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The f-sectional curvature of f-Kaehlerian manifolds

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Abstract. Analogously to the Kaehlerian case, we introduce the notion of f-Kaehlerian manifold, where f is a metric polynomial structure, and consider the sectional curvature by planes invariant by f.

0. Let (M, g) be a connected Riemannian manifold and let f be a $(1, 1)$-tensor field on M satisfying the condition $g(fX, fY) = g(X, Y)$ for any tangent vectors X and Y. The Riemannian connection defined by the metric tensor g will be denoted by ∇. If $\nabla f = 0$ then (M, g, f) will be called an *f-Kaehlerian manifold*. Of course, if (M, g, f) is an f-Kaehlerian manifold, then f is a metric polynomial structure in the sense of [3]. An f-Kaehlerian manifold will be called of *type* I, II, III or IV if the metric polynomial structure f is of type I, II, III or IV, respectively, i.e., the minimal polynomial $P(\xi)$ of f is

(I) $P(\xi) = (\xi^2+2a_1\xi+1)\dots(\xi^2+2a_s\xi+1)$,

(II) $P(\xi) = (\xi-1)(\xi^2+2a_1\xi+1)\dots(\xi^2+2a_{s-1}\xi+1)$,

(III) $P(\xi) = (\xi+1)(\xi^2+2a_1\xi+1)\dots(\xi^2+2a_{s-1}\xi+1)$,

(IV) $P(\xi) = (\xi-1)(\xi+1)(\xi^2+2a_1\xi+1)\dots(\xi^2+2a_{s-2}\xi+1)$,

where $a_i^2 < 1$ and $a_i \neq a_j$ for $i \neq j$.

Recall that the almost product structure $D = (D_1, \dots, D_s)$ associated with f has projectors $P_1, \dots, P_s$ which are polynomials in f, [3]. It follows that D is f-invariant and parallel with respect to ∇.

Since the connection ∇ is without torsion, f and D are integrable.

The almost product structure D is orthogonal, i.e., distributions $D_1, \dots, D_s$ are mutually orthogonal, [5].

For the tensor field f we associate the $(1, 1)$-tensor field defined by

$$J = \sum_{i=1}^{s} \frac{f+a_i I}{\sqrt{1-a_i^2}} P_i,$$

where I is the identity tensor field on M. It is easy to check that J satisfies the equation $J^3 - J = 0$ and J is an almost complex structure if only f is of type I. For an f-Kaehlerian manifold of type I its metric is Hermitian with respect to J, [5], and consequently, (M, g, J) is Kaehlerian.

It would be of some interest to note that what is called an *f-structure*, i.e. a $(1, 1)$-tensor field satisfying the equation $f^3 - f = 0$, may be considered as a metric polynomial structure. Namely, defining $\tilde{f}$ by $\tilde{f} = P + f$, where P denotes the projector on the distribution on which f vanishes, $\tilde{f}$ is a metric polynomial structure and $\nabla f = 0$ if and only if $\nabla \tilde{f} = 0$. For instance, if f is a $\mathscr{C}$-structure (see [2]), M is $\tilde{f}$-Kaehlerian.

As usual, R will denote the curvature tensor of ∇, i.e. the tensor field of type $(1, 3)$ defined by $R(X, Y)Z = [\nabla_X \nabla_Y]Z - \nabla_{[X,Y]}Z$ for any vector fields X, Y, Z.

R will also denote the Riemannian curvature tensor field, i.e. the tensor field of type $(0, 4)$ defined as $R(X, Y, Z, W) = g(R(Z, W)Y, X)$ for vectors X, Y, Z, W.

If X and Y form an orthonormal basis of a plane p in the tangent space $T_x M$, then the sectional curvature $K(p)$ of p is given by $g(R(X, Y)Y, X)$. If (M, g, f) is an f-Kaehlerian manifold and p is an f-invariant plane in $T_x M$, then the curvature $K(p)$ will be called the *f-sectional curvature* by p. If f is an almost complex structure, then the f-sectional curvature is the holomorphic sectional curvature.

By an *integral manifold* of a distribution we shall always mean a connected integral manifold.

If X is a vector field and T is a distribution, then the notation $X \in T$ means that $X_x \in T_x$ for every point x belonging to the domain of X.

