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Existence theorems for discrete boundary problems

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1. In recent years an intensive development of boundary problems for difference equations may be observed, generally as numerical aids to problems for differential equations. In this situation it seems to be natural and useful to give certain theorems on the existence of solutions of boundary problems for difference equations. It is the aim of this paper to present such theorems. They will be discrete analogues of the Lasota [5] and the Lasota-Opial [7], [8] theorems.

In section 2 we establish the notation and introduce the notions. Section 3 contains two general theorems on the existence of solutions of boundary difference problems under the assumption that a homogeneous boundary problem for a certain difference equation with a multi-valued right-hand side has only a trivial solution. Next, in section 4, as applications of these theorems we give the existence theorems for the so-called aperiodic problem. In section 5 we show how the assumption which gives the possibility of applying Lasota's existence theorem (see [5]) to the aperiodic differential problem allows us to prove the existence of solutions of the approximating difference aperiodic problem and the convergence of these solutions to a solution of the differential aperiodic problem. Finally (section 6) we state a theorem on the existence of solutions for a discrete boundary problem with a multi-valued right-hand side which corresponds to an analogous Lasota-Opial theorem [8], we discuss the possibilities of constructing similar theorems and we give as an illustration two examples: a Nicoletti type problem and a problem of periodic solutions generalizing Halanay's result [3].

2. Let R^l be the l-dimensional Euclidean space. $\mathrm{n}(R^l)$ $(\mathrm{cf}(R^l))$ denotes the family of all non-empty (non-empty, closed and convex) subsets of R^l. For $p \in R^m$ $|p|$ denotes the usual Euclidean norm of p and $|A| = \sup_{p \in A} |p|$ for $A \subset R^m$. Let $N = \{0, 1, \dots, n\}$ be a topological space with the discrete topology. For $u = (u_0, \dots, u_n) \in (R^m)^{n+1}$ $\|u\| = \sup_{i \in N} |u_i|$ and $\Delta u_i = u_{i+1} - u_i$ for $i = 0, 1, \dots, n-1$, $\Delta u_n = 0$, $\Delta u = (\Delta u_0, \dots, \Delta u_n)$. $\delta(p, A)$ denotes the Euclidean distance of $p \in R^l$ from $A \subset R^l$. The mapping

$F\colon R^l \to n(R^s)$ is called *upper semicontinuous* if for all sequences $\{p_k\} \subset R^l$, $\{q_k\} \subset R^s$ the conditions $p_k \to p_0$, $q_k \to q_0$, $q_k \in F(p_k)$ imply $q_0 \in F(p_0)$. The mapping $F\colon R^l \to \mathrm{cf}(R^s)$ is called *continuous* if it is continuous in the Hausdorff metric. $\{x\}$ is a set consiting only of x.

3. We make the following assumptions:

(i) the mapping $F\colon N \times R^m \to \mathrm{cf}(R^m)$ is continuous, $F(i, \cdot)$ is homogeneous for every $i \in N$, i.e.

$$F(i, \lambda p) = \lambda F(i, p), \quad \lambda \in R, \; p \in R^m$$

and

$$\sum_{i=0}^{n} \sup_{|p|=1} |F(i, p)| < +\infty; \tag{3.1}$$

(ii) the mapping $f\colon N \times R^m \to R^m$ is continuous and

$$\lim \frac{1}{k} \sum_{i=0}^{n} \sup_{|p|<k} \delta\big(f(i, p), F(i, p)\big) = 0; \tag{3.2}$$

(iii) the mapping $L\colon (R^m)^{n+1} \to R^m$ is continuous and homogeneous.

We shall consider two boundary problems for $u \in (R^m)^{n+1}$.

The first: the difference equation

$$\Delta u_i = f(i, u_i) \quad (i = 0, 1, \ldots, n-1) \tag{3.3}$$

with the boundary condition

$$Lu = r \quad (r \in R^m). \tag{3.4}$$

The second: the difference equation with a multi-valued right-hand side

$$\Delta u_i \in F(i, u_i) \quad (i = 0, 1, \ldots, n-1) \tag{3.5}$$

with the homogeneous boundary condition

$$Lu = 0. \tag{3.6}$$

THEOREM 3.1. *Suppose that* F, f, L *satisfy conditions* (i), (ii), (iii). *If problem* (3.5), (3.6) *has only the trivial solutions* $u = 0$, *then there exists at least one solution of problem* (3.3), (3.4).

