore, by the Genchev–Dzhrbashyan theorem

$$\|P_B f\|^2 = \int_{-\infty}^{\infty} \left| h \cdot \frac{2}{1+q} \right|^2 \frac{e^{-4\pi(-\pi)t} - e^{-4\pi(\pi)t}}{4\pi t} dt = \int_{-\infty}^{\infty} |h|^2 \frac{4}{(1+q)^2} \frac{q^{-1}-q}{4\pi t} dt.$$

On the other hand,

$$\|f\|^2 = \int_{-\infty}^{\infty} |h|^2 \frac{e^{-4\pi(-2\pi)t} - e^{-4\pi(0)t}}{4\pi t} dt = \int_{-\infty}^{\infty} |h|^2 \frac{q^{-2}-1}{4\pi t} dt.$$

Now, by the usual argument (see [12]) we conclude that $\cos^2\gamma(A, B)$ is equal to the maximal value attained by the ratio of integrands. Therefore we want to find the maximum of the function

$$\frac{4}{(1+q)^2} \frac{q^{-1}-q}{q^{-2}-1} = \frac{4q}{(1+q)^2}, \quad q \in (0, \infty).$$

This maximum equals 1 and is attained at $q = 1$. This completes the proof that

$$\gamma(A, B) = 0. \tag{4c}$$

3. Alternating projections. Let us recall the procedure of alternating projections in a domain $D = A \cup B$ (see also [10]). For $f \in L^2(D)$ it describes the Bergman projection $P_D f \in L^2 H(D)$ as the $L^2(D)$-limit of the following sequence. The first term f_1 is obtained by modifying f on the set B, by replacing the values of f by the values of $P_B f$, the Bergman projection of f in B. Then f_2 is obtained by modifying f_1 on A, by replacing the values of f_1 by the values of $P_B f_1$, the Bergman projection of f_1 in A. Next f_2 is modified over B to yield f_3, f_3 is modified over A to yield f_4, and so forth. When D is the punctured plane we have $L^2 H(D) = \{0\}$, hence f_n, $n = 1, 2, \ldots$, converges in $L^2(D)$ to 0. We can now give a more detailed description of this procedure. With no real loss of generality we may assume that $f \in L^2(A)$, and that h is the Genchev transform of the image of f in the tube I (over $(-2\pi, 0)$). We have already seen that the image of $f_1 = P_B f$ in the tube II (over $(-\pi, \pi)$) has the Genchev transform equal to

$$h_1 = h \cdot 2/(1+q). \tag{5}$$

Now we want to consider $f_2 = P_A f_1$, and the Genchev transform h_2 of its image in the tube I (over $(-2\pi, 0)$). To find h_2 we need to carry out a reasoning analogous to the one which was already presented in Section 2. The role of Fig. 1c is now played by Fig. 2.

To describe $P_A f_1^+$ and $P_A f_1^-$ we study the Genchev transforms h_{IV} and h_I of their images in the tubes IV (over $(0, 2\pi)$) and I (over $(-2\pi, 0)$). Again, by Theorem 6 and Corollary 4 of [12],

$$h_I = h_1 \cdot \frac{q}{1+q}, \quad h_{IV} = h_1 \cdot \frac{1}{1+q} \quad (q = e^{-4\pi^2 t}).$$

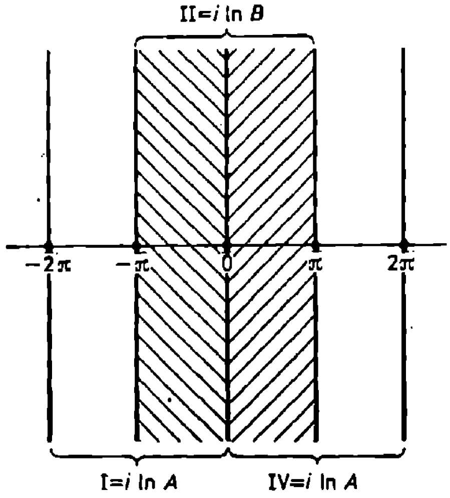

Fig. 2

We find next the image of $P_A f_1^+$ in the tube I (or rather the corresponding Genchev transform h^I) using the knowledge of the image in the tube IV. According to remarks in Section 1, the transform h^I is obtained by pulling h_{IV} to the left; more precisely,

$$h^I = q h_{IV} = h_1 \cdot \frac{q}{1+q}.$$

The sum

$$h_I + h^I = h_1 \cdot \frac{2q}{1+q} = h \cdot \frac{4q}{(1+q)^2}$$

is the Genchev transform of the image of $P_A f_1 = P_A f_1^+ + P_A f_1^-$ in the tube I. Since $f_2 = P_A f_1$ we obtain by the Genchev–Dzhrbashyan theorem

$$\|f_2\|^2 = \int_{-\infty}^{\infty} |h|^2 \frac{(4q)^2}{(1+q)^4} \frac{e^{-4\pi(-\pi)t} - e^{-4\pi(\pi)t}}{4\pi t} dt.$$

An obvious induction now yields

$$\|f_{2k}\|^2 = \int_{-\infty}^{\infty} |h|^2 \left[\frac{4q}{(1+q)^2}\right]^{2k} \frac{e^{4\pi^2 t} - e^{-4\pi^2 t}}{4\pi t} dt.$$

Now the obvious inequality

$$0 < \frac{4q}{(1+q)^2} < 1 \quad \text{for almost all } t \in \boldsymbol{R}$$

together with the Lebesgue dominated convergence theorem show that $\|f_{2k}\|^2$, $k = 1, 2, \ldots$, converges to zero. This implies that $\|f_n\|^2$, $n = 1, 2, \ldots$, converges to zero (since by construction the latter sequence is nonincreasing). Thus $P_D f = 0$, as claimed.