1. PROPOSITION 1. *Let $T = (T_1, \ldots, T_m)$ be an orthogonal almost product structure parallel with respect to ∇. Then $R(X, Y, Z, W) = 0$ provided that two of the tangent vectors X, Y, Z, W belong to different distributions of T.*

Proof. Since the almost product structure T is parallel with respect to ∇, $\nabla_X Y \in T_i$, $1 \leqslant i \leqslant m$, provided $Y \in T_i$. Hence if $Y \in T_i$ then $R(Z, W)Y \in T_i$ for any tangent vectors Z, W. Let $X \in T_i$, $Y \in T_j$ and $i \neq j$. Since $R(X, Y, Z, W) = g(R(Z, W)Y, X)$ and T_i is orthogonal to T_j, we have $R(X, Y, Z, W) = 0$. But $R(X, Y, Z, W) = R(Z, W, X, Y)$, and so if $Z \in T_i$ and $W \in T_j$, where $i \neq j$, then $R(X, Y, Z, W) = 0$. Now, it is sufficient to consider the case where $X, Y \in T_i$, $Z, W \in T_j$ and $i \neq j$. It is known that R satisfies the identity

$$R(X, Y, Z, W) = -R(X, Z, W, Y) - R(X, W, Y, Z).$$

But $R(X, Z, W, Y) = 0$ and $R(X, W, Y, Z) = 0$ by the first part of the proof. Hence $R(X, Y, Z, W) = 0$.

PROPOSITION 2. *Let f be a metric polynomial structure on M. If f is of type* I, II, *or* III, *then for every f-invariant plane $p \subset T_x M$, there exists an $1 \leqslant i \leqslant s$ such that $p \subset D_{ix}$.*

If f is of type IV and p is an f-invariant plane in $T_x M$, then $p \subset D_{1x} \oplus D_{2x}$ or there exists an $3 \leqslant i \leqslant s$ such that $p \subset D_{ix}$.

Proof. It is sufficient to consider a metric polynomial structure of type IV.

We set $T_1 = D_1 \oplus D_2$, $T_2 = D_3 \oplus \ldots \oplus D_s$. Let $p \subset T_x M$ be an f-invariant plane. If there is a vector $Y \in p$, $Y \neq 0$ and $Y \in T_{2x}$, then Y and fY belong to $p \cap T_{2x}$. Vectors Y and fY are linearly independent and p is a 2-dimensional vector space; hence $p \subset T_{2x}$.

Suppose now that there is a vector $X \in p$ such that

$$X = X_1 + X_2 + X_3, \quad \text{where} \quad X_1 \in D_{1x}, \quad X_2 \in D_{2x}, \ X_3 \in T_{2x} \quad \text{and} \quad X_1 + X_2 \neq 0. \tag{1}$$

Then we have

$$f(X) = X_1 - X_2 + f(X_3). \tag{2}$$

By equalities (1) and (2) and by the obvious fact that $X + f(X) \in p$, we have

$$2X_1 + X_3 + f(X_3) \in p. \tag{3}$$

Since $f(2X_1 + X_3 + f(X_3)) \in p$, we obtain

$$2X_1 + f(X_3) + f^2(X_3) \in p. \tag{4}$$

By equalities (3) and (4), we have $X_3 - f^2(X_3) \in p$. But $X_1 + X_2 \neq 0$, and thus $X_1 + X_2 + X_3 \notin T_{2x}$. This means that $p \not\subset T_{2x}$ and by the first part of the proof $X_3 - f^2(X_3) = 0$. Therefore, by the definition of T_2, we see that $X_3 = 0$. Consequently $p \subset T_{1x}$.

Suppose now that $p \subset T_{2x}$. Notice first that if $p \cap D_{ix} \neq \{0\}$, then $p \subset D_{ix}$. In fact, let $X \neq 0$ and $X \in p \cap D_{ix}$. The vectors X and $f(X)$ are linearly independent, and hence X, $f(X)$ is a basis in p. Therefore $p \subset D_{ix}$. Let X be a vector such that $X \in p$, $X = X_3 + \ldots + X_s$, $X_3 \in D_{3x}, \ldots, X_s \in D_{sx}$, and there exist $i \neq j$, $3 \leqslant i, j \leqslant s$ such that $X_i \neq 0$, $X_j \neq 0$. It is obvious that if a plane is f-invariant then it is also f^{-1}-invariant. Hence $f^{-1}(X_3) + \ldots \ldots + f^{-1}(X_s) \in p$. Since

