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Harmonic functions in four variables with rational and algebraic p_4-associates

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Abstract. Integral representations for harmonic functions in four variables are investigated by means of an operator bearing close resemblence to the Whittaker-Bergman operator. The cases where the p_4-associate is algebraic is considered by means of the theory of double integrals on algebraic three-folds. When the p_4-associate is rational one obtains particulary interesting representations by considering the connections with Weierstrass integrals of the first, second, and third kinds defined over a Riemann surface. In addition, a residue theorem is given for a class of harmonic vectors $U \equiv (u_1, u_2, u_3, u_4)$ satisfying the relations

$$\varepsilon_{mnrs}\frac{\partial u_r}{\partial x_s} = 0, \qquad \frac{\partial u_\nu}{\partial x_\nu} = 0,$$

which are analogous to the vanishing of the curl and divergence in three-dimensions.

I. Introduction. The solutions of Laplace's equation in four variables,

$$\Box u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0, \tag{1}$$

may be generated by means of an integral operator $p_4[f]$, which bears a close resemblance to the Whittaker-Bergman operator $p_3[f]$ ([1], [2], [3], [4], [7], [8], [10], [11], [12], [13], [17]). The operator $p_4[f]$ transforms analytic functions of three complex variables into harmonic functions of four variables ([8], [12], [9]),

$$u(X) = p_4[f] = -\frac{1}{4\pi^2}\int\int_{\mathfrak{D}} f(\tau, \eta, \xi)\frac{d\eta}{\eta}\frac{d\xi}{\xi}, \tag{2}$$

where

$$\tau = x_1\left(1+\frac{1}{\eta\xi}\right) + ix_2\left(1-\frac{1}{\eta\xi}\right) + x_3\left(\frac{1}{\xi}-\frac{1}{\eta}\right) + ix_4\left(\frac{1}{\xi}+\frac{1}{\eta}\right),$$

$$\|X-X^0\| < \varepsilon, \qquad X \equiv (x_1, x_2, x_3, x_4), \qquad X^0 \equiv (x_1^0, x_2^0, x_3^0, x_4^0),$$

$\mathfrak{D} = C \times \Gamma$ is the product of a contour C in the ξ-plane, and a contour Γ in the η-plane, and $\varepsilon > 0$ is taken sufficiently small. We restrict $\mathfrak{D}$ further by insisting for each choice of $f(\tau, \eta, \xi)$ the integrand is absolutely integrable ([3], [15]); here the double integral may be regarded as an iterated integral, and the orders of integration may be interchanged.

In order to realize how the operator p_4 transforms analytic functions in harmonic functions we may introduce the homogeneous, harmonic polynomials of degree n which are defined as follows ([6], [8], [9], [12]):

$$\tag{3} \begin{aligned} \tau^n &= \left\{x_1\left(1+\frac{1}{\eta\xi}\right)+ix_2\left(1-\frac{1}{\eta\xi}\right)+x_3\left(\frac{1}{\xi}-\frac{1}{\eta}\right)+ix_4\left(\frac{1}{\xi}+\frac{1}{\eta}\right)\right\}^n \\ &= \sum_{k,l=0}^{n} H_n^{k,l}(X)\,\xi^{-k}\eta^{-l}. \end{aligned}$$

In view of (3), these polynomials have an integral representation

$$\tag{4} H_n^{k,l}(X) = -\frac{1}{4\pi^2}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{|\eta|=1}\frac{d\eta}{\eta}\tau^n\eta^k\xi^l,$$

where k, l are integers from 0 to n. Because of this representation, and because the $H_n^{k,l}(X)$ form a complete set of harmonic polynomials, we may generate any harmonic function regular in the small of the origin,

$$\tag{5} u(X) = \sum_{n=0}^{\infty}\sum_{k,l=0}^{n} a_{nkl}H_n^{kl}(X),$$

from an analytic function,

$$\tag{6} f(\tau, \eta, \xi) = \sum_{n=0}^{\infty}\sum_{k,l=0}^{n} a_{nkl}\tau^n\eta^k\xi^l,$$

by means of the operator $p_4[f]$. We shall (following Kreyszig [12] and Bergman [4]) refer to (6) as the *normalized associated function of* $u(X)$ *with respect to* p_4, or more concisely as the p_4-associate of $u(X)$.

II. Algebraic associates. In this section we consider the case, where

$$\tag{7} \eta^{-1}\xi^{-1}f(\tau, \eta, \xi) = \frac{p_1(S; \tau, \eta, \xi)}{p_2(S; \tau, \eta, \xi)} \equiv \frac{P_1(X; S; \eta, \xi)}{P_2(X; S; \eta, \xi)}$$

is a rational function of S, τ, η, ξ, and where S, τ, η, ξ are connected by an algebraic equation [1]

$$\tag{8} \Omega(S; \tau, \eta, \xi) \equiv A_0(\tau, \eta, \xi)S^n + A_1(\tau, \eta, \xi)S^{n-1} + \dots + A_n(\tau, \eta, \xi) = 0,$$

[1] This case is an extension to four variables of the case considered by Professor Bergman in [5].

where the $A_\nu(\tau, \eta, \xi)$ are polynomials in τ, η, ξ. If we multiply (8) by a suitable power of η and ξ, we obtain

$$(9) \qquad \mathfrak{Q}(X; S; \eta, \xi) \equiv A_0(X; \eta, \xi) S^n + A_1(X; \eta, \xi) S^{n-1} + \dots + A_n(X; \eta, \xi) = 0\,,$$

where the $A_\nu(X; \eta, \xi)$ are polynomials in η, ξ, and x_1, x_2, x_3, x_4.