4. The question of stability. Let us assume that $D = A \cup B$ and

(6) $$D_s = A_s \cup B_s, \quad s \in S,$$

is a family of domains such that in some specified sense of convergence

$$D_s \to D, \quad A_s \to A, \quad B_s \to B,$$

as $s \to s_0$. If

$$\lim_{s \to s_0} \gamma(A_s, B_s) = \gamma(A, B)$$

we say that the pair (A, B) is *stable* with respect to the family (6), otherwise it is *unstable*. That unstability can in fact occur was shown by Jakóbczak and Mazur [7]. In the present section we shall be concerned with a family (6) where D_s, $s \in (0, \infty)$, is the complex plane without the closed segment $[-s, s]$; furthermore,

(6a) $$A_s = \{z \in C;\ z \notin [-s, \infty]\}, \quad B_s = \{z \in C;\ z \notin (-\infty, s]\}, \quad s \in (0, \infty).$$

We shall show that $\gamma(A_s, B_s) = 0$ for all s; this obviously implies that the pair (1) is stable with respect to the family (6a) when s converges to 0. We begin with some auxiliary results:

LEMMA 1. *The L^2-angle $\gamma(A_s, B_s)$ is independent of $s \in (0, \infty)$.*

Proof. This is obvious in view of the fact that L^2-angle is invariant under biholomorphic transformations.

LEMMA 2. *For every $\varepsilon > 0$ there exists $f \in L^2H(A)$ such that $\|f\| = 1$ and $\|P_B f\| > 1-\varepsilon$.*

Proof. In view of (4b) the above statement is equivalent to (4c).

LEMMA 3. *For every $f \in L^2(C)$ the inequalities*

(7) $$\|P_{B_s} f - P_B f\| < \varepsilon,$$

(8) $$\|P_{D_s} f - P_D f\| < \varepsilon,$$

are valid for all sufficiently small s. (Note that $P_D f = 0$.)

Proof. This follows from $D_s \nearrow D$, $B_s \nearrow B$ and Theorem 4 in [10].

Now we are prepared for:

THEOREM 1. *For every $s \in (0, \infty)$*

$$\gamma(A_s, B_s) = 0.$$

Proof. Let ε be an arbitrary number in $(0, \frac{1}{3})$ and let $f \in L^2H(A)$ be as in Lemma 2. Assume that s is so small that (7) and (8) are valid. Let us introduce

$f_s(z) = f(z-s)$. Then $\|f_s\| = 1, f_s \in L^2 H(A_s)$. Moreover, we may assume that s is so small that

$$\|f_s - f\| < \varepsilon. \tag{9}$$

(A proof of (9) can be found in [4], p. 15.) Note that (8) and (9) imply

$$\|P_{D_s} f_s\| \leqslant \|P_{D_s} f\| + \|P_{D_s}(f_s - f)\| < 2\varepsilon. \tag{10}$$

Similarly (7) and (9) imply

$$\begin{aligned} \|P_{B_s} f_s\| &= \|P_{B_s}(f_s - f) + (P_{B_s} f - P_B f) + P_B f\| \\ &\geqslant \|P_B f\| - \|P_{B_s}(f_s - f)\| - \|P_{B_s} f - P_B f\| > 1 - 3\varepsilon. \end{aligned} \tag{11}$$

The function f_s is not necessarily orthogonal to $L^2 H(D_s)$, therefore we introduce the function $f_s^* := f_s - P_{D_s} f_s$. In view of (10) this is a "small" modification. By (11)

$$\begin{aligned} \|P_{B_s} f_s^*\| &\geqslant \|P_{B_s} f_s\| - \|P_{B_s} P_{D_s} f_s\| \\ &= \|P_{B_s} f_s\| - \|P_{D_s} f_s\| > 1 - 3\varepsilon - 2\varepsilon = 1 - 5\varepsilon. \end{aligned}$$

Moreover,

$$\|f_s^*\| \geqslant \|f_s\| - \|P_{D_s} f_s\| > 1 - 2\varepsilon, \qquad \|f_s^*\| \leqslant \|f_s\| + \|P_{D_s} f_s\| < 1 + 2\varepsilon.$$

This implies that $f_s^* \in L^2 H(A_s) \setminus \{0\}$, is orthogonal to $L^2 H(D_s)$ and

$$\frac{\|P_{B_s} f_s^*\|}{\|f_s^*\|} > \frac{1 - 5\varepsilon}{1 + 2\varepsilon}.$$

Therefore by (4a) for all sufficiently small s

$$\cos \gamma(A_s, B_s) \geqslant (1 - 5\varepsilon)/(1 + 2\varepsilon).$$

By Lemma 1 this inequality is valid for all s. When ε approaches 0 we see that

$$\cos \gamma(A_s, B_s) = 1.$$

Hence the proof is complete.

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