$$f^{-1}(X_k) = -f(X_k) - 2a_{k-2} X_k, \quad 3 \leqslant k \leqslant s,$$

we obtain

$$-f(X_3) - 2a_1 X_3 - \ldots - f(X_s) - 2a_{s-2} X_s \in p.$$

But $f(X_3) + \ldots + f(X_s) \in p$ and thus $a_1 X_3 + \ldots + a_{s-2} X_s \in p$. Since $X_i \neq 0$, $X_j \neq 0$ and $a_{i-2} \neq a_{j-2}$, vectors $X_3 + \ldots + X_s$ and $a_1 X_3 + \ldots + a_{s-2} X_s$ are linearly independent. Consequently, there are numbers β and γ such that

$$\begin{aligned} f(X_3) + \ldots + f(X_s) &= \beta(X_3 + \ldots + X_s) + \gamma(a_1 X_3 + \ldots + a_{s-2} X_s) \\ &= (\beta + \gamma a_1) X_3 + \ldots + (\beta + \gamma a_{s-2}) X_s. \end{aligned}$$

Since $X_k \in D_{kx}$ and $f(X_k) \in D_{kx}$, $f(X_k) = (\beta + \gamma a_{k-2}) X_k$ for every $k = 3, \ldots, s$. On the other hand, vectors X_i and $f(X_i)$ are linearly independent. Hence the contradiction and this completes the proof.

It is trivial that if a plane p is included in D_{ix} and f is not a multiple of indentity on D_i then p is f-invariant if and only if p is J-invariant.

Observe also that if a plane p is f-invariant and $p \subset D_{1x} \oplus D_{2x}$, then $p \subset D_{1x}$ or $p \subset D_{2x}$, or p is spanned by vectors $X_1 \in D_{1x}$ and $X_2 \in D_{2x}$. In fact, if $p \not\subset D_{1x}$, and $p \not\subset D_{2x}$, then there is a vector $X \in p$, $X = X_1 + X_2$, where $0 \neq X_1 \in D_{1x}$, $0 \neq X_2 \in D_{2x}$. Since p is f-invariant, $X_1 - X_2 \in p$. Therefore, $X_1 \in p$ and $X_2 \in p$. This means that X_1, X_2 is a basis in p. Conversely, if $0 \neq X_1 \in D_{1x}$, $0 \neq X_2 \in D_{2x}$, then a plane spanned by vectors X_1, X_2 is f-invariant. Notice also that if $\dim D_i \geqslant 2$, $i = 1, \ldots, s$, then there exists an f-invariant plane in D_{ix}, $x \in M$.

The following versions of Schur's theorems are known:

THEOREM 3. *Let (M, g) be a connected Riemannian manifold of dimension $n \geqslant 3$. If the sectional curvature $K(p)$, where p is a plane in $T_x M$ depends only on x, then M is a space of constant curvature.*

THEOREM 4. *Let (M, g, J) be a connected Kaehlerian manifold of complex dimension $n \geqslant 2$. If the holomorphic sectional curvature $K(p)$, where p is a plane in $T_x M$, depends only on x, then M is a space of constant holomorphic sectional curvature.*

We now prove an f-Kaehlerian analogue of Schur's theorem. We shall first prove the following

THEOREM 5. *Let (M, g, f) be a connected f-Kaehlerian manifold. If for some $1 \leqslant i \leqslant s$ $\dim D_i \geqslant 3$ and if the f-sectional curvature $K(p)$, where p is an f-invariant plane in D_{ix}, depends only on x, then there exists a number c such that $K(p) = c$ for every f-invariant plane $p \subset D_x$ and $x \in M$.*

Proof. Since M is connected, it suffices to show that for every $x \in M$ there is a neighbourhood U and a number c such that $K(p) = c$ for every f-invariant plane $p \subset D_{iy}$ and $y \in U$. In the proof of this statement we shall use the following

LEMMA. *Let (M, g, f) be an f-Kaehlerian manifold. Let $T = (T_1, \ldots, T_m)$ be an almost product structure on M the projectors of which are polynomials in f. Then every integral manifold N_i of distribution T_i, $i = 1, \ldots, m$, with the metric tensor g_i which is the restriction of g to N_i and with the tensor field f_i which is the restriction of f to N_i is an f_i-Kaehlerian manifold. Moreover, for any plane $p \subset T_x N_i$, $x \in N_i$, $K_i(p) = K(p)$, where $K_i(p)$ is the sectional curvature by p on N_i.*