The equation $\mathfrak{Q}(X; S; \eta, \xi) = 0$ defines for each fixed value of X an algebraic surface in S, η, ξ. The surface may also be represented parametrically as

$$(10) \qquad S = \chi(X; \alpha, \beta), \qquad \eta = \varphi(X; \alpha, \beta), \qquad \xi = \psi(X; \alpha, \beta)\,,$$

where χ, φ, ψ are algebraic functions in α, β. Using this representation we consider the sets of equations

$$(11) \qquad \begin{aligned} \chi(X; \alpha, \beta) &= \chi(X; \alpha', \beta')\,, \\ \varphi(X; \alpha, \beta) &= \varphi(X; \alpha', \beta')\,, \\ \psi(X; \alpha, \beta) &= \psi(X; \alpha', \beta') \end{aligned}$$

and

$$(12) \qquad \begin{aligned} \chi(X; \alpha, \beta) &= \chi(X; \alpha', \beta') = \chi(X; \alpha'', \beta'')\,, \\ \varphi(X; \alpha, \beta) &= \varphi(X; \alpha', \beta') = \varphi(X; \alpha'', \beta'')\,, \\ \psi(X; \alpha, \beta) &= \psi(X; \alpha', \beta') = \psi(X; \alpha'', \beta'')\,. \end{aligned}$$

There is an ∞^1 of solutions to (11) and this will be in general a line through which pass two nappes of the surface. This line is called a *double curve.* There will be only a limited number of solutions to (12); these are the *triple points* of the surface, through which passes three double lines and three nappes of the surface. The double curves, and triple points are singularities of the surface. If by a birational transformation the surface is mapped into another algebraic surface, whose only singularities are a double curve and its triple points, the singularities are called *ordinary singularities* ([16]).

We now suppose for $X = X^0$ (an initial point) the algebraic surface $\mathfrak{Q}(X^0; S; \eta, \xi) = 0$ has just ordinary singularities. We consider integrals on $\mathfrak{Q} = 0$ of the form,

$$(13) \qquad U(X) = -\frac{1}{4\pi^2} \int\!\!\int_{\mathfrak{D}} \frac{P_1(X; S; \eta, \xi)}{P_2(X; S; \eta, \xi)}\, d\eta\, d\xi\,,$$

where $X \in N(X^0)$, and $N(X^0)$ is a sufficiently small neighborhood of X^0 such that $\mathfrak{Q}(S; X^0; \eta, \xi) = 0$ has only ordinary singularities. It is convenient sometimes to write

$$(14) \qquad \frac{P_1(X; S; \eta, \xi)}{P_2(X; S; \eta, \xi)} = \frac{R(X; S; \eta, \xi)}{\partial \mathfrak{Q}(X; S; \eta, \xi)/\partial S}\,,$$

where $R(X; S; \eta, \xi)$ is rational in S, η, ξ. In Picard and Simart ([16]) it is stated that the necessary and sufficient conditions for an integrand (14) to always correspond to a finite valued integral (13) is that $R(X; S; \eta, \xi) \equiv Q(X; S; \eta, \xi)$ be a polynomial of degree $n-4$, and that $Q(X; S; \eta, \xi)$ pass through the double curve of $\mathfrak{Q} = 0$. In this case, we call the integral (13) a *double integral of the first kind* ([16]). There is a number $p_g(X)$ associated with the surface $\mathfrak{Q} = 0$, called the *geometric genus* or *Flachengeschlecht*, which plays a role similar to the Riemannian genus of an algebraic curve. This number $p_g(X)$ is the number of linearly independent double integrals of the first kind, that is the number of linearly independent polynomials of degree $n-4$, $Q_k(X; S; \eta, \xi)$, which pass through the double curve of $\mathfrak{Q} = 0$. It is understood that $p_g = p_g(X)$ is essentially independent of X (²). If $\mathfrak{B}$ is the set of points for which the surface $\mathfrak{Q} = 0$ only has ordinary singularities, the geometric genus is given by the simple formula ([16])

$$p_g = \frac{(n-1)(n-2)(n-3)}{6}. \tag{15}$$

This follows from the fact, that p_g is invariant under a birational transformation, and that p_g is the number of arbitrary constants in the most general surface of degree $n-4$ with no singular points. Consequently, the most general integrals of the first kind representing $u(X)$ may be expressed as

$$u(X) = -\frac{1}{4\pi^2}\sum_{k=1}^{p_g}\iint_{\mathfrak{D}} A_k \frac{Q_k(X; S, \eta, \xi)}{\partial\mathfrak{Q}(X; S, \eta, \xi)/\partial S}\, d\eta\, d\xi . \tag{16}$$

We say that the integral (13) is a *double integral of the second kind over the algebraic surface* ([5], [9]), if

$$\iint_{\mathfrak{D}} \frac{P_1(X; S, \eta, \xi)}{P_2(X; S, \eta, \xi)}\, d\eta\, d\xi - \iint_{\mathfrak{D}} \left(\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right) d\eta\, d\xi$$

remains finite for all $\mathfrak{D}$. (U and V are rational functions.) Furthermore, if $\mathfrak{Q} = 0$ is an algebraic surface with just ordinary singularities it is known that the integrals of the second kind representing $u(X)$ have the form

$$u(X) = -\frac{1}{4\pi^2}\iint_{\mathfrak{D}} \frac{P(X; S, \eta, \xi)}{\partial\mathfrak{Q}(X; S, \eta, \xi)/\partial S}\, d\eta\, d\xi + \frac{1}{4\pi^2}\iint_{\mathfrak{D}} \left(\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right) d\eta\, d\xi , \tag{17}$$

where $P(X; S, \eta, \xi)$ is a polynomial which vanishes on the double curve of $\mathfrak{Q} = 0$. This result suggests the definition ([15]) that a set of integrals

(²) See footnote (¹), p. 335.

of the second kind are distinct if any linear combination is not equal to the form

$$\iint_{\mathfrak{D}} \left(\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right) d\eta\, d\xi \,.$$

From this definition follows the fundamental theorem on double integrals of the second kind ([15]):