Proof. Put $E = (R^m)^{n+1} \times R^m$ with the norm $\|(u, p)\| = \|u\| + |p|$, $u \in (R^m)^{n+1}$, $p \in R^m$. We define two mappings, $h\colon E \to E$ and $H\colon E \to \mathrm{cf}(E)$, in the following way:

$$h(u, p) = (\bar{u}, \bar{p}), \quad \bar{u}_i = \sum_{j=0}^{i-1} f(j, u_j) + p, \quad \bar{p} = Lu + p - r;$$

$(\bar{\bar{u}}, \bar{\bar{p}}) \in H(u, p)$ if there exists a v such that

$$v_j \in F(j, u_j), \quad \bar{\bar{u}}_i = \sum_{j=0}^{i-1} v_j + p, \quad \bar{\bar{p}} = Lu + p.$$

We show that

1° $$\lim_{\|(u,p)\| \to \infty} \frac{\varrho\big(h(u,p), H(u,p)\big)}{\|(u,p)\|} = 0,$$

where ϱ is the distance between a point and a set,

2° H is the homogeneous mapping,

3° H and h are completely continuous,

4° $(u, p) \in H(u, p)$ implies $(u, p) = (0, 0)$.

Then applying Theorem 1.1 from [5] we find that there exists a fixed point (u, p) of h, which means that u is the solution of problem (3.3), (3.4).

Now we prove 1°-4°. (3.2) implies 1°. 2° is evident. If $(u, p) \in H(u, p)$, then $\Delta u_i \in F(i, u_i)$, $p = Lu + p$.

Thus, by our assumptions, we immediately deduce that $(u, p) = (0, 0)$. By (3.1) and by

$$\sum_{i=0}^{n} \bigcup_{\|u\|=1} F(i, u_i) = \bigcup_{\|u\|=1} \sum_{i=0}^{n} F(i, u_i)$$

we infer that the closure of $\bigcup_{\|u\|=1} H(u, p)$ is compact.

It is also evident that H is upper semi-continuous. Thereby H is completely continuous.

On the other hand, by the continuity of f, we find that $\bigcup_{\|(u,p)\|=1} \{h(u, p)\}$ is also relatively compact. Then 3° is fulfilled.

We make the following successive assumptions:

(iv) the mapping F: $N \times R^m \to s(R^m)$ is upper semi-continuous;

(v) the mapping f: $N \times R^m \to R^m$ is continuous and

$$f(i, p) - f(i, q) \in F(i, p - q) \tag{3.7}$$

for $i \in N$, $p, q \in R^m$;

(vi) the mapping L: $(R^m)^{n+1} \to R^m$ is linear.

THEOREM 3.2. *Suppose that* F, f, L *satisfy* (iv), (v), (vi) *and that* $u = 0$ *is the unique solution of problem* (3.5), (3.6). *Then there exists exactly one solution of problem* (3.3), (3.4).

Proof. The uniqueness of the solutions of problem (3.3), (3.4) easily follows from the linearly of L, condition (3.7) and the uniqueness of the solutions of problem (3.5), (3.6). Therefore, we prove only the existence.

Consider the mapping $T\colon E\to E$ (E is as in the proof of Theorem 3.1) such that $(v,q)=T(u,p)$ is given by the formulae

$$v_i = u_i-\sum_{j=0}^{i-1} f(j,u_j)-p\,,\quad q=Lu\,. \tag{3.8}$$

The mapping T is completely continuous, of course. It is sufficient to show that $T(E)=E$. Then the point $(0,r)$ belongs to E, which means that there exists a solution of (3.3), (3.4). To this end we prove that the mapping T is a so-called ε-mapping in the narrow sense (see [2]), i.e. T has the following property: for some constants $\varepsilon>0$, $\delta>0$ the condition