Proof of Lemma. Since the almost product structure is f-invariant, the restriction f_i of f to an integral manifold N_i is well defined. Let $Q_1, \ldots, Q_m$ be projectors of T. As polynomials in f, Q_i, $i = 1, \ldots, m$, are parallel with respect to ∇. Therefore, if ∇^i denotes the Riemannian connection defined by g_i, $\nabla^i_X Y$

$= \nabla_X Y$ for any vector fields X, Y on N_i. Using this fact, we infer that for any vector fields X, Y on N_i

$$(\nabla^i_X f_i) Y = \nabla^i_X (f_i Y) - f_i (\nabla^i_X Y) = \nabla_X (f Y) - f(\nabla_X Y) = (\nabla_X f) Y = 0.$$

Hence (N_i, g_i, f_i) is an f_i-Kaehlerian manifold.

If R_i denotes the curvature tensor of ∇^i, then $R_i(X, Y)Z = R(X, Y)Z$ for all vector fields X, Y, Z on N_i, because

$$R_i(X, Y)Z = [\nabla^i_X \nabla^i_Y] Z - \nabla^i_{[X,Y]} Z = [\nabla_X \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

Consequently, if p is a plane in $T_x N_i$, then $K_i(p) = g_i(R_i(X, Y) Y, X) = g(R(X, Y) Y, X) = K(p)$, where X, Y is an orthonormal basis in p. This finishes the proof of the lemma.

Now we go back to the proof of our theorem.

The tensor field f is 0-deformable, and so there is a matrix $F \in GL(n, \boldsymbol{R})$ such that the Jordan canonical form of f_x is equal to F for all $x \in M$. Since f is integrable, for every x of M there exists a chart $(U, \varphi = (x^1, \dots, x^n))$ such that $d_y \varphi \circ f_y \circ d_{\varphi(y)} \varphi^{-1} = F$ for every $y \in U$. A chart chosen in this way is called a *chart associated with the integrable tensor field f*. The same chart is also associated with the integrable almost product structure D, because its projectors are polynomials in f.

Let $x \in M$ and let $(U, \varphi = (x^1, \dots, x^n))$ be a chart associated with the integrable tensor field f and, moreover, let $\varphi(x) = 0 \in \boldsymbol{R}^n$. Then there are vector subspaces $\boldsymbol{R}^{(i)}$, $i = 1, \dots, s$, in $\boldsymbol{R}^n$ satisfying the equality $\boldsymbol{R}^{(i)} = d_y \varphi(D_{iy})$ for $y \in U$. The projection in $\boldsymbol{R}^n$ onto $\boldsymbol{R}^{(i)}$ will be denoted by $P^{(i)}$. Of course, U can be chosen so that $\varphi(U) = V_1 \times \dots \times V_s$, where V_i, $i = 1, \dots, s$, are neighbourhoods of 0 in $\boldsymbol{R}^{(i)}$. Then $M_{iy} = \varphi^{-1}\big(\sum_{j \neq i} P^{(i)}(\varphi(y)) + V_i\big)$ is an integrable manifold of D_i through $y \in U$. In particular, $M_{ix} = \varphi^{-1}(V_i)$ and we shall denote $U_i = M_{ix}$.

If f is a multiple of identity on D_i then (U_i, g_i) has a constant sectional curvature by Theorem 3. By Theorem 4, if f is not a multiple of identity on D_i, then $\left(U_i, g_i, J_i = \dfrac{f_i + a_i I}{\sqrt{1 - a_i^2}}\right)$ is a Kaehlerian manifold with a constant holomorphic sectional curvature. Hence, by the remarks made after Proposition 2, it is seen that (U_i, g_i, f_i) is an f_i-Kaehlerian manifold of constant sectional curvature, say c.

Consider mappings

$$\tilde{\varphi}^r = \varphi^{-1} \circ P^{(r)} \circ \varphi \colon U \to U_r, \qquad r = 1, \dots, s.$$

The restriction $\tilde{\varphi}^r|_{M_{r_y}} \colon M_{r_y} \to U_r$ is a diffeomorphism. It is also an isometry (with respect to the restrictions of g to M_{r_y} and U_r). In fact, taking $X_i = \partial/\partial x^i$, $X_j = \partial/\partial x^j \in D_r$, we have $d_z \tilde{\varphi}^r(X_i) = X_{i(\tilde{\varphi}^r(z))}$, $d_z \tilde{\varphi}^r(X_i) = X_{j(\tilde{\varphi}^r(z))}$ and $(P^{(r)} \circ \varphi)(\tilde{\varphi}^r(z)) = (P^{(r)} \circ \varphi)(z)$. Therefore, it suffices to show that if X_k