For an algebraic surface $\Omega = 0$ there are ϱ_0 distinct double integrals of the second kind, $I_1, I_2, \ldots, I_{\varrho_0}$, such that any double integral of the second kind may be written in the form

$$a_1 I_1 + a_2 I_2 + \ldots + a_{\varrho_0} I_{\varrho_0} + \iint_{\mathfrak{D}} \left(\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right) d\eta\, d\xi \,,$$

where the a_k are constants and U, V are rational in S, η, ξ. Furthermore, if $\Omega = 0$ is simply connected the number ϱ_0 is given by the formula

$$\varrho_0 = N - 4p - (n-1) - (\varrho - 1) \,,$$

where N is the class of the surface, p is the genus of an arbitrary plane section, and n is the degree. $(p-1)$ is the number of particular irreducible algebraic curves $\{C_i\}_{i=1}^{\varrho-1}$, which may be drawn on $\Omega = 0$, such that there does not exist a total differential of the third kind, having for its "logarithmic curves" one or more of the curves $[\{C_i\}_{i=1}^{\varrho-1}$ + the curve at $\infty]$, but if another arbitrary curve C_ϱ is added to the set there exists an integral having one or more curves from this new set as its "logarithmic curves".

The class N of a surface defined by the equation $\Omega(X;\, S, \eta, \xi) = 0$ is the number of values of $\xi = \xi_i$, $i = 1, 2, \ldots, N$, for which the genus of the Riemann surface, $E\{\Omega = 0\} \cap \{\xi = \xi_i\}$, is less than p. It is clear, that the genus will be diminished by one when the plane $\xi = \xi_i$ becomes tangent to the surface $\Omega = 0$ at a simple point; such a point constitutes in general a double point of the surface. From the above discussion we realize that if the integral representing $u(X)$ is of the second kind one may write

$$(18)\quad u(X) = \sum_{k=1}^{\varrho_0} a_k I_k(X) + \frac{1}{4\pi^2} \iint_{\mathfrak{D}} \left(\frac{\partial U(X;\, S, \eta, \xi)}{\partial \eta} + \frac{\partial V(X;\, S, \eta, \xi)}{\partial \xi}\right) d\eta\, d\xi \,,$$

where the $\{I_k(X)\}$ are a unique linearly independent set of integrals of the second kind over $\Omega = 0$.

III. Rational associates. When $f(\tau, \eta, \xi)$ is a rational function of τ, η, ξ it is possible to obtain representation formulae for $u(X)$ in terms of certain classical functions. In certain special cases we shall be able to reproduce the representations obtained by Bergman ([1], [3], [17]) and Mitchell ([13]) for harmonic functions of three variables with algebraic associates. For instance let us suppose

$$f(\tau, \eta, \xi) \equiv \frac{q_1(\tau, \eta, \xi)}{q_2(\tau, \eta, \xi)} \eta\xi = \frac{Q_1(X; \eta, \xi)}{Q_2(X; \eta, \xi)} \tag{19}$$

and let us consider

$$u(X) = -\frac{1}{4\pi^2} \iint_{\mathfrak{D}} \frac{Q_1(X; \eta, \xi)}{Q_2(X; \eta, \xi)} d\eta\, d\xi, \qquad \mathfrak{D} = \Gamma \circ C.$$

The singularity manifold of the integrand may be represented as

$$\begin{cases} Z^4 \equiv E\{Q_2(X; \eta, \xi) = 0\}, & \text{or as} \\ Z^4 \equiv E\{\xi = A_\nu(X; \eta);\ \nu = 1, 2, \dots, m\}. \end{cases} \tag{20}$$

In general for each fixed $\eta^0 \in \Gamma$, there will be μ roots, $\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_\mu}$, inside C and $m-\mu$ roots, $\xi_{k_{\mu+1}}, \dots, \xi_{k_m}$, outside C. As η varies over Γ the roots $\xi_\nu(\eta)$ move in the ξ-plane, and some may cross over C. If we restrict X such that,

$$X \notin \mathcal{M}^3 \equiv E\Big\{X \Big| \prod_{0 \leqslant k \leqslant j \leqslant m} [A_j(X; \eta) - A_k(X; \eta)] = 0;\ \eta \in \Gamma\Big\}, \tag{21}$$

then there cannot be more than a first order pole of the integrand on the domain of integration, and in this case the integrand is absolutely integrable. One then has

$$u(X) = -\frac{1}{4\pi^2} \iint_{\mathfrak{D}} \frac{Q_1(X; \eta, \xi)}{Q_2(X; \eta, \xi)} d\eta\, d\xi = \frac{1}{2\pi i} \sum_{\mu=1}^{m} \int_{\Gamma_\mu} \frac{Q_1(X; \eta, \xi_\mu)}{\partial Q_2(X; \eta, \xi_\mu)/\partial \xi} d\eta, \tag{22}$$

where Γ_μ is that subset of the path Γ for which $\xi_\mu = A_\mu(X; \eta)$ lies inside of C. Consequently, one may express each residue of the integrand (19) as an Abelian integral

$$u(X) = \frac{1}{2\pi i} \int_{\Gamma^*} \frac{Q_1(X; \eta, \xi)}{\partial Q_2(X; \eta, \xi)/\partial \xi} d\eta, \tag{23}$$

where $\Gamma^* = \sum_{\mu=1}^{m} \Gamma_\mu$ is taken over the Riemann surface, $\mathfrak{R}(X)$, defined by $Q_2(X; \eta, \xi) = 0$.

