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On the approximation of continuous functions by solutions to partial complex differential equations

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Abstract. Let G be a bounded domain in the complex z-plane, $z = x+iy$ and $\bar{z} = x-iy$. Denote by $\mathfrak{C}_{\boldsymbol{R}}(\bar{G})$ the space of all real-valued functions defined and continuous in $\bar{G}$. Analogously the corresponding space of all complex-valued functions is denoted by $\mathfrak{C}_{\boldsymbol{C}}(\bar{G})$. Further, by $\boldsymbol{R}[X_1, \ldots, X_n]$ and $C[X_1, \ldots, X_n]$ are denoted the rings of all polynomials in $X_1, \ldots, X_n$, where their coefficients are real and complex, respectively. Then the Weierstrass–Stone theorem (see, for instance, Aleksandrov [2]) states that $\boldsymbol{R}[x, y]$ is dense in $\mathfrak{C}_{\boldsymbol{R}}(\bar{G})$ and, analogously, $C[z, \bar{z}]$ is dense in $\mathfrak{C}_{\boldsymbol{C}}(\bar{G})$. On the other hand, the function g defined by $g(z) = z$ is holomorphic. The aim of the present paper is to deduce sufficient conditions under which solutions to general partial complex differential equations generate rings which are dense in $\mathfrak{C}_{\boldsymbol{C}}(\bar{G})$.

1. Approximation of continuous functions by given solutions. Let g be continuous and univalent in $\bar{G}$. Then $\operatorname{Re} g(z_1) \neq \operatorname{Re} g(z_2)$ or $\operatorname{Im} g(z_1) \neq \operatorname{Im} g(z_2)$ for any two points different from each other. Thus the Weierstrass–Stone theorem implies that $\boldsymbol{R}[\operatorname{Re} g, \operatorname{Im} g]$ is dense in $\mathfrak{C}_{\boldsymbol{R}}(\bar{G})$. Therefore the following lemma holds:

Lemma. *The ring $C[g, \bar{g}]$ is dense in $\mathfrak{C}_{\boldsymbol{C}}(\bar{G})$.*

Example. Let $w = g_0(z)$ be the basic homeomorphic solution of the Beltrami equation

$$\frac{\partial w}{\partial \bar{z}} = q(z) \frac{\partial w}{\partial z} \tag{1}$$

in the whole plane, $|q(z)| \leqslant q_0 < 1$ (cf., for instance, Ahlfors [1] or Vekua [6]). Then every solution of (1) may be represented as a superposition $\Phi \circ g$, where Φ is holomorphic. Since $\Phi \circ g_0$ is univalent iff Φ is univalent, the following assertion results from the lemma:

For every holomorphic Φ univalent in the image of $\bar{G}$ defined by g_0 the ring $C[\Phi \circ g_0, \overline{\Phi \circ g_0}]$ is dense in $\mathfrak{C}_{\boldsymbol{C}}(\bar{G})$.

Now regard the partial complex differential equation

(2) $$\partial w/\partial \bar{z} = F(z, w, \partial w/\partial z),$$

where the right-hand side fulfils the Lipschitz condition

(3) $$|F(z_2, w_2, h_2) - F(z_1, w_1, h_1)| \leqslant l(|z_2 - z_1|^\lambda + |w_2 - w_1| + |h_2 - h_1|),$$

where λ is a given Hölder exponent, $0 < \lambda < 1$. This condition ensures that the superposition $F(\cdot, w, h)$ is Hölder-continuous with exponent λ in $\bar{G}$ if $w = w(z)$ and $h = h(z)$ are Hölder-continuous with the same exponent. Let $w = g(z)$ be a given solution of (2). If T_G and Π_G are the well-known double-integral operators with the singularities $1/(\zeta - z)$ and $1/(\zeta - z)^2$, respectively (cf. Vekua [6]), then

(4) $$\Phi = w - T_G F(\cdot, w, \partial w/\partial z)$$

is holomorphic and $\partial w/\partial z$ satisfies the singular integro-differential equation

(5) $$\partial w/\partial z = \Phi' + \Pi_G F(\cdot, w, \partial w/\partial z).$$

The norm in the space of functions Hölder-continuous in $\bar{G}$ is denoted by $\|\cdot\|$. The norms of operators T_G and Π_G in the space of Hölder-continuous functions are denoted by $\|T_G\|$ and $\|\Pi_G\|$, respectively.