$= \partial/\partial x^k \in D_m$, where $m \neq r$, then $X_k(g(X_i, X_j)) = 0$. Since $\nabla_{X_k} X_j = \nabla_{X_k} X_i = 0$, [5], and g is parallel with respect to ∇, we have

$$X_k(g(X_i, X_j)) = \nabla_{X_k}(g(X_i, X_j)) = g(\nabla_{X_k} X_i, X_j) + g(X_i, \nabla_{X_k} X_j) = 0.$$

The map $\tilde{\varphi}^r$ has the following property: $d_y \tilde{\varphi}^r \circ f_y = f_{\tilde{\varphi}^r(y)} \circ d_y \tilde{\varphi}^r$ for $y \in U$. We have

$$\begin{aligned} d_y \tilde{\varphi}^r \circ f_y &= d_{P^{(r)}(\varphi(y))} \varphi^{-1} \circ P^{(r)} \circ d_y \varphi \circ f_y \\ &= d_{P^{(r)}(\varphi(y))} \varphi^{-1} \circ F \circ P^{(r)} \circ d_y \varphi \\ &= (d_{P^{(r)}(\varphi(y))} \varphi^{-1} \circ F \circ d_{\tilde{\varphi}^r(y)} \varphi) \circ (d_{P^{(r)}(\varphi(y))} \varphi^{-1} \circ P^{(r)} \circ d_y \varphi) \\ &= f_{\tilde{\varphi}^r(y)} \circ d_y \tilde{\varphi}^r. \end{aligned}$$

This means that if $p \subset D_{r_y}$ is an f-invariant plane in D_{r_y} then $d_y \tilde{\varphi}^r(p)$ is also f-invariant. By virtue of Lemma, and by the fact that $\tilde{\varphi}^r|_{M_{r_y}}$ is an isometry, it follows that $K(p) = K(d_y \tilde{\varphi}(p)) = c$. Hence the proof is completed.

COROLLARY 6. *Let (M, g, f) be an f-Kaehlerian manifold. Assume that there is an $1 \leqslant i \leqslant s$ such that $\dim D_i \geqslant 3$, where $D = (D_1, \ldots, D_s)$ is the almost product structure associated with f. If the f-sectional curvature $K(p)$, where p is an f-invariant plane in $T_x M$, depends only on x, then the f-sectional curvature is constant on M. Moreover, if f is of type* IV, *the f-sectional curvature vanishes on M.*

Proof. The first part of the assertion immediately follows from Theorem 5. Now let f be of type IV and let $X_1 \in D_{1x}$, $X_2 \in D_{2x}$ be unit vectors. As we have already remarked, X_1, X_2 span an f-invariant plane p in $T_x M$. Since D_1 and D_2 are orthogonal, $K(p) = R(X_1, X_2, X_1, X_2)$. By Proposition 1, $K(p) = 0$, and this completes the proof.

2. An f-Kaehlerian manifold (M, g, f) will be called *δ-f-pinched* if there is a positive number A such that $\delta A \leqslant K(p) \leqslant A$, for all planes p invariant by f. If f is an almost complex structure then a δ-f-pinched manifold is said to be *δ-holomorphically pinched.* Notice also that a δ-f-pinched f-Kaehlerian manifold with $\delta > 0$ is not of type IV.

We now give a number of implications of the pinching assumptions.

THEOREM 7. *For a δ-f-pinched f-Kaehlerian manifold (M, g, f), with $\delta > 1/2$, the decomposition of the tangent space $T_x M = \bigoplus_{i=1}^{s} D_{ix}$ associated with f is the de Rham decomposition of $T_x M$ with respect to g.*

Moreover, if there are vectors in $T_x M$ which are fixed by the restricted linear holonomy group $\psi(x)$, then f is of type II *or* III, *D_{1x} is a set of all such vectors and* $\dim D_1 = 1$.