Before proceeding further, we should like to make some remarks about the topology of $\mathfrak{R}(X)$. In general, the genus $\varrho(X)$ of $\mathfrak{R}(X)$ will be constant, and $\mathfrak{R}(X)$ will have m-sheets over the η-plane. However, we note as Bergman has done in the case of three variables that there will be certain exceptional points. To locate these exceptional points we express

$$Q_2(X; \eta, \xi) \equiv \sum_{\nu=0}^{m} q_\nu(X; \eta)\xi^\nu, \tag{24}$$

where the $q_\nu(X; \eta)$ are polynomials in X, η. We designate the following sets:

$$
\begin{aligned}
&\mathfrak{S}_1^2 \equiv E\{X \mid q_n(X; \eta) = 0\}, \\
(25)\qquad &\mathfrak{S}_2^2 \equiv E\Big\{X \mid \Delta(X; \eta) \equiv \prod_{0 \leqslant k < j \leqslant m} [A_j(X; \eta) - A_k(X; \eta)] \equiv 0\Big\}, \\
&\mathfrak{S}_3^2 \equiv \Big[E\{X \mid \Delta(X; \eta) = 0\} \cap E\Big\{X \,\Big|\, \frac{\partial \Delta}{\partial \eta} = 0\Big\}\Big] \cup \\
&\qquad\qquad \cup \Big[E\{X \mid q_n(X; \eta) = 0\} \cap E\Big\{X \,\Big|\, \frac{\partial q_n}{\partial \eta} = 0\Big\}\Big],
\end{aligned}
$$

and

$$\mathfrak{S}^2 \equiv \mathfrak{S}_1^2 \cup \mathfrak{S}_2^2 \cup \mathfrak{S}_3^2 \text{ (}^3\text{)}.$$

As Mitchell has pointed out ([13]) when $X \in \mathfrak{S}_1^2, \mathfrak{R}(X)$ has less than n-sheets; when $X \in \mathfrak{S}_2^2$, $Q_2(X; \eta, \xi)$ has a repeated irreducible factor involving ξ; if $X \in \mathfrak{S}_3^2$, $\Delta(X; \eta)$ or $q_n(X; \eta)$ has a multiple root and two branch points coincide.

Now, as Bergman ([1], [3], [17]) has done in the case of algebraic p_3-associates, we make use of the Weierstrass decomposition formula ([18], p. 264) for algebraic functions defined over a Riemann surface

$$
\begin{aligned}
(26)\qquad F(X; \eta, \xi) &= \sum_{\nu=1}^{r} c_\nu(X) H(X; \eta_\nu, \xi_\nu; \eta, \xi) - \\
&- \sum_{\alpha=1}^{p} [g_\alpha^*(X) H_\alpha(X; \eta, \xi) - g_\alpha(X) H_\alpha^*(X; \eta, \xi)] + \frac{d}{d\eta}\Big[\sum_{\nu=1}^{r} F_\nu(X; \eta, \xi)\Big],
\end{aligned}
$$

$$\sum_{\nu=1}^{r} c_\nu(X) = 0,$$

where $H_\alpha(X; \eta, \xi)$, $H_\alpha^*(X; \eta, \xi)$, $H(X; \eta_\nu, \xi_\nu; \eta, \xi)$ are Weierstrass integrands of the first, second, and third kind respectively. The $F_\nu(X; \eta, \xi)$ are rational functions of η, ξ, p is the genus, $r = r(X)$ is the number of infinity points of $F(X; \eta, \xi)$, and $c_\nu(X)$, $g_\alpha(X)$, $g_\alpha^*(X)$, are algebraic functions of X. The number of infinity points of $F \equiv Q_1/Q_2$ is the same as the number of zeros of the coefficient $q_\nu(X; \eta)$.

In the situations discussed by Bergman ([1], [3], [17]) and Mitchell ([13]) for integrals of the type (23), Γ^* was always a closed curve over $\mathfrak{R}(X)$. With our case Γ^* is in general a sum of m disjoint segments on $\mathfrak{R}(X)$. The special case where Γ^* becomes a closed curve corresponds to when each of the roots $\xi = A_\mu(X; \eta)$ $(\mu = 1, 2, \ldots, m)$, remains in-

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(3) The superscripts on the sets $\mathfrak{S}_k^2$ indicate that the sets are of real dimension two. This convention for superscripts on sets will be used throughout this paper.

side C for all $\eta \in \Gamma$. In that case we obtain representations identical to those of Bergman ([1], [3], [17]) and Mitchell ([13]).

In order to evaluate (23) we must introduce certain integrals on $\mathfrak{R}(X)$ associated with the integrands of the first, second, and third kind:

$$\tag{27} \begin{gathered} \int_C H_\alpha(X;\eta,\xi)d\eta, \qquad \int_C H_\alpha^*(X;\eta,\xi)d\eta \qquad (\alpha=1,2,\ldots,p), \\ \int_C H(X;\eta_\nu,\xi_\nu;\eta,\xi)d\eta \qquad (\nu=1,2,\ldots,r), \end{gathered}$$

and their periods taken over the p cycles $K_\beta(X)$, and p conjugate cycles $K_\beta^*(X)$

$$\tag{28} \begin{matrix} 2\omega_{\alpha\beta}(X), & 2\eta_{\alpha\beta}(X), & \Omega_\beta(X;\eta_\nu,\xi_\nu) & (\alpha=1,2,\ldots,p), \\ 2\omega_{\alpha\beta}^*(X), & 2\eta_{\alpha\beta}^*(X), & \Omega_\beta^*(X;\eta_\nu,\xi_\nu) & (\beta=1,2,\ldots,p). \end{matrix}$$

In addition we introduce integrals taken over open paths on $\mathfrak{R}(X)$:

$$\tag{29} \begin{gathered} J_\alpha(X;\eta,\xi)=\int_{(a_0,b_0)}^{(\eta,\xi)} H_\alpha(X;\eta',\xi')d\eta', \\ J_\alpha^*(X;\eta,\xi)=\int_{(a_0,b_0)}^{(\eta,\xi)} H_\alpha^*(X;\eta',\xi')d\eta' \qquad (\alpha=1,2,\ldots,p) \end{gathered}$$

and

$$\tag{30} \begin{aligned} &J(X;\eta,\xi;\eta_\nu,\xi_\nu;\eta_0,\xi_0) \\ &\qquad =\int_{(a_0,b_0)}^{(\eta,\xi)} \{H(X;\eta_\nu,\xi_\nu;\eta',\xi')-H(X;\eta_0,\xi_0;\eta',\xi')\}d\eta'. \end{aligned}$$

