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Tsuji points and conformal mapping

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Abstract. Let E be a (non-degenerate) continuum in the complex plane and let φ be the conformal mapping of $C\backslash E$ onto the exterior of the unit circle. Then, Leja [1] showed that the coefficients of the Laurent expansion of φ'/φ can be easily computed by means of the Fekete points of E. (It should be noted here, that a great part of the Fekete point theory is due to F. Leja.) Tsuji [4] "modified" these points to be suitable for the "hyperbolic case", where E is a subset of the open unit disk. Under additional assumptions on E we here establish a theorem qualitatively corresponding to Leja's result.

1. Introduction. Let E be a compact subset of the complex plane. If

$$\max_{\eta_1,\ldots,\eta_n\in E} \prod_{j=1}^{n} \prod_{\substack{k=1\\ j\neq k}}^{n} |\eta_j-\eta_k| = \prod_{j=1}^{n} \prod_{\substack{k=1\\ j\neq k}}^{n} |\eta_{nj}-\eta_{nk}|,$$

then $\eta_{n1},\ldots,\eta_{nn}$ are called *n-th Fekete points*. If in addition E is a (non-degenerate) continuum, let φ be the conformal mapping of $C\backslash E$ onto the exterior of the unit circle (normalized in the usual manner).

We have the expansion

$$\frac{\varphi'(w)}{\varphi(w)} = \sum_{k=0}^{\infty} \frac{s_k}{w^{k+1}} \quad \text{for } |w| > R,$$

R sufficiently large.

Leja [1] proved

$$s_k = \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \eta_{n\nu}^k, \quad k = 0, 1, 2, \ldots$$

(For further estimates see Pommerenke [3].)

Here we want to establish a corresponding result for the case of a doubly connected region.

In the following let E be a continuum in the open unit disk D. For this case Tsuji [4] introduced another point system replacing the Euclidean by the hyperbolic distance:

If

$$\max_{z_1,\ldots,z_n\in E} \prod_{j=1}^{n} \prod_{\substack{k=1\\ j\neq k}}^{n} \left|\frac{z_j-z_k}{1-\bar{z}_j z_k}\right| = \prod_{j=1}^{n} \prod_{\substack{k=1\\ j\neq k}}^{n} \left|\frac{z_{nj}-z_{nk}}{1-\bar{z}_{nj} z_{nk}}\right|$$

we call $z_{n1},\ldots,z_{nn}$ *n-th Tsuji points.*

Now, let $0\in E$ and let E be non-degenerate. We denote by g the conformal mapping of $D\backslash E$ onto the annulus $\{w\colon \varrho<|w|<1\}$, where ϱ is the hyperbolic capacity of E; g is uniquely determined by $g(1)=1$.

For $R<|\zeta|<R^{-1}$ (R close to 1) we have the Laurent expansion

$$\log\frac{g(\zeta)}{\zeta} = \alpha_0+\sum_{k=1}^{\infty} \alpha_k\zeta^k-\bar{\alpha}_k\zeta^{-k}, \qquad \text{where } \operatorname{Re}\alpha_0=0.$$

(Note that g maps the unit circle onto the unit circle.)

With these notations we obtain the following

THEOREM. *Let E be bounded by a closed analytic Jordan curve and let* $z_{n\nu}$, $\nu=1,\ldots,n$ *be n-th Tsuji points; then there exists a constant L independent of n, such that*

$$\left|\bar{\alpha}_k\cdot k-\frac{1}{n}\sum_{\nu=1}^{n} z_{n\nu}^k\right| \leqslant L\cdot k\cdot\frac{\log n}{n}, \qquad k=1,2,\ldots$$

The proof will show that the smoothness condition on ∂E is needed only in part (d) of the following lemma and probably can be weakened.

2. Proof of the theorem. Let f be the inverse function of g. By the reflection principle we can extend f to a function analytic and univalent for $\varrho<|w|<\varrho^{-1}$. If $z\in E$, then

$$\varphi(w,z):=\log\frac{f(w)-z}{w\big(1-\bar{z}f(w)\big)}$$

is analytic in w for $\varrho<|w|<\varrho^{-1}$ and we have the expansion

$$\varphi(w,z)=A_0(z)+\sum_{k=1}^{\infty} A_k(z)\,w^k-\overline{A_k(z)}\,w^{-k} \qquad (z\in E), \tag{1}$$

where $\operatorname{Re}A_0(z)=0$.

Now, we can prove

LEMMA. *Let E satisfy the conditions of the above theorem, and let* $\{z_{n\nu}\}$, $\nu=1,\ldots,n$; $n=2,3,\ldots$ *be a point system on E with the following properties:*

There exists a non-negative α such that

$$\text{(i)} \qquad \max_{z\in E} \prod_{\nu=1}^{n} \left|\frac{z-z_{n\nu}}{1-\bar{z}_{n\nu} z}\right| \leqslant \varrho^n \cdot n^\alpha,$$

$$\text{(ii)} \qquad \prod_{\substack{\nu=1\\ \nu\neq k}}^{n} \left|\frac{z_{nk}-z_{n\nu}}{1-\bar{z}_{n\nu} z_{nk}}\right| \geqslant \varrho^n n^{-\alpha},$$

then

$$\sum_{k=1}^{\infty} \left|\sum_{\nu=1}^{n} A_k(z_{n\nu})\right|^2 \varrho^{-2k} \leqslant M \cdot (\log n)^2,$$

where M is not depending on n.