Proof. We shall first show that if f is not a multiple of identity on D_i and, for every $h \in \psi(x)$, $h(X) = X$, then $X = 0$. As usual, let M_{ix} denote the integral manifold of D_i through x and g_i the restriction of the metric tensor g to M_{ix}. Then (M_{ix}, g_i, J_i) is Kaehlerian, where

$$J_{i_y} = \frac{f|_{D_{iy}} + a_i I|_{D_{iy}}}{\sqrt{1-a_i^2}} \quad \text{for} \quad y \in M_{ix}.$$

Furthermore, M_{ix} is δ-holomorphically pinched. It is known, see [1], that a $\frac{1}{2}$-holomorphically pinched Kaehlerian manifold has positive definite Ricci tensor. On the other hand, we have the following

THEOREM ([4], vol. II, p. 173). *Let (M, g, J) be a Kaehlerian manifold. If the Ricci tensor is non-degenerate at some point of M, then the restricted linear holonomy group $\psi(x)$ at x contains J_x.*

Hence $J_{ix} \in \psi^i(x)$, where $\psi^i(x)$ is the restricted linear holonomy group at x of the Riemannian connection ∇^i. Therefore, the tensor field $\bar{J}_i$ defined as $JP_i + \sum_{i \neq j} P_j$ belongs to $\psi(x)$. But $\bar{J}_i$ does not fix any non-zero vector of D_{ix}, and hence $X = 0$.

Considering again D_i on which f is not a multiple of identity, suppose that there is a non-trivial subspace S_x in D_{ix} invariant by $\psi(x)$. Let S'_x be the orthogonal complement of S_x in $T_x M$. S'_x is also invariant by $\psi(x)$. Let S and S' denote distributions on M obtained from S_x and S'_x by parallel displacement to each point of M. Choose vectors $X_1 \in S_x$, $X_2 \in S'_x$ such that $g(X_1, X_1) = g(X_2, X_2) = \frac{1}{2}$. Let p be the plane spanned by vectors $X_1 + X_2$, $JX_1 + JX_2$. We have $J(X_1 + X_2) = \bar{J}_i(X_1 + X_2) = \bar{J}_i(X_1) + \bar{J}_i(X_2)$. Since $\bar{J}_{ix} \in \psi(x)$, $\bar{J}_i(X_1) \in S_x$ and $\bar{J}_{ix}(X_2) \in S'_x$. It is clear that $\bar{J}_i(X_1) = J(X_1)$ and $\bar{J}_i(X_2) = J(X_2)$. Therefore, $\sqrt{2}X_1$, $\sqrt{2}J(X_1)$ is an orthonormal basis for an f-invariant plane $p_1 \subset S_x$ and $\sqrt{2}X_2$, $\sqrt{2}J(X_2)$ is an orthonormal basis for an f-invariant plane $p_2 \subset S'_x$. By virtue of Proposition 1, we obtain

$$\begin{aligned} K(p) &= R(X_1 + X_2, JX_1 + JX_2, X_1 + X_2, JX_1 + JX_2) \\ &= R\left(\frac{1}{\sqrt{2}}(\sqrt{2}X_1), \frac{1}{\sqrt{2}}(\sqrt{2}JX_1), \frac{1}{\sqrt{2}}(\sqrt{2}X_1), \frac{1}{\sqrt{2}}(\sqrt{2}JX_1)\right) + \\ &\quad + R\left(\frac{1}{\sqrt{2}}(\sqrt{2}X_2), \frac{1}{\sqrt{2}}(\sqrt{2}JX_2), \frac{1}{\sqrt{2}}(\sqrt{2}X_2), \frac{1}{\sqrt{2}}(\sqrt{2}JX_2)\right) \\ &= \tfrac{1}{4}K(p_1) + \tfrac{1}{4}K(p_2). \end{aligned}$$

On the other hand, $K(p) \geqslant \delta A > \frac{1}{2}A$ for some constant A and $K(p_1) \leqslant A$, $K(p_2) \leqslant A$. Hence $\frac{1}{2}A < K(p) \leqslant \frac{1}{2}A$, which is a contradiction. This means that $\psi(x)$ is irreducible on D_{ix}.

Now suppose that $\psi(x)$ is reducible on D_{1x} and f is a multiple of identity on D_1. As in the previous case, there is an almost product structure $(S, S', D_2, \ldots, D_s)$ satisfying the assumptions of Proposition 1. Therefore, for any vectors $X_1 \in S_x$, $X_2 \in S'_x$, $R(X_1, X_2, X_1, X_2) = 0$. So $K(p) = 0$ for the f-invariant plane spanned by X_1 and X_2. Hence a contradiction.

The second part of the assertion follows immediately from the above proof.