The integral (30) has the following primitive period system:

$$\tag{31} 2\pi i, \qquad \log\frac{E_\beta^*(X;\eta_\nu,\xi_\nu)}{E_\beta^*(X;\eta_0,\xi_0)}, \qquad \log\frac{E_\beta(X;\eta_\nu,\xi_\nu)}{E_\beta(X;\eta_0,\xi_0)} \qquad (\beta=1,2,\ldots,p),$$

where the functions E_β, E_β^* are defined as

$$\tag{32} \begin{aligned} E_\beta(X;\eta,\xi)&=\exp\{\Omega_\beta(X;\eta,\xi)\}, \\ E_\beta^*(X;\eta,\xi)&=\exp\{\Omega_\beta^*(X;\eta,\xi)\}. \end{aligned}$$

Lastly, we shall need the integrals of the third kind

$$\tag{33} \Omega(X;\eta,\xi;\eta_\nu,\xi_\nu;\eta_0,\xi_0)=\int_{(\eta_0,\xi_0)}^{(\eta_\nu,\xi_\nu)} H(X;\eta,\xi;\eta',\xi')d\eta',$$

and the functions

$$\tag{34} E(X;\eta,\xi;\eta_\nu,\xi_\nu;\eta_0,\xi_0)=\exp\{\Omega(X;\eta,\xi;\eta_\nu,\xi_\nu;\eta_0,\xi_0)\}.$$

According to Weierstrass ([18], pp. 373, 374, 398) certain relations exist between the integrands $H_\beta(X;\eta,\xi)$, $H_\beta^*(X;\eta,\xi)$, $H(X;\eta_\nu,\xi_\nu;\eta,\xi)$ and the E-functions,

$$(35)\qquad \begin{aligned} H_\beta(X;\eta,\xi) &= \frac{d}{d\eta}J_\beta(X;\eta,\xi)\\ &= \frac{1}{\pi i}\sum_{\alpha=1}^{p}\Big\{\omega_{\beta\alpha}\frac{d}{d\eta}\log E_\alpha^*(X;\eta,\xi)-\omega_{\beta\alpha}^*\frac{d}{d\eta}\log E_\alpha(X;\eta,\xi)\Big\},\\ H_\beta^*(X;\eta,\xi) &= \frac{d}{d\eta}J_\beta^*(X;\eta,\xi)\\ &= \frac{1}{\pi i}\sum_{\alpha=1}^{p}\Big\{\eta_{\beta\alpha}\frac{d}{d\eta}\log E_\alpha^*(X;\eta,\xi)-\eta_{\beta\alpha}^*\frac{d}{d\eta}\log E_\alpha(X;\eta,\xi)\Big\},\end{aligned}$$

and

$$(36)\quad H(X;\eta_\nu,\xi_\nu;\eta,\xi)-H(X;\eta_0,\xi_0;\eta,\xi) = \frac{d}{d\eta}\log E(X;\eta,\xi;\eta_\nu,\xi_\nu;\eta_0,\xi_0)-$$
$$-\frac{1}{2\pi i}\sum_{\alpha=1}^{p}\Big\{\log\frac{E_\alpha^*(X;\eta_\nu,\xi_\nu)}{E_\alpha^*(X;\eta_0,\xi_0)}\frac{d}{d\eta}\log E_\alpha(X;\eta,\xi)-$$
$$-\log\frac{E_\alpha(X;\eta_\nu,\xi_\nu)}{E_\alpha(X;\eta_0,\xi_0)}\frac{d}{d\eta}\log E_\alpha^*(X;\eta,\xi)\Big\}.$$

The function $F(X;\eta,\xi)$ may now be expressed in terms of derivatives of the E-functions by subtracting from it the sum $\sum_{\nu=1}^{r} c_\nu H(X;\eta_\nu,\xi_\nu;\eta,\xi)$,

$$(37)\qquad F(X;\eta,\xi)-\sum_{\nu=1}^{r}c_\nu H(X;\eta_\nu,\xi_\nu;\eta,\xi)$$
$$=\sum_{\nu=1}^{r}c_\nu\frac{d}{d\eta}\log E(X;\eta,\xi;\eta_\nu,\xi_\nu;\eta_0,\xi_0)+$$
$$+\sum_{\alpha=1}^{p}\Big\{C_\alpha^*(X)\frac{d}{d\eta}\log E_\alpha(X;\eta,\xi)-C_\alpha(X)\frac{d}{d\eta}\log E_\alpha^*(X;\eta,\xi)\Big\}+$$
$$+\frac{d}{d\eta}\Big[\sum_{\nu=1}^{r}F_\nu(X;\eta,\xi)\Big],$$

where

$$
\begin{aligned}
(38)\quad & C^*_\alpha(X) \\
&= -\frac{1}{2\pi i}\left(\sum_{\nu=1}^{r} c'_\nu \log\frac{E^*_\alpha(X;\,\eta_\nu,\,\xi_\nu)}{E^*_\alpha(X;\,\eta_0,\,\xi_0)} - 2\sum_{\beta=1}^{p}[\omega^*_{\beta\alpha}(X)g^*_\beta(X) - \eta^*_{\beta\alpha}(X)g_\beta(X)]\right), \\
& C_\alpha(X) \\
&= \frac{1}{2\pi i}\left(\sum_{\nu=1}^{r} c_\nu \log\frac{E_\alpha(X;\,\eta_\nu,\,\xi_\nu)}{E_\alpha(X;\,\eta_0,\,\xi_0)} - 2\sum_{\beta=1}^{p}[\omega_{\beta\alpha}(X)g^*_\beta(X) - \eta_{\beta\alpha}(X)g_\beta(X)]\right).
\end{aligned}
$$