Proof. We use a modification of a method due to Pommerenke [3]. For simplicity we drop the index n in $z_{n\nu}$ and we denote by c_k ($k = 1, 2, \ldots$) positive constants not depending on n. Let $r_n := (1-n^{-2})^{-1}$ for $n \geqslant 2$; then

$$H_n(w) := \sum_{\nu=1}^{n} \varphi(r_n w, z_\nu) \quad \text{is analytic for } \varrho/r_n < |w| < (\varrho \cdot r_n)^{-1}.$$

(a) $\operatorname{Re} H_n(w)$ vanishes for $|w| = 1/r_n$ and for $|w| = \varrho/r_n$, (i) leads us to $\operatorname{Re} H_n(w) \leqslant \alpha \log n$.

So, by the maximum principle, we have for $|w| = \varrho$

$$\operatorname{Re} H_n(w) \leqslant \alpha \log n.$$

(b) The Lagrange interpolation formula shows

$$\sum_{j=1}^{n} \left(\prod_{\substack{\nu=1\\ \nu\neq j}}^{n} \frac{z-z_\nu}{z_j-z_\nu} \prod_{\nu=2}^{n} (1-\bar{z}_\nu z_j)\right) = \prod_{\nu=2}^{n} (1-\bar{z}_\nu z).$$

If we set $\psi_n(z) := \prod_{\nu=1}^{n} \frac{z-z_\nu}{1-\bar{z}_\nu z}$, we get by use of (ii)

$$1 \leqslant |\psi_n(z)| \cdot \frac{n^{1+\alpha} \cdot c_1}{\operatorname{dist}(z, E)\,\varrho^n}, \qquad z \in D \setminus E,$$

hence

$$|\psi_n(f(r_n w))| \geqslant \frac{c_2 \varrho^n}{n^{1+\alpha}} \operatorname{dist}(f(r_n w), E), \qquad |w| = \varrho.$$

Now, we can compose $f = f_1 \circ f_2$, where f_1 is univalent on the exterior of the unit circle, and f_2 is a conformal mapping of the annulus $\{w\colon \varrho < |w| < 1\}$ onto a doubly connected region which is bounded by the unit circle and a closed analytic Jordan curve. So we can apply a distortion theorem to f_1 and we get

$$\operatorname{dist}(f(r_n w), E) \geqslant c_2 (r_n - 1)^2, \qquad |w| = \varrho,$$

and we conclude

$$\operatorname{Re} H_n(w) = \log|\psi_n(f(r_n w))| - n(\log r_n + \log \varrho) \geqslant -c_4 \log n, \qquad |w| = \varrho.$$

(c) An easy computation shows

$$\int_0^{2\pi} (\operatorname{Re} H_n(\varrho e^{it}))^2 \, dt = \pi \sum_{k=1}^{\infty} \varrho^{-2k} r_n^{-2k} (1 - (r_n \cdot \varrho)^{2k})^2 \cdot \Big| \sum_{\nu=1}^{n} A_k(z_\nu) \Big|^2.$$

Since $r_n \to 1$ for $n \to \infty$, we have for $n \geqslant n_0$

$$1 - (r_n \varrho)^{2k} \leqslant (1 - \varrho^2)/2 \qquad \text{for } k = 1, 2, \ldots$$

and with (a) and (b) we get

$$\sum_{k=1}^{\infty} \varrho^{-2k} r_n^{-2k} \Big| \sum_{\nu=1}^{n} A_k(z_\nu) \Big|^2 \leqslant c_4 (\log n)^2$$

and

$$\sum_{k=1}^{n^2} \Big| \sum_{\nu=1}^{n} A_k(z_\nu) \Big|^2 \varrho^{-2k} \leqslant \sum_{k=1}^{n^2} \Big| \sum_{\nu=1}^{n} A_k(z_\nu) \Big|^2 r_n^{2n^2 - 2k}$$
$$\leqslant r_n^{2n^2} \cdot c_4 (\log n)^2 \leqslant c_5 (\log n)^2.$$

(d) Since the boundary of E is a closed analytic Jordan curve (only here we need the assumption on the boundary of E), we conclude with some simple considerations

$$A_k(f(\omega)) = b_k(\omega) + \bar{\omega}^k / k, \qquad k = 1, 2, \ldots, \ |\omega| = \varrho,$$

where $|b_k(\omega)| \leqslant c_6 \cdot r^k$ for some r, $r < \varrho$. Hence

$$\sum_{k=n^2+1}^{\infty} \Big| \sum_{\nu=1}^{n} A_k(z_\nu) \Big|^2 \varrho^{-2k} \leqslant c_7$$

which completes the proof.

Remark. It was proved in [2] that the Tsuji points fulfil the conditions (i) and (ii) of the above lemma.

If we insert $w = g(\zeta)$ in the expansion (1), we have for $|\zeta| = 1$:

$$A_0(z) + \sum_{k=1}^{\infty} A_k(z) g(\zeta)^k - \overline{A_k(z)} \, g(\zeta)^{-k} = -\log \frac{g(\zeta)}{\zeta} + \sum_{k=1}^{\infty} \frac{1}{k} ((\zeta \cdot \bar{z})^k - (z/\zeta)^k).$$

If we multiply this equation by ζ^{j-1} and integrate around the unit circle (with respect to ζ), we get on the right-hand side the expression $2\pi i(\bar{\alpha}_j - z^j/j)$.

By a trivial estimate we conclude

$$\left|j\bar{\alpha}_j - \frac{1}{n}\sum_{\nu=1}^{n} z_{n\nu}^j\right| \leqslant 2\frac{j}{n}\sum_{k=1}^{\infty}\left|\sum_{\nu=1}^{n} A_k(z_{n\nu})\right| \leqslant L\cdot j\cdot\frac{\log n}{n}.$$

References

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