Remark. If an f-Kaehlerian manifold is of type I and is δ-f-pinched, with $\delta > 0$, then the associated Kaehlerian manifold is $\frac{1}{s^2}\delta$-holomorphically pinched. In fact, let $X \in T_x M$ and $g(X, X) = 1$. Then $X = \sum_{i=1}^{s} X_i$, where $X_i \in D_{ix}$. Assume that $g(X_i, X_i) = B_i > 0$, $i = 1, \ldots, k \leqslant s$ and $\sum_{i=1}^{k} B_i = 1$. We have

$$R(X, JX, X, JX) = \sum_{i=1}^{s} R(X_i, JX_i, X_i, JX_i)$$
$$= \sum_{i=1}^{k} B_i^2 R\left(\frac{1}{\sqrt{B_i}} X_i, J\left(\frac{1}{\sqrt{B_i}} X_i\right), \frac{1}{\sqrt{B_i}} X_i, J\left(\frac{1}{\sqrt{B_i}} X_i\right)\right).$$

It is clear that $\sum_{i=1}^{k} B_i^2 \leqslant 1$ and there is $1 \leqslant i_0 \leqslant k$ such that $B_{i_0} \geqslant \frac{1}{s}$. Hence $\frac{1}{s^2}\delta A \leqslant R(X, JX, X, JX) \leqslant A$ for some constant A and every vector $X \in T_x M$.

The following Kaehlerian version of Myers theorem is known

THEOREM 8. *A complete δ-holomorphically pinched Kaehlerian manifold, with $\delta > 0$, is compact and simply connected.*

By this theorem and by the last Remark it can be seen that a complete δ-f-pinched f-Kaehlerian of type I with $\delta > 0$ is compact and simply connected. As for the general case, this assertion is not true. For instance, let M be a $0 < \delta$-holomorphically pinched Kaehlerian manifold. Then $\boldsymbol{R} \times M$, with the product metric and with the metric polynomial structure M which is a direct product of the identity (1, 1)-tensor field on $\boldsymbol{R}$ and the almost complex structure on M, is a $0 < \delta$-f-pinched f-Kaehlerian manifold and it is not compact.

However, using the same idea as in the proof of the theorem of Myers, we obtain

THEOREM 9. *A complete f-Kaehlerian manifold whose f-sectional curvature is bounded away from zero and for which* $\dim D_1 \geqslant 2$ *is compact.*

Proof. We outline the proof given in [4], vol. I, p. 88, making the necessary changes.

The theorem is to be proved for f-Kaehlerian manifolds of type II or III.

Recall that if τ is a geodesic in M, then two points x and y on τ are said to be *conjugate to each other along* τ if there exists a non-zero Jacobi field X along τ which vanishes both at x and at y.

Next we shall use the following theorems:

THEOREM 10. *Let $\tau = x_t$, $a \leqslant t \leqslant b$, be a geodesic in M such that x_a has no conjugate point along $\tau = x_t$ for $a \leqslant t \leqslant b$. If X is a piecewise differentiable vector field along τ vanishing at x_a and x_b and perpendicular to τ, then*

$$\int_a^b \big(g(\nabla X, \nabla X) - g(R(X, \dot\tau)\dot\tau, X)\big)dt \geqslant 0.$$

THEOREM 11. *Let $\tau = x_t$, $a \leqslant t \leqslant b$, be a geodesic. If there is a conjugate point x_c, where $a < c < b$, of x_a, then τ is not a minimizing geodesic joining x_a to x_b.*

Let x and y be arbitrary points of M and let $\tau = x_t$, $a \leqslant t \leqslant b$, be a minimizing geodesic joining x to y. Such a geodesic exists because (M, g) is complete. By Theorem 11, x_a has no conjugate point along τ for $a < t < b$. Let $a < c < b$. The vector field $\dot\tau$ can be decomposed into the sum $\dot\tau = X_1 + \ldots + X_s$, where $X_i \in D_i$. Of course, vector fields $X_1, \ldots, X_s$ are parallel along τ. Let Y_1 be a vector field belonging to D_1 parallel along τ, perpendicular to $\dot\tau$ and such that $g(Y_1, Y_1) = g(X_1, X_1)$. Now define $Y = Y_1 + JX_2 + \ldots + JX_s$. Then Y is parallel along τ and $g(Y, Y) = 1$. If $K(p) \geqslant \delta > 0$ for some number δ and for any f-invariant plane p, then $R(Y, \dot\tau, Y, \dot\tau) \geqslant \frac{1}{s^2}\delta$. Now let $\alpha(t)$ be a non-zero function such that $\alpha(a) = \alpha(c) = 0$. By virtue of Theorem 10, we have

$$\begin{aligned} 0 &\leqslant \int_a^c \big(g(\nabla\alpha Y, \nabla\alpha Y) - R(\alpha Y, \dot\tau, \alpha Y, \dot\tau)\big)dt \\ &= \int_a^c \big(g(\alpha' Y, \alpha' Y) - \alpha^2 R(Y, \dot\tau, Y, \dot\tau)\big)dt \\ &= \int_a^c \left(\alpha'^2 - \frac{1}{s^2}\delta\alpha^2\right)dt. \end{aligned}$$