We return to the evaluation of (23), and recall that Γ_μ is that subset of Γ for which the root $\xi_\mu = A_\mu(X;\,\eta)$ lies inside C. The root ξ_μ crosses C a finite number of times, providing C is sufficiently smooth, since $f(\tau,\eta,\xi)$ is a rational function. Furthermore, we assume that ξ_μ lies inside C for η contained in the union of intervals,

$$
(39)\qquad \Gamma_\mu = \bigcup_{j=1}^{k_\mu} (\eta_\mu^{2j-1},\,\eta_\mu^{2j}), \qquad \eta^l_\mu = \eta^l_\mu(X) \in \Gamma .
$$

Consequently, one has the result:

THEOREM 1. *If the p_4-associate of $u(X)$ is a rational function of τ,η,ξ then whenever $X \notin \mathfrak{S}^2$, $u(X)$ may be represented as*

$$
\begin{aligned}
(40)\quad u(X) = \sum_{\mu=1}^{m}\sum_{j=1}^{k_\mu}\Bigg\{&\sum_{\nu=1}^{r} c_\nu(X)\log\frac{E(X;\,\eta_\mu^{2j},\,\xi_\mu^{2j};\,\eta_\nu,\,\xi_\nu;\,\eta_0,\,\xi_0)}{E(X;\,\eta_\mu^{2j-1},\,\xi_\mu^{2j-1};\,\eta_\nu,\,\xi_\nu;\,\eta_0,\,\xi_0)} + \\
&+\sum_{\alpha=1}^{p}\left[C^*_\alpha(X)\log\frac{E_\alpha(X;\,\eta_\mu^{2j},\,\xi_\mu^{2j})}{E_\alpha(X;\,\eta_\mu^{2j-1},\,\xi_\mu^{2j-1})} - C_\alpha(X)\log\frac{E^*_\alpha(X;\,\eta_\mu^{2j},\,\xi_\mu^{2j})}{E^*_\alpha(X;\,\eta_\mu^{2j-1},\,\xi_\mu^{2j-1})}\right] + \\
&+\sum_{\nu=1}^{r}[F_\nu(X;\,\eta_\mu^{2j},\,\xi_\mu^{2j}) - F_\nu(X;\,\eta_\mu^{2j-1},\,\xi_\mu^{2j-1})]\Bigg\}.
\end{aligned}
$$

THEOREM 2. *If in Theorem 1 the roots $\xi_\mu(X;\,\eta)$ all lie inside for all $\eta\in\Gamma$, then the representation* (40) *reduces to the form*

$$
\begin{aligned}
(41)\quad u(X) = \sum_{\beta=1}^{p}\Bigg\{&\sum_{\nu=1}^{r}[c_\nu(X)\Omega_\beta(X;\,\eta_\nu,\,\xi_\nu) + c'_\nu(X)\Omega^*_\beta(X;\,\eta_\nu,\,\xi_\nu)] - \\
&-\sum_{\alpha=1}^{p}[g^*_\alpha(X)\,\omega_{\alpha\beta}(X) - g^{*\prime}_\alpha(X)\,\omega_{\alpha\beta}(X) - g_\alpha(X)\,\eta_{\alpha\beta}(X) - g'_\alpha(X)\,\eta^*_{\alpha\beta}(X)]\Bigg\} + \\
&+\operatorname{Res}\sum_{\nu=1}^{r}F_\nu(X;\,\eta,\,\xi).
\end{aligned}
$$

Theorem 2 is essentially Bergman's result for three-dimensional harmonic functions, and the reader is referred to his paper [5] on this subject, and also to his book [4].

IV. A residue theorem for harmonic vectors. In this section we introduce the harmonic vector $U(X) \equiv (u_1, u_2, u_3, u_4)$, $\square u_k = 0$, where the components are defined as follows:

$$(42)\qquad \begin{aligned} u_k(X) &= -\frac{1}{4\pi^2} \int\limits_{|\xi|=1} \int\limits_{|\eta|=1} f(\tau, \eta, \xi) N_k(\eta, \xi) \frac{d\eta}{\eta} \frac{d\xi}{\xi}, \\ N(\eta, \xi) &= \frac{1}{\eta\xi} (\eta\xi + 1, i[\eta\xi - 1], \eta - \xi, i[\eta + \xi]) . \end{aligned}$$

$U(X)$ satisfies conditions similar to the vanishing of the divergence and curl, namely

$$(43)\qquad \frac{\partial u_\nu}{\partial x_\nu} = 0, \qquad \varepsilon_{mnrs} \frac{\partial u_r}{\partial x_s} = 0 ,$$

where repeated indices indicate summation and ε_{mnrs} is a permutation symbol.

We are interested in integrals of the form

$$(44)\qquad -4\pi^2 \int\limits_{\mathfrak{J}'} U(X) \circ dX = \int\limits_{\mathfrak{J}'} dX \circ \iint\limits_{\mathfrak{D}} f(\tau, \eta, \xi) N(\eta, \xi) \frac{d\eta d\xi}{\eta\xi},$$

where $\mathfrak{J}'$ is a smooth oriented curve, such that $\mathfrak{J}' \cap \mathfrak{S}^2 = 0$, and $f(\tau, \eta, \xi)$ is a rational function of τ, η, ξ. We shall assume that $\mathfrak{J}'$ intersects

$$\mathcal{M}^3 \equiv E\{X \mid \Delta(X; \eta) = 0; \eta \in \Gamma\}$$

and

$$\mathcal{N}^3 \equiv E\{X \mid q_n(X; \eta) = 0; \eta \in \Gamma\}$$

in a discrete set of points $\{X_k\}_{k=1}^{s}$, and that these points subdivide $\mathfrak{J}$ into open segments $\mathfrak{J}'_k \equiv \mathfrak{J}(X_{k-1}, X_k)$, $\mathfrak{J}' \equiv \bigcup\limits_{k=1}^{s} \overline{\mathfrak{J}}'_k$. If $X^0 \in \mathfrak{J}'_k$, Γ does not meet any of the branch points of $\mathfrak{R}(X)$, because these points are contained in the union of the sets,

$$E\{\eta \mid \Delta(X^0; \eta) = 0\} \cup E\{\eta \mid q_n(X^0; \eta) = 0\} .$$

Furthermore, for $X^0 \in \mathfrak{J}'_k$ there are no poles of

$$F(X^0; \eta, \xi) \equiv \frac{Q_1(X^0; \eta, \xi)}{\partial Q_2(X^0; \eta, \xi)/\partial \xi} \qquad \text{on} \qquad Q_2(X^0; \eta, \xi) = 0 ,$$

since the discriminant $\Delta(X^0; \eta)$ is computed by eliminating ξ between $Q_2 = 0$ and $\partial Q_2/\partial \xi = 0$.

