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Determination of differential concomitants of the first class of a pair of contravariant vectors in a two-dimensional space

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Let there be given a pair of fields of contravariant vectors $u^i(\xi^1, \xi^2)$, $v^i(\xi^1, \xi^2)$, $i = 1, 2$, defined in a neighbourhood of a point $\underset{0}{\xi} = (\underset{0}{\xi^1}, \underset{0}{\xi^2})$ and differentiable at the point $\underset{0}{\xi}$. Assume that in $\underset{0}{\xi}$

$$\det(u^i, v^i) \neq 0 \tag{1}$$

and let us denote the derivatives of the functions u^i, v^i by $\partial_j u^i, \partial_j v^i$.

Definition (see [1]). A geometric object σ^λ with the transformation formula

$$\sigma^{\lambda'} = F^{\lambda'}(\sigma^\lambda; A^{i'}_i, A^{i'}_{ij}) \qquad (\lambda' = 1', \ldots, m') \tag{2}$$

is called the *differential concomitant of the first class of the object* (u^i, v^i), if there exist functions φ^λ which satisfy the functional equations

$$\varphi^{\lambda'}(u^{i'}, v^{i'}, \partial_{j'} u^{i'}, \partial_{j'} v^{i'}) = F^{\lambda'}\big(\varphi^\lambda(u^i, v^i, \partial_j u^i, \partial_j v^i);\ A^{i'}_i, A^{i'}_{ij}\big) \qquad (\lambda' = 1', \ldots, m'), \tag{3}$$

where

$$A^{i'}_i = \frac{\partial \xi^{i'}}{\partial \xi^i}, \quad A^{i'}_{ij} = \frac{\partial \xi^{i'}}{\partial \xi^i \partial \xi^j}, \quad \det(A^{i'}_i) \neq 0 \tag{4}$$

and

$$\begin{aligned} u^{i'} &= A^{i'}_i u^i, \\ v^{i'} &= A^{i'}_i v^i, \\ \partial_{j'} u^{i'} &= A^{i'}_{ij} A^j_{j'} u^i + A^{i'}_i A^j_{j'} \partial_j u^i, \\ \partial_{j'} v^{i'} &= A^{i'}_{ij} A^j_{j'} v^i + A^{i'}_i A^j_{j'} \partial_j v^i, \qquad A^j_{j'} = \frac{\partial \xi^j}{\partial \xi^{j'}}. \end{aligned} \tag{5}$$

In this note we shall find the general solution of (3) with given $F^{\lambda'}$ and unknown $\varphi^{\lambda'}$. We shall use the same method as in paper [2] (see also [3], [4]).

On the basis of paper [3] we formulate without proof the following fact: A necessary and sufficient condition for the existence of solution of (3) is that the object σ^λ be of the type $[m, 2, k]$, $k \leqslant 2$ (see [1]).

The set

$$X = \{(u^i, v^i, \partial_j u^i, \partial_j v^i) \colon \det(u^i, v^i) \neq 0\}$$

may be treated as the fibre of an abstract geometric object with the transformation formula (5). The transitive fibres of this object are 10-dimensional surfaces in 12-dimensional space. One of the generators of the fibre X (the definition is given in [2]) is the two-dimensional surface given by the parametric equations

$$(6) \qquad \begin{aligned} u^i &= \delta_1^i , \\ v^i &= \delta_2^i , \\ \partial_j u^i &= \delta_j^i t_1 , \\ \partial_j v^i &= \delta_j^i t_2 , \qquad (t_1, t_2) \in R^2 . \end{aligned}$$

From (5) and (6) we get the following transformation of the fibre X onto itself (see (4)):

$$\begin{aligned} u^{i'} &= A_i^{i'} \delta_1^i , \\ v^{i'} &= A_i^{i'} \delta_2^i , \\ \partial_{j'} u^{i'} &= A_{ij}^{i'} A_{j'}^j \delta_1^i + A_i^{i'} A_{j'}^j \delta_j^i t_1 , \\ \partial_{j'} v^{i'} &= A_{ij}^{i'} A_{j'}^j \delta_2^i + A_i^{i'} A_{j'}^j \delta_j^i t_2 , \qquad (A_{j'}^j) = (A_j^{j'})^{-1} , \end{aligned}$$

the inverse transformation of which is the following:

$$t_i = \frac{1}{\begin{vmatrix} u^{1'} & u^{2'} \\ v^{1'} & v^{2'} \end{vmatrix}} \left(\begin{vmatrix} u^{1'} & u^{2'} \\ w^{1'} & w^{2'} \end{vmatrix} \delta_i^1 + \begin{vmatrix} v^{1'} & v^{2'} \\ w^{1'} & w^{2'} \end{vmatrix} \delta_i^2 \right) ,$$

$$(7) \qquad A_i^{i'} = u^{i'} \delta_i^1 + v^{i'} \delta_i^2 ,$$

$$A_{ij}^{i'} = (u^{j'} \partial_{j'} u^{i'} - u^{i'} t_1) \delta_i^1 \delta_j^1 + (u^{j'} \partial_{j'} v^{i'} + u^{i'} t_2) \delta_i^1 \delta_j^2 + (v^{j'} \partial_{j'} v^{i'} - v^{i'} t_2) \delta_i^2 \delta_j^2 ,$$

where

$$w^{i'} = u^{j'} \partial_{j'} v^{i'} - v^{j'} \partial_{j'} u^{i'} .$$

Putting in (3) the right-hand-side expressions of (6) instead of the variables $u^i, v^i, \partial_j u^i, \partial_j v^i$ and using the notation

$$\Phi^\lambda(t_1, t_2) = \varphi^\lambda(\delta_1^i, \delta_2^i, \delta_j^i t_1, \delta_j^i t_2) \qquad (\lambda = 1, \ldots, m)$$

we obtain

$$(8) \qquad \varphi^{\lambda'}(u^{i'}, v^{i'}, \partial_{j'} u^{i'}, \partial_{j'} v^{i'}) = F^{\lambda'}\big(\Phi^\lambda(t_1, t_2); A_i^{i'}, A_{ij}^{i'}\big) \qquad (\lambda' = 1', \ldots, m') .$$

If we now apply formulae (7) to formulae (8), then we obtain the general solution of the given functional equations (3), where functions Φ^λ, defined on R_2, are arbitrary.

Remark 2 contained in [2] is also true here.

References

[1] J. Aczél und S. Gołąb, *Funktionalgleichungen der Theorie der geometrischen Objekte*, Warszawa 1960.

[2] S. Topa, *Determination of differential concomitants of the first class of a pair of covariant vectors in a two-dimensional space*, Ann. Polon. Math., ce volume, p. 337-341.

[3] — *On a generalization of homogeneous functions*, Publ. Math. Debrecen 13, 1-4 (1966), pp. 289-300.

[4] — *Pewna nowa metoda rozwiązywania równań funkcyjnych uogólnionej funkcji pseudo-jednorodnej i funkcji prawie-jednorodnej wprowadzonych przez V. Alaci'ego*, Rocznik Nauk Dydakt. WSP w Krakowie, Matematyka, 25 (1966), pp. 195-213.

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