$$\|T(\bar{u},\bar{p})-T(\bar{\bar{u}},\bar{\bar{p}})\|\leqslant\delta \Rightarrow \|(\bar{u},\bar{p})-(\bar{\bar{u}},\bar{\bar{p}})\|<\varepsilon$$

is fulfilled. Suppose, on the contrary, that T is not an ε-mapping. This means, by the continuity of f and by the fact that T is injective, that there exist two sequences $(\bar{u}^k,\bar{p}^k)$, $(\bar{\bar{u}}^k,\bar{\bar{p}}^k)$ such that

$$\|\bar{u}^k-\bar{\bar{u}}^k\|=1\,, \tag{3.9}$$

$$\|\bar{v}^k-\bar{\bar{v}}^k\|+|\bar{q}^k-\bar{\bar{q}}^k|\to 0\,, \tag{3.10}$$

where $T(\bar{u}^k,\bar{p}^k)=(\bar{v}^k,\bar{q}^k)$, $T(\bar{\bar{u}}^k,\bar{\bar{p}}^k)=(\bar{\bar{v}}^k,\bar{\bar{q}}^k)$. Putting $u^k=\bar{u}^k-\bar{\bar{u}}^k$, $v^k=\bar{v}^k-\bar{\bar{v}}^k$, $q^k=\bar{q}^n-\bar{\bar{q}}^k$, by a straightforward calculation, from (3.5) we obtain

$$\Delta(u^k-v^k)_i=f(i,\bar{u}_i^k)-f(i,\bar{\bar{u}}_i^k)\in F(i,u_i^k)\,, \tag{3.11}$$
$$q^k=Lu^k \text{ and } \|u^k\|=1\,.$$

Passing to a suitable subsequence, if necessary, we may assume that the sequence $\{u^k\}$ converges to u. By the upper semi-continuity of F and by (3.10), (3.11), we have

$$\Delta u_i\in F(i,u_i)\,,\qquad Lu=0\,.$$

This implies $u=0$, which contradicts (3.9). Thus, T is a ε-mapping in the narrow sense. The known theorem on ε-mappings in the narrow sense (see [2], p. 62) yields $T(E)=E$.

4. In the case where equation (3.5) reduces to a difference inequality

$$|\Delta u_i|\leqslant\omega(i,|u_i|)\qquad (i=0,1,\ldots,n-1) \tag{4.1}$$

we have following two theorems as simple corollaries to the above ones. Let $\omega\colon N\times\langle 0,+\infty)\to\langle 0,+\infty)$ and $f\colon N\times R^m\to R^m$ be continuous mappings. Put

$$F(i,p)=\{q\in R^m\colon |q|\leqslant\omega(i,|p|)\}\,.$$

From Theorem 3.1 easily follows

THEOREM 4.1. *Suppose that* $\omega(i, \cdot)$ *is homogeneous for every* $i \in N$, L *satisfies* (iii) *and*

$$|f(i, p)| \leqslant \omega(i, |p|) + a_i \quad (a_i \in \langle 0, +\infty)) .$$

If problem (4.1), (3.6) *has only the trivial solution* $u = 0$, *then there exists at least one solution of problem* (3.3), (3.4).

As an application of theorem 3.2 we obtain

THEOREM 4.2. *Suppose that* L *satisfies* (vi) *and*

$$|f(i, p) - f(i, q)| \leqslant \omega(i, |p-q|) .$$

If problem (4.1), (3.6) *has only the trivial solution* $u = 0$, *then problem* (3.3), (3.4) *has exactly one solution.*

In order to illustrate the above theorems we consider the aperiodic problem, i.e. the problem of seeking solutions of equation (3.3) satisfying the boundary condition

$$u_0 + \lambda u_n = r \quad (r \in R^n, \ \lambda \in R) . \tag{4.2}$$

Consider the difference inequality

$$|\Delta u_i| \leqslant \mu_i |u_i| \quad (i = 0, \dots, n-1) \tag{4.3}$$

with the condition

$$u_0 + \lambda u_n = 0 . \tag{4.4}$$

Setting $z_i = |u_i|$ we obtain

$$|\Delta z_i| \leqslant \mu_i z_i .$$

Hence, for $i = 0, \dots, n-1$ we have the inequalities

$$\max\big(0, (1-\mu_i) z_i\big) \leqslant z_{i+1} \leqslant (1+\mu_i) z_i . \tag{4.5}$$