Consequently we consider the integral

$$4\pi^2 \sum_{k=1}^{s} \int_{\mathfrak{Z}'_k} U(X) \circ dX = \sum_{k=1}^{s} \int_{\mathfrak{Z}'_k} dX \circ \iint_{\mathfrak{D}} f(\tau, \eta, \xi) N(\eta, \xi) \frac{d\eta}{\eta} \frac{d\xi}{\xi}$$

$$(45) \qquad = 2\pi i \sum_{k=1}^{s} \int_{\mathfrak{Z}'_k} dX \circ \int_{\Gamma^*} \frac{Q_1(X; \eta, \xi)}{\partial Q'_2(X; \eta, \xi)/\partial \xi} N'(\eta, \xi) d\eta,$$

$$\left(\Gamma^* = \sum_{\mu=1}^{m} \Gamma_\mu,\ Q'_2 = \eta\xi Q_2, \text{ and } N' = \eta\xi N\right)$$

(46)

$$= 2\pi i \sum_{k=1}^{s} \int_{\mathfrak{Z}'_k} dX \circ \left\{ \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu(X)} \left[\sum_{\nu=1}^{r(X)} c_\nu(X) \log E(X; \eta_\mu^{2j}, \xi_\mu^{2j}; \eta_\nu, \xi_\nu; \eta_0, \xi_0) + \right.\right.$$

$$+ \sum_{\alpha=1}^{p} C^*_\alpha(X) \log \frac{E_\alpha(X; \eta_\mu^{2j}, \xi_\mu^{2j})}{E_\alpha(X; \eta_\mu^{2j-1}, \xi_\mu^{2j-1})} - \sum_{\alpha=1}^{p} C_\alpha(X) \log \frac{E^*(X; \eta_\mu^{2j}, \xi_\mu^{2j})}{E^*_\alpha(X; \eta_\mu^{2j-1}, \xi_\mu^{2j-1})} +$$

$$\left.\left. + \sum_{\nu=1}^{r(X)} [F_\nu(X; \eta_\mu^{2j}, \xi_\mu^{2j}) - F_\nu(X; \eta_\mu^{2j-1}, \xi_\mu^{2j-1})] \right]\right\},$$

where $c_\nu(X)$, $C_\alpha(X)$, $C^*_\alpha(X)$, etc. are vector functions whose components are terms similar to those of (38), arising from the Weierstrass decomposition of the components $F_r \equiv \dfrac{Q_1 N'_r}{\partial Q'_2/\partial \xi}$, on the Riemann surface defined by $Q'_2 = 0$.

Because the integrand in (45) is absolutely integrable we may interchange the X and (η, ξ) orders of integration. Then in order to facilitate evaluation it is convenient to make the transformation from the X-space to the τ-plane by $\tau = \tau(X; \eta, \xi)$, where η, ξ are held fixed. We are led to consider the integral

$$(47) \quad I_2 \equiv \iint_{\mathfrak{D}} \frac{d\eta}{\eta} \frac{d\xi}{\xi} \int_{\mathfrak{Z}(\eta,\xi)} f(\tau, \eta, \xi) d\tau = \iint_{\mathfrak{D}} \frac{d\eta}{\eta} \frac{d\xi}{\xi} \int_{\mathfrak{Z}(\eta,\xi)} \frac{q_1(\tau, \eta, \xi)}{q_2(\tau, \eta, \xi)} d\tau,$$

where $\mathfrak{Z}(\eta, \xi)$ is the image of $\mathfrak{Z}'$ under the mapping $X \to \tau$ for fixed η, ξ. Since, $q_2(\tau, \eta, \xi)$ is a polynomial, we may decompose it as

$$(48) \qquad q_2(\tau, \eta, \xi) = [\tau - \tau_1(\eta, \xi)]^{l_1} [\tau - \tau_2(\eta, \xi)]^{l_2} \ldots [\tau - \tau_k(\eta, \xi)]^{l_k}$$

for all $(\eta, \xi) \notin E\{(\eta, \xi) | \prod_{0 \leqslant \mu < \nu \leqslant k} [\tau_\mu(\eta, \xi) - \tau_\nu(\eta, \xi)] = 0\}$, that is $q_2(\tau, \eta, \xi)$ will have the factorization (48) with the exception of those (η, ξ) lying on a certain set of $n(n+1)/2$ analytic surfaces $\xi = \psi_k(\eta)$. Hence, we obtain,

$$I_2 = \iint_{\mathfrak{D}} \frac{d\eta}{\eta}\frac{d\xi}{\xi} \sum_{j=1}^{k} \frac{n(\mathfrak{J};\tau_j)}{(l_j-1)!}\cdot\frac{\partial^{l_j-1}}{\partial\tau^{l_j-1}}\left\{[\tau-\tau_j(\eta,\xi)]^{l_j}\frac{q_1(\tau,\eta,\xi)}{q_2(\tau,\eta,\xi)}\right\}_{\tau=\tau_j},$$

where $n(\mathfrak{J};\tau_j)$ is the winding number, or index, of $\mathfrak{J}(\eta,\xi)$ with respect to τ_j. It is clear, that for certain subsets in $(\eta,\xi)\in\mathfrak{D}$, $n(\mathfrak{J};\tau_j)$ will be zero, and for other subsets of $\mathfrak{D}$ it will take on positive integer values. In the special case, where $\mathfrak{J}'$ is chosen such that $\mathfrak{J}(\eta,\xi)$ winds about τ_j at most once, and where $l_j\equiv 1$, we may write