Suppose that the holomorphic function Φ fulfils the condition

(6) $$\inf_G |\Phi'| > (1 + \|\Pi_G\|)\|F(\cdot, w, \partial w/\partial z)\|.$$

Taking into consideration relations (2) and (5) from the last inequality, one easily gets

(7) $$|\partial w/\partial z|^2 - |\partial w/\partial \bar{z}|^2 > 0$$

such that the given solution $w = g(z)$ defines a locally univalent orientable mapping of $\bar{G}$ into the complex plane. Let φ be a uniformizer of the Riemann surface defined by the given solution $w = g(z)$ (concerning the uniformization of Riemann surfaces see Nevanlinna [4]). Then the composite function $\varphi \circ g$ is globally univalent in $\bar{G}$. Applying the lemma we prove the following theorem:

THEOREM 1. *Let $w = g(z)$ be a given solution to the differential equation (2). Suppose that the holomorphic function Φ that corresponds to the given solution $w = g(z)$ in the sense of (4) fulfils inequality (6). Then $C[\varphi \circ g, \overline{\varphi \circ g}]$ is dense in $\mathfrak{C}_C(\bar{G})$.*

EXAMPLE. In view of (4) in the case $F = 0$ the solutions of (2) are identical with holomorphic functions Φ. Conditions (6) is fulfilled iff $\Phi' \neq 0$ everywhere in $\bar{G}$.

2. Construction of solutions by which continuous functions can be approximated. Let Φ be a given function continuous in $\bar{G}$ and holomorphic in G. Regard the operator defined by

$$W = \Phi + T_G F(\cdot, w, h), \quad H = \Phi' + \Pi_G F(\cdot, w, h). \tag{8}$$

If (w, h) is a fixed point of this operator, then w turns out to be a solution of the differential equation (2). Thus operator (8) allows us to construct a mapping between holomorphic functions and solutions of the differential eqation (2). In the space of all Hölder-continuous pairs (w, h) we define a bicylinder $\mathfrak{D}$ by

$$\mathfrak{D} = \{(w, h): \|w - \Phi\| \leqslant d_1, \ \|h - \Phi'\| \leqslant d_2\},$$

where d_1, d_2 are certain positive constants. In order to construct fixed points of the operator defined by (8) we will investigate its behaviour in the bicylinder $\mathfrak{D}$. Accordingly, we suppose that the right-hand side $F(z, w, h)$ of (2) fulfils the Lipschitz condition (3) in the bicylinder $\mathfrak{D}$. Further we assume that the right-hand side $F(z, w, h)$ fulfils the Hölder–Lipschitz condition

$$\|F(\cdot, w, h) - F(\cdot, \tilde{w}, \tilde{h})\| \leqslant L_1 \|w - \tilde{w}\| + L_2 \|h - \tilde{h}\| \tag{9}$$

in $\mathfrak{D}$. Denote $\|F(\cdot, \Phi, \Phi')\|$ by $\tilde{M}$. Then for $(w, h) \in D$ the estimate

$$\begin{aligned} \|F(\cdot, w, h)\| &\leqslant \|F(\cdot, w, h) - F(\cdot, \Phi, \Phi')\| + \|F(\cdot, \Phi, \Phi')\| \\ &\leqslant L_1 d_1 + L_2 d_2 + \hat{M} \end{aligned}$$

holds. Using this estimate, one easily gets the following results on the operator defined by (8):

(a) It maps $\mathfrak{D}$ into itself if

$$\begin{aligned} &\|\Phi\| + \|T_G\| (L_1 d_1 + L_2 d_2 + M) \leqslant d_1, \\ &\|\Phi'\| + \|\Pi_G\| (L_1 d_1 + L_2 d_2 + M) \leqslant d_2. \end{aligned} \tag{10}$$

(b) It is contractive if

$$\|T_G\| (L_1 + L_2) < 1, \quad \|\Pi_G\| (L_1 + L_2) < 1. \tag{11}$$

Provided that inequalities (10) and (11) are fulfilled the Banach fixed-point theorem leads to the solution looked for. This solution w is uniquely determined in $\mathfrak{D}$.

Additionally, assume that the given holomorphic function fulfils the inequality

$$\inf_G |\Phi'| > (1 + \|\Pi_G\|)(L_1 d_1 + L_2 d_2 + \tilde{M}). \tag{12}$$

This inequality implies that the constructed solution $w = g(z)$ fulfils inequality (7). Therefore the solution w defines a locally univalent mapping of $\bar{G}$ into the complex plane. Once again using a uniformizer φ of the resulting Riemann surface, we find that the superposition $\varphi o g$ turns out to be univalent. Applying the lemma, one gets the following theorem for right-hand sides $F(z, w, h)$ fulfilling inequalities (3), (9), and (11):

THEOREM 2. *Let Φ be a given holomorphic function that fulfils inequalities* (10) *and* (12).

Then the ring $C[\varphi \circ g, \overline{\varphi \circ g}]$ is dense in $\mathfrak{C}_C(\bar{G})$, where $w = g(z)$ is the solution of (2) *generated by Φ.*

Connections between L_1, L_2, M and d_1, d_2 may be discussed in the some way as in [5].

References

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