Hence

$$\begin{aligned} (4.6) \quad & (1+\mu_0) \dots (1+\mu_{n-1}) z_0 \geqslant z_n \\ & \geqslant \begin{cases} (1-\mu_0) \dots (1-\mu_{n-1}) z_0 & \text{if} \quad 1-\mu_i > 0 \quad \text{for} \quad i = 0, \dots, n-1 , \\ 0 & \text{in the other case} . \end{cases} \end{aligned}$$

We prove that if u satisfies the difference inequality (4.3) and the condition (4.4) and if

$$\begin{cases} |\lambda| > \big((1-\mu_0) \dots (1-\mu_{n-1})\big)^{-1} & \text{if} \quad 1-\mu_i > 0 \quad \text{for} \quad i = 0, \dots, n-1 \\ \text{or} & \\ |\lambda| < \big((1+\mu_0) \dots (1+\mu_{n-1})\big)^{-1} , & \end{cases} \tag{4.7}$$

then $u_i = 0$ for all $i \in N$.

In fact, suppose that $u_i \neq 0$ for some $i \in N$. By (4.5) and (4.4) we have either $z_i = 0$ for all $i \in N$ or $z_i \neq 0$. Hence $z_n \neq 0$. By (4.6) we have

$$\big((1+\mu_0)\dots(1+\mu_{n-1})\big)^{-1} \leqslant \frac{z_0}{z_n} \leqslant \big((1-\mu_0)\dots(1-\mu_{n-1})\big)^{-1};$$

the right inequality holds only if $1-\mu_i > 0$ for $i = 0, \dots, n-1$. This contradicts (4.7).

Using Theorems 4.1 and 4.2 for $\omega(i, p) = \mu_i|p|$, we obtain from the preceding considerations

THEOREM 4.3. *Suppose that* $f\colon N \times R^m \to R^m$ *is continuous and satisfies*

$$|f(i, p)| \leqslant \mu_i|p| + \nu_i \qquad (\nu_i \in \langle 0, +\infty)).$$

If (4.7) *is fulfilled, then there exists at least one solution of problem* (3.3), (4.2). *If, in addition,* f *satisfies the Lipschitz inequality*

$$|f(i, p) - f(i, q)| \leqslant \mu_i|p-q|,$$

this solution is unique.

5. Now we deal with the approximation problem for the differential equation

(5.1) $$x' = g(t, x)$$

and the aperiodic boundary condition

(5.2) $$x(a) + \lambda x(b) = r \qquad (r \in R^m),$$

where $g\colon \langle a, b\rangle \times R^m \to R^m$.

It is known (see [5]) that if g satisfies the condition

(5.3) $$|g(t, p) - g(t, q)| \leqslant \sigma(t)|p-q|,$$

where $\sigma\colon \langle 0, +\infty) \to \langle 0, +\infty)$ is continuous and if

(5.4) $$\big|\ln|\lambda|\big| > \int_a^\beta \sigma(t)\,dt,$$

then there exists exactly one solution of problem (5.1), (5.2).

Side by side with equation (5.1) we consider the difference equation

(5.5) $$\Delta u_i = hg(t_i, u_i)$$

and the boundary condition

(5.6) $$u_0 + \lambda u_n = r$$

defining the sequence $(t_0, \dots, t_n)$ and the mesh h by the formulae

$$t_0 = a, \quad \Delta t_i = t_{i+1} - t_i = h = (b-a)/n, \quad i = 0, \dots, n-1.$$

We may write

THEOREM 5.1. *Suppose that g is continuous and satisfies inequality* (5.3) *and that* (5.4) *is satisfied. Under the above assumptions* 1° *for sufficiently great n there exists exactly one solution u^n of difference problem* (5.5), (5.6), 2° $\lim |u_i^n - x(t_i)| = 0$ *uniformly in i, where x is the solution of problem* (5.1), (5.2).