Taking $\alpha(t) = \sin \pi\left(\frac{t-a}{c-a}\right)$, we obtain $c - a \leqslant \pi/\sqrt{(1/s^2)\delta}$ for every $a < c < b$. Consequently, $b - a \leqslant \pi/\sqrt{(1/s^2)\delta}$. This means that M is bounded. On the other hand, M is complete. Hence it is compact.

Remark. Theorem 7 holds under weaker assumptions than those given above. Namely, we may assume that there are positive numbers $A_1, \ldots, A_s$ and numbers $\delta_1, \ldots, \delta_s$ such that if an f-invariant plane p is contained in D_i, then $\delta_i A_i \leqslant K(p) \leqslant A_i$ and we shall say that (M, g, f) is $(\delta_1, \ldots, \delta_s)$-$f$-

pinched. If $\delta_i > 1/2$, $i = 1, \ldots, s$, in the case where f is of type I or $\delta_1 > 0$ and $\delta_i > 1/2$, $i = 2, \ldots, s$, in the case where f is of type II or III, then we obtain the assertion of Theorem 7 without changing the proof.

Note that, since $A_1, \ldots, A_s > 0$, by replacing the metric tensor g by $\bar{g}$ defined as

$$\bar{g}(X_1 + \ldots + X_s, Y_1 + \ldots + Y_s) = \frac{1}{A_1} g(X_1, Y_1) + \ldots + \frac{1}{A_s} g(X_s, Y_s),$$
$$\text{where} \quad X_i, Y_i \in D_i,$$

we obtain an f-Kaehlerian manifold $(M, \bar{g}, f)$. Moreover, if (M, g, f) is $(\delta_1, \ldots, \delta_s)$-$f$-pinched and f is not of type IV, then $(M, \bar{g}, f)$ is δ-f-pinched, where $\delta = \min(\delta_1, \ldots, \delta_s)$. Hence, if $\delta_1, \ldots, \delta_s > 0$ and f is of type I, then $(M, \bar{g}, f)$ is $0 < \delta$-f-pinched and consequently M is simply connected and compact providing (M, g) is complete.

Assume now that (M, g, f) is a complete, simply connected f-Kaehlerian manifold. Let M_i, $i = 1, \ldots, s$, denote the maximal integral manifold of D_i through a fixed point of M. It is known (as in the case of the de Rham decomposition, [4], vol. I, Chapter IV, § 6) that there is an isometry $F = (p_1, \ldots, p_s)\colon M \to M_1 \times \ldots \times M_s$ having the following property: if a vector field X equals $X_1 + \ldots + X_s$, where $X_i \in D_{ix}$, then $d_x p_i(X)$ is obtained by the composition of parallel displacements of X_i, first along a curve in M with respect to ∇, and then along a curve in M_i with respect to ∇^i. Since $\nabla f = 0$ and, by virtue of Lemma in the proof of Theorem 5, $\nabla^i f_i = 0$, we have $dF \circ f|_{D_i} = f_i \circ dF$ and consequently $dF \circ f = (f_1 \times \ldots \times f_s) \circ dF$. Therefore, a complete, simple connected f-Kaehlerian manifold is a "product" of integral manifolds of the almost product structure associated with f.

References

[1] M. Berger, *Pincement riemannien et pincement holomorphe*, Ann. Scuola Norm. Sup. Pisa 14 (1960), p. 151–159.

[2] D. E. Blair, *Geometry of manifolds with structural group $U(n) \times O(s)$*, J. Differential Geometry 4 (1970), p. 155–169.

[3] J. Bureš, J. Vanžura, *Metric polynomial structure*, Kodai Math. Sem. Rep. 27 (1976), p. 345–352.

[4] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, vols I and II, Interscience Publishers 1963, 1969.

[5] B. Opozda, *A theorem on metric polynomial structures*, Ann. Polon. Math. 41 (1983), p. 139–147.

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