$$\text{(49)}\qquad I_2 = \iint_{\mathfrak{D}} \frac{d\eta}{\eta}\frac{d\xi}{\xi}\left\{\sum_{j=1}^{k} n(\mathfrak{J};\tau_j)\frac{q_1(\tau,\eta,\xi)}{\partial q_2(\tau,\eta,\xi)/\partial\tau}\right\}$$

where $\tau_j(\eta,\xi)$ is a root of $q_2(\tau,\eta,\xi)\equiv\sum_{\nu=0}^{k} a_\nu(\eta,\xi)\tau^\nu = 0$. The index $n(\mathfrak{J};\tau_j)$ may be thought of here as a set function, since when $(\eta,\xi)\in\mathfrak{D}_j^*\subset\mathfrak{D}$, $n(\mathfrak{J};\tau_j)=1$, whereas for $(\eta,\xi)\in\mathfrak{D}-\mathfrak{D}_j^*$, $n(\mathfrak{J};\tau_j)=0$. Now, for each fixed $\eta=\eta^0\in\Gamma$, there are certain values of $\xi\in C$ for which τ_j lies inside $\mathfrak{J}(\eta,\xi)$. Let us suppose that the segments of C for which this occurs may be expressed as

$$\text{(50)}\qquad C_j^* \equiv \bigcup_{i=1}^{t_j(\eta^0)} C_{ji}(\eta^0) \equiv \bigcup_{i=1}^{t_j(\eta^0)} C(\xi_{2i-1}^j[\eta^0],\xi_{2i}^j[\eta^0]),$$

where the ξ_{2i-1}^j, ξ_{2i}^j are the end points of the segments on C. As η^0 varies along Γ the intervals $C_{ji}(\eta)$ and their number vary. If for a particular $\eta=\eta'$ there is no subset of C for which τ_j lies inside $\mathfrak{J}$ we shall set $C_j^*=0$; consequently, we may write

$$\text{(51)}\qquad I_2 = \sum_{j=1}^{k}\frac{q_1(\tau_j,\eta,\xi)}{\partial q_2(\tau_j,\eta,\xi)/\partial\tau}\frac{d\xi}{\xi}\frac{d\eta}{\eta} = \sum_{j=1}^{k}\int_\Gamma\frac{d\eta}{\eta}\int_{C_j^*}\frac{q_1(\tau_j,\eta,\xi)}{\partial q_2/\partial\tau}\frac{d\xi}{\xi}.$$

If $q_2(s,\eta,\xi)$ is of degree m and $q_1(s,\eta,\xi)$ is of degree $m-4$, the integral I_2 may be represented as an integral of the first kind provided $q_1(s,\eta,\xi)=0$ passes through the double lines of $q_2(s,\eta,\xi)=0$. Here we may represent $q_2(s,\eta,\xi)$ as a linear combination of $p_g=\binom{m-1}{3}$ independent polynomials $Q_\nu(s,\eta,\xi)$ $(\nu=1,2,\dots,p_g)$ and consequently $q_1\Big/\frac{\partial q_2}{\partial s}$ may be represented in terms of the integrands of the first kind $H_\mu(s,\eta,\xi)$ $=Q_\mu\Big/\eta\xi\frac{\partial q_2}{\partial s}$. Consequently,

$$\text{(52)}\qquad I_2 = \sum_{j=1}^{k}\sum_{\mu=1}^{p_g} a_\mu\int_\Gamma\int_{C_j^*} H_\mu(s,\eta,\xi)\,d\xi\,d\eta,$$

and we have the result:

THEOREM 3. *Let $U(X) \equiv p_4[fN]$ be a harmonic vector defined as above with the rational associate $f(\tau, \eta, \xi) = q_1(\tau, \eta, \xi)/q_2(\tau, \eta, \xi)$. Furthermore, let $q_2(\tau, \eta, \xi)$ be of degree m, $q_1(s, \eta, \xi)$ be of degree $m-4$, and let $q_1 = 0$ pass through the double line of $q_2 = 0$. Then the integral of $U(X) \circ dX$ taken about a smooth, oriented curve $\mathfrak{J}'$ may be represented as follows:*

$$\int_{\mathfrak{J}'} U(X) \circ dX \tag{53}$$

$$= \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu} \int_{\mathfrak{J}'} dX \circ \Bigg\{ \sum_{\nu=1}^{r} c_\nu(X) \log \frac{E(X;\, \eta_\mu^{2j},\, \xi_\mu^{2j};\, \eta_\nu,\, \xi_\nu;\, \eta_0,\, \xi_0)}{E(X;\, \eta_\mu^{2j-1},\, \xi_\mu^{2j-1};\, \eta_\nu,\, \xi_\nu;\, \eta_0,\, \xi_0)} +$$

$$+ \sum_{\alpha=1}^{p} \left[C_\alpha^*(X) \log \frac{E_\alpha(X;\, \eta_\mu^{2j},\, \xi_\mu^{2j})}{E_\alpha(X;\, \eta_\mu^{2j-1},\, \xi_\mu^{2j-1})} - C_\alpha(X) \log \frac{E_\alpha^*(X;\, \eta_\mu^{2j},\, \xi_\mu^{2j})}{E_\alpha^*(X;\, \eta_\mu^{2j-1},\, \xi_\mu^{2j-1})} \right] \Bigg\}$$

$$= \sum_{j=1}^{k} \sum_{\mu=1}^{p_g} a_\mu \int_{\Gamma} \int_{C_j^*} H_\mu(s, \eta, \xi)\, d\xi\, d\eta\,,$$

where s is a root of $q_2(s, \eta, \xi) = 0$. Furthermore, the integral has always a finite value for each compact $\mathfrak{J}'$.

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