Proof. We define f^n: $N \times R^m \to R^m$ as follows:

$$f^n(i, p) = hg(t_i, p)$$

and put $\sigma = \sigma(t_i)$. Thereby

$$|f^n(i, p) - f^n(i, q)| \leqslant h\sigma_i |p - q| .$$

It is clear that

$$-h(\sigma_0 + \ldots + \sigma_{n-1}) \leqslant \ln\big((1 + h\sigma_0)\ldots(1 + h\sigma_{n-1})\big)^{-1}$$

or

$$h(\sigma_0 + \ldots + \sigma_{n-1}) \geqslant \ln\big((1 - h\sigma_0)\ldots(1 - h\sigma_{n-1})\big)^{-1} ,$$

where all $1 - h\sigma_i \neq 0$ $(i = 0, \ldots, n-1)$.

Passing to the limit we have

$$-\int_a^\beta \sigma(t)\,dt \leqslant \lim_{n\to\infty} \ln\big((1 + h\sigma_0)\ldots(1 + h\sigma_{n-1})\big)^{-1}$$

or

$$\int_a^\beta \sigma(t)\,dt \geqslant \lim_{n\to\infty} \ln\big((1 - h\sigma_0)\ldots(1 - h\sigma_{n-1})\big)^{-1} .$$

By (5.4) this shows that for sufficiently large n

$$|\lambda| < \big((1 + h\sigma_0)\ldots(1 + h\sigma_{n-1})\big)^{-1}$$

or

$$|\lambda| > \big((1 + h\sigma_0)\ldots(1 - h\sigma_{n-1})\big)^{-1} .$$

Hence, in view of Theorem 4.3 using to f^n, we find that for sufficiently large n there exists exactly one solution of (5.5), (5.6), i.e. 1° is fulfilled.

Now put $x_i = x(t_i)$. There exist points $s_i \in (t_i, t_{i+1})$ such that $\Delta x_i = hx'(s_i)$. (5.1) for $t = s_i$ may be written in the form

$$\Delta x_i = hg\big(s_i, x(s_i)\big)$$

or

$$\Delta x_i = hg(t_i, x_i) + r_i ,$$

where $r_i = h\big(g\big(s_i, x(s_i)\big) - g\big(t_i, x(t_i)\big)\big)$.

We note that

$$\lim a^n/|h| = \lim a^n \cdot n = 0 , \tag{5.8}$$

where $a^n = \max\limits_i |r_i|$.

Putting $v_i^n = u_i^n - x_i$ and $w_i^n = |v_i^n|$ we obtain from (5.2), (5.6), (5.5) and (5.7)

$$v_0^n + \lambda v_n^n = 0 , \tag{5.9}$$

$$|\Delta v_i^n| \leqslant \big|h\big(g(t_i, u_i) - g(t_i, x_i)\big)\big| + |r_i| \leqslant |h\sigma_i v_i^n| + a^n ,$$

$$|\Delta w_i^n| \leqslant h\sigma_i w_i^n + a^n . \tag{5.10}$$

On the other hand, we have $w_i^n \leqslant w_0^n + \sum_{j=1}^{i-1} w_j^n$ and by (5.10)

$$w_i^n \leqslant w_0^n + \sum_{j=1}^{i} h\sigma_j w_j^n + na^n .$$

Applying Gronwall's Lemma (see [1], p. 455) we obtain

$$w_i^n \leqslant (w_0^n + na^n)\exp\Big(h\sum_{j=0}^{n}\sigma_j\Big) ; \tag{5.11}$$

in a similar way we get

$$w_i^n \leqslant (w_n^n + na^n)\exp\Big(h\sum_{j=0}^{n}\sigma_j\Big) . \tag{5.12}$$

From (5.4) we deduce that for sufficiently large n either

$$|\lambda| < \exp\Big(-h\sum_{i=0}^{n}\sigma_i\Big) \tag{a}$$

or

$$|\lambda| > \exp\Big(h\sum_{i=0}^{n}\sigma_i\Big) . \tag{b}$$

In case (a) (5.11) and (5.9) imply

$$w_0^n \leqslant \frac{na^n \exp\big(h\sum_{i=0}^{n}\sigma_i\big)}{1 - |\lambda|\exp\big(h\sum_{i=0}^{n}\sigma_i\big)}$$

and by (5.8) $\lim_{n\to\infty} w_0^n = 0$. In view of (5.11) $\lim_{n\to\infty} w_i^n = 0$ uniformly in i.

