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Periodic solutions of $x''+f(x)x'^{m}+g(x) = 0$

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1. Recently Sędziwy [4] has shown that the equation

$$x''+f(x)x'^{2n}+g(x) = \mu p(t)$$

has at least one periodic solution with period ω provided $n \geqslant 1$ is an integer, all coefficients are continuous, $xg(x) > 0$ for $x \neq 0$,

$$\lim_{|x|\to\infty} \int_0^x g(u)du \to \infty, \quad \limsup_{x\to-\infty} g(x)/x = a, \quad 0 < b \leqslant f(x) \leqslant c < \infty$$

for all x, $p(t+\omega) = p(t)$ and $|\mu|$ is sufficiently small. This is achieved by showing that these hypotheses imply that

$$x''+f(x)x'^{2n}+g(x) = 0 \tag{1}$$

has periodic solutions with distinct positive periods.

In this paper we will show that (1) has numerous periodic solutions under different hypotheses than those of Sędziwy. However, we are unable to conclude that periodic solutions have distinct periods and for this reason do not secure the corresponding theorem for the forced equations. In a second theorem we give similar hypotheses to infer that solutions of

$$x''+f(x)x'^{2n-1}+g(x) = 0 \tag{2}$$

are bounded and remark that not only is the periodicity of solutions impossible to conclude but, under these hypotheses, so is the oscillation of solutions.

Equations of the form (1) and (2) have been studied recently by the author [5], [6], [7], DiAntonio [1] and others. These equations, which are physically significant (cf., for example, [2], [7], [8] and references therein), provide a rich source of interesting dynamical systems when considered in the x, x'-phase plane. Equations

$$x''+F(x')+G(x) = E(t)$$

describing the motion of a particle with one degree of freedom are extensively studied and frequently provide examples of periodic phenomena (cf., for example, the numerous papers of Rolf Reissig on this subject of which [3] is an instance).

2. In the sequel we will always assume that the functions $f(x)$, $g(x)$ are continuous for all real x.

THEOREM 1. *If $xg(x) > 0$ for $x \neq 0$, $0 < b \leqslant f(x)$ for some real b, $|g(x)| > \varepsilon > 0$ for x sufficiently large and if $\lim_{x\to 0} g(x)/x = a \neq 0$, equation* (1) *has periodic solutions. In particular, any solution of* (1) *with initial conditions $x(0) = -m^2$, $x'(0) = 0$ is periodic.*

Proof. Let $x = x(t)$ satisfy (1) and set $v = dx/dt$ to secure the equation

$$v\frac{dv}{dx} + f(x)v^{2n} + g(x) = 0. \tag{3}$$

Now, in (3) set $z = v^2$ to secure

$$\frac{dz}{dx} = -2f(x)z^n - 2g(x) \tag{4}$$

and consider the vector field determined in the x, z-plane by equation (4). The points at which $dz/dx = 0$ lie on a curve $z^n = -g(x)/f(x)$. If n is odd, the curve is in the second and fourth quadrants. If n is even the curve is symmetric with respect to the x-axis and lies in the second and third quadrants but in both cases the curve contains the point $(x = 0$, $z = 0)$. Let $z = F(x)$ denote the second quadrant portion of this curve in both cases. In particular, F is a single-valued function defined for $x \leqslant 0$, $F(0) = 0$ and $F(x) > 0$ for $x < 0$.

We now select $m > 0$ and consider the point P_0 $(x_0 = -m^2, z_0 = 0)$ in the x, z-plane. At P_0, $dz/dx > 0$ and since $dz/dx > 0$ for $x < 0$, $z < F(x)$ it is clear that the trajectory determined by P_0 rises to meet $z = F(x)$ at a point P_1 $(x_1 < 0, z_1 > 0)$, where $dz/dx = 0$.

For points in the region A $(x < 0, z > F(x))$ one has $dz/dx < 0$ and the trajectory falls to meet the z-axis at a point P_2 $(x_2 = 0, z_2 > 0)$ unless it turns into the origin (in which case $z_2 = 0$). That this is not possible and that $z_2 > 0$ is seen by examining the vector field in, and on the boundary of, A.