In case (b), (5.11) and (5.9) imply

$$w_n^n \leqslant \frac{na^n \exp\big(h\sum_{i=0}^{n}\sigma_i\big)}{|\lambda| - \exp\big(h\sum_{i=0}^{n}\sigma_i\big)} ,$$

and consequently $\lim_{n\to\infty} w_n^n = 0$ and by (5.12) $\lim_{n\to\infty} w_i^n = 0$ uniformly in i.

This completes the proof of theorem 5.1.

6. Now we deal with the difference equation

$$\Delta u_i \in A_i u_i + F(i, u_i) \qquad (i = 0, \ldots, n-1) \tag{6.1}$$

together with condition (3.6), and with the linear difference equation

$$\Delta u_i = A_i u_i \tag{6.2}$$

together with condition (3.4). $A = (A_0, \ldots, A_n)$, A_i is a $m \times m$ matrix.

We make the following assumption:

(vii) the mapping F: $N \times R^m \to \mathrm{cf}(R^m)$ is upper semi-continuous.

THEOREM 6.1. *Suppose that* F, L *satisfy* (vii), (vi) *and*

$$|F(i, p)| \leqslant a_i + \beta_i |p| . \tag{6.3}$$

If $u = 0$ *is the unique solution of the linear homogeneous problem* (6.2), (3.4), *then there exists a number* $\bar{\beta} > 0$ *(depending only on* A *and* L*) such that for every* F *satisfying* (6.3) *with* $\sum_{i=0}^{n} \beta_i < \bar{\beta}$ *problem* (6.1), (3.6) *has at least one solution on for every* $r \in R^m$.

Proof. For an arbitrary $b \in (R^m)^{n+1}$ the unique solution of the linear problem

$$\Delta u_i = A_i u_i + b_i , \quad Lu = r$$

has the form

$$u = \Gamma b + Hr ,$$

where Γ maps $(R^m)^{n+1}$ into itself, H maps R^m into $(R^m)^{n+1}$. The mappings Γ and H are linear and continuous.

For every $u \in (R^m)^{n+1} \mathcal{F}(u)$ denotes the set of all $v \in (R^m)^{n+1}$ such that $v_i \in F(i, u_i)$. It is sufficient to prove the existence of a u satisfying

$$u \in \Gamma \mathcal{F}(u) + Hr .$$

T: $u \to \Gamma \mathcal{F}(u) + Hr$ is a mapping $(R^m)^{n+1}$ into $\mathrm{cf}\big((R^m)^{n+1}\big)$. For $z \in T(u)$, under our assumptions, we have

$$\|z\| \leqslant \|\Gamma\| (\bar{a} + \bar{\beta} \|u\|) + \|Hr\| , \qquad \bar{a} = \sum_{i=0}^{n} a_i . \tag{6.4}$$

Assuming that $\bar{\beta} \|\Gamma\| < 1$ and putting

$$K_\varrho = \{u \in (R^m)^{n+1} \colon \|u\| < \varrho\} , \qquad \varrho = (\bar{a} \|\Gamma\| + \|Hr\|)(1 - \bar{\beta} \|\Gamma\|)^{-1}$$

we obviously verify that $T(K_\varrho) \subset K_\varrho$.

Moreover, the closure of $T(K_\varrho) = \Gamma \mathcal{F}(K_\varrho) + Hr$ is compact. We shall apply the Kakutani fixed point theorem [4] to the mapping T and to the convex envelope of $T(K_\varrho)$ but we must also prove the upper semicontinuity of T. To this end we observe that if

$$w^k \to w^0, \quad u^k \to u^0, \quad w^k \in T(u^k) = \Gamma\mathcal{F}(u^k) + Hr,$$

then there exists a sequence $\{v^{\nu k}\}$ such that

$$w^{\nu k} = \Gamma v^{\nu k} + Hr, \quad v^{\nu k} \in \mathcal{F}(u^{\nu k}), \quad v^{\nu k} \to v^0 \in T(u^0).$$

This follows from the fact that the mapping $u \to \mathcal{F}(u)$ is upper semicontinuous and from (6.3). Finally,

$$\lim_{k\to\infty} w^{\nu k} = \lim_{k\to\infty} w^k = w^0 \in \mathcal{F}(u^0).$$

By way of illustration we take for equation (3.5) the Nicoletti type conditions, i.e. the conditions

$$u_{k_j,j} = r_j \quad (r_j \in R,\ k_j \in N,\ j = 1, \dots, m), \tag{6.5}$$

where $u_i = (u_{i1}, \dots, u_{im})$.

Since the corresponding homogeneous problem

$$\Delta u_i = 0, \quad u_{k_j,j} = 0$$

has the unique solution $u = 0$, applying Theorem 6.1 (for $A = 0$ and $Lu = (u_{k_11}, \dots, u_{k_mm})$) we obtain the existence of a solution of problem (3.5), (6.3) for F satisfying (6.3) with $\bar{\beta}$ sufficiently small.

Analysing the proof of Theorem 6.1 we see that condition (6.3) may be replaced by other conditions which give similar results. For example we consider the difference equation (6.3) with the boundary condition

$$Lu = u_0 - u_n = 0. \tag{6.6}$$

This is a problem of periodic solutions for a difference equation with a multi-valued right-hand side. We formulate a theorem which is a generalization of Halanay's theorem (see [3], Theorem 2).

Let Γ be a mapping $(R^m)^{n+1}$ into itself defining in the following way

$$\Gamma b = a, \quad a_i = \sum_{j=0}^{n} G_{ij} b_j,$$

$$G_{ij} = \begin{cases} B_{i-1}\dots B_0 A_{n-1}\dots A_0 B_{i-1}\dots B_{j-1} + B_{i-1}\dots B_{j-1} & \text{for } i > j, \\ B_{i-1}\dots B_0 A_{n-1}\dots A_0 B_{n-1}\dots B_{j-1} & \text{for } i \leqslant j, \end{cases}$$

$B_i = I - A_i$, I — the unit matrix.

THEOREM 6.2. *Suppose that F satisfies* (vii) *and* $|F(i,p)| \leqslant \beta(|p|)$, *where β is non-decreasing and* $\beta(\varrho_0)/\varrho_0 \leqslant 1/\|\Gamma\|$ *for some* ϱ_0. *If* $u = 0$ *is the unique solution of problem* (6.2), (6.6), *then there exists at least one solution of problem* (6.1), (6.6).

The proof of the above theorem is similar to the proof of Theorem 6.1. In particular, evaluation (6.4) has now the form

$$\|z\| \leqslant \|\Gamma\| \frac{\beta(\varrho_0)}{\|\Gamma\|} \leqslant \varrho_0 \quad \text{when} \quad \|u\| \leqslant \varrho_0 .$$

Observe that for $F(i,p) = \{f(i,p)\}$ we get exactly Halanay's result.

References

[1] F. V. Atkinson, *Discrete and continuous boundary problems*, New York-London 1964.

[2] A. Granas, *The theory of compact vector fields and some of its applications to topology of functional spaces*, Rozprawy Matematyczne 30, Warszawa 1962,

[3] A. Halanay, *Solutions périodiques et presque-périodiques des systèmes d'équations aux différence finies*, Arch. Rat. Mech. Anal. 12 (1963), pp. 134-149.

[4] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. Journ. 8 (1941), pp. 457-459.

[5] A. Lasota, *Une généralisation du premier théorème de Fredholm et ses applications à la théorie des équations différentielles ordinaires*, Ann. Polon. Math. 18 (1966), pp. 65-77.

[6] — *A discrete boundary-value problem*, ibidem 20 (1968), pp. 183-190.

[7] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinairy differential equations*, Bull. Acad. Polon. Sci., Sér., sci, math. astr. et phys., 13 (1965), pp. 781-786.

[8] — — *On the existence of solutions of linear problems for ordinairy differential equations*, ibidem 14 (1966), pp. 371-276.

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