First, we observe that

$$F'(0^-) = \lim_{x\to 0^-} \frac{F(x)}{x} = \lim_{x\to 0^-} \frac{1}{x}\left(\frac{-g(x)}{f(x)}\right)^{1/n} = -\infty$$

since $\lim_{x\to 0} g(x)/x = a \neq 0$ and $0 < b \leqslant f(x)$. On the other hand,

$$\lim_{x\to 0} \frac{dz}{dx} = -2f(0)z^n$$

which is bounded from below in any neighborhood of the origin. Thus, the trajectory does not turn to the origin and so must fall to a cutting of the z-axis at $z_2 > 0$.

Continuing into the first quadrant, $x > 0, z > 0$, one has $dz/dx < 0$ and so the trajectory is falling toward the x-axis. It cannot turn toward the origin because $dz/dx < 0$. Since the trajectory does not turn toward the origin, it either cuts the x-axis or $z \to p \geqslant 0$ and $dz/dx \to 0$ as $x \to \infty$. The latter case is impossible since as $x \to \infty$,

$$\frac{dz}{dx} \leqslant -2bp^n - \varepsilon \leqslant -\varepsilon < 0$$

to contradict $dz/dx \to 0$. Thus, the trajectory cuts the x-axis at some point P_3 $(x_3 > 0, z_3 = 0)$.

For the initial conditions $x_0 = -m^2, z_0 = 0$ one secures a solution $z(x)$ of (4) such that $z(-m^2) = z(x_3) = 0, z(x) > 0$ for $-m^2 < x < x_3$ and $v^2 = z(x)$ is a simple closed curve in the x, v-plane representing the path of a periodic orbit of the system

$$x' = v, \; v' = -f(x)v^{2n} - g(x) \tag{5}$$

to prove the theorem.

From the form of the system of equations (5) one could have seen immediately that if the vector (x', v') corresponds to the point (x, v), then the vector $(-x', v')$ corresponds to the point $(x, -v)$ and so the trajectories are symmetric with respect to the x-axis. One can even see, initially, that the motion is in a clockwise direction about any simple closed trajectory with the singularity $(0, 0)$ in its interior. The substance of the proof was to show that some trajectories are closed by showing that they meet the x-axis twice.

By contrast with Theorem 1, for m odd we have the following theorem, where, as before, $f(x)$, $g(x)$ are assumed continuous for all real x.

THEOREM 2. *If in equation* (2) *we assume that* $xg(x) > 0$, $f(x) \geqslant 0$ *for all real* x *and*

$$\int_0^x g(u)\,du \to \infty \quad \text{as } |x| \to \infty,$$

then any solution of (2) *defined for all large* t *is bounded.*

With these hypotheses, which for $f(x)$, $g(x)$ are comparable and weaker than those of Theorem 1, it is not possible to conclude that solutions even oscillate as is easily seen from linear cases. For example, if $f(x) = p > 0$ and $g(x) = qx, q > 0$, then for $n = 1$ the hypotheses of the theorem are satisfied but no solution oscillates if $p^2 - 4q > 0$. In fact, one can see from the linear case the importance of the even exponent in the Sędziwy theorem since the linear equation, above, satisfies all other hypotheses of his theorem.

If one replaces x'^{2n-1} in (2) by an odd-like function, the proof of Theorem 2 is no more difficult. By doing so one includes numerous other significant equations. For these reasons we will prove the following theorem to which Theorem 2 is a corollary.

THEOREM 3. *If in $x''+f(x)M(x')+g(x) = 0$ we assume that $f(x)$, $g(x)$ are continuous, $xg(x) > 0$ for $x \neq 0, f(x) \geqslant 0$ for all real x,*

$$\int_0^x g(u)\,du \to \infty \qquad as\ |x| \to \infty$$

and $zM(z) \geqslant 0$ for all real z, then any solution of the equation valid for all large t is bounded.

Proof. Write the given equation as the system

$$x' = v, \qquad v' = -f(x)M(v)-g(x)$$

and consider the positive definite scale function

$$E^2 = 2\int_0^x g(u)\,du+v^2.$$

Since

$$EE' = g(x)x'+vv' = -vf(x)M(v) \leqslant 0$$

all solutions are bounded and the theorem is proved.

References

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