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On the tangency of sets in metric spaces

by J. Grochulski, T. Konik and M. Tkacz (Częstochowa)

Abstract. The present paper deals with the connections between tangency relations of sets in metric space (E, ϱ) and (E, ϱ'), the tangency relations under consideration being defined by functions ϱ_i $(i = 0, 1, \ldots, 7)$ introduced in the introduction.

In Section 1 the so-called *condition of rings* for a metric space (E, ϱ) is introduced and certain conditions for the occurrence of tangency of sets defined by the functions ϱ_6 and ϱ_7 in metric spaces (E, ϱ) and (E, ϱ') are established under the assumption that

$$m\varrho(x, y) \leqslant \varrho'(x, y) \leqslant M\varrho(x, y) \quad \text{for } x, y \in E,$$

where $0 < m \leqslant M$. In Section 2 we consider the connection between tangency relations defined by the functions ϱ_i $(i = 0, 1, \ldots, 7)$ in metric space (E, ϱ) and (E, ϱ') if metrics ϱ and ϱ' satisfy the condition $\varrho'(x, y) = f(\varrho(x, y))$, for $x, y \in E$, where f is an increasing real function such that $f(r) \xrightarrow[r\to 0_+]{} 0$.

Introduction. In the present paper we consider connections between tangency relations of sets in metric spaces (E, ϱ) and (E, ϱ'). In paper [6] W. Waliszewski has introduced the following definition of the tangency relation in a space (E, l):

$$T_l(a, b, k, p) = \Big\{(A, B)\colon (A \cup B) \subset E \text{ and } (A, B) \text{ is } (a, b)$$
$$- \text{clustered at } p \in E \text{ and } \frac{1}{r^k}\, l\big(A \cap S_l(p, r)_{a(r)},\ B \cap S_l(p, r)_{b(r)}\big) \xrightarrow[r\to 0_+]{} 0\Big\},$$

where k is a positive real number, l is a real non-negative function defined on the Cartesian product $E_0 \times E_0$ (E_0 is a family of all non-empty subsets of the set E) and a, b are certain non-negative real functions defined in the right-hand side neighbourhood of 0 such that $a(r) \xrightarrow[r\to 0_+]{} 0$, $b(r) \xrightarrow[r\to 0_+]{} 0$.

We say that a pair of sets (A, B) are (a, b)-*clustered* at a point p of the space (E, l) if 0 is a cluster point of the set of all the real numbers $r > 0$ such that the sets $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{b(r)}$ are non-empty. By definition (see [6]), $S_l(p, r)_u$ denotes the neighbourhood of the sphere $S_l(p, r)$ with centre at p and radius r, defined as the union $\bigcup_{q \in S_l(p,r)} K_l(q, u)$, where $K_l(q, u)$ is the open ball with centre at q and radius u in the space (E, l).

If $(A, B) \in T_l(a, b, k, p)$, then we say that the set A is *(a, b)-tangent* of order k to the set B at the point p.

In the present paper we shall consider the functions ϱ_i $(i = 0, 1, \ldots, 7)$ being special cases of the function l (see [6]). These functions are induced by the metric ϱ and are defined as follows:

$$\begin{aligned}
\varrho_0(A, B) &= \sup\{\varrho(x, B);\ x \in A\},\\
\varrho_1(A, B) &= \max\{\varrho_0(A, B),\ \varrho_0(B, A)\},\\
\varrho_2(A, B) &= \min\{\varrho_0(A, B),\ \varrho_0(B, A)\},\\
\varrho_3(A, B) &= \inf\{\operatorname{diam}_\varrho(\{x\}\cup B);\ x \in A\},\\
\varrho_4(A, B) &= \max\{\varrho_3(A, B),\ \varrho_3(B, A)\},\\
\varrho_5(A, B) &= \min\{\varrho_3(A, B),\ \varrho_3(B, A)\},\\
\varrho_6(A, B) &= \inf\{\varrho(x, B);\ x \in A\},\\
\varrho_7(A, B) &= \operatorname{diam}_\varrho(A\cup B)
\end{aligned}$$

for $A, B \in E_0$, where $\varrho(x, B) = \inf\limits_{y\in B} \varrho(x, y)$ and $\operatorname{diam}_\varrho A$ denotes the diameter of the set A in the metric space (E, ϱ).

In Section 1 we investigate connections between tangency relations defined by the functions ϱ_6 and ϱ_7 in the metric spaces (E, ϱ) and (E, ϱ') if the metrics ϱ and ϱ' are connected by a certain inequality. In Section 2 we consider connections between tangency relations defined by the functions ϱ_i $(i = 0, 1, \ldots, 7)$ in the metric spaces (E, ϱ) and (E, ϱ') in the case where one metric is equal to the other composed with a certain real function.

1. Let E be any set and let ϱ be a metric on E. We say that the metric space (E, ϱ) *satisfies the condition of rings at the point* $p \in E$ if there exists a real number $\mu > 0$ such that

(1) $$S_\varrho(p, r)_u = \{x \in E;\ r-u < \varrho(p, x) < r+u\} \quad \text{for } r, u \in (0, \mu).$$

If the space (E, ϱ) satisfies the condition of rings at any point $p \in E$, then we say that the space satisfies the condition of rings. Let us consider two metric spaces (E, ϱ) and (E, ϱ') and assume that the metrics ϱ and ϱ' satisfy the condition

(2) there exist real numbers m, M $(0 < m \leqslant M)$ such that $m\varrho(x, y) \leqslant \varrho'(x, y) \leqslant M\varrho(x, y)$ for $x, y \in E$.

Let us put $\sigma = \min(m, 1/M, m/M)$, $\eta = 1/\sigma$.

Let $F_{m,M}$ be the class of increasing non-negative real functions defined in a certain right-hand side neighbourhood of 0 fulfilling the conditions:

(i) $$a(r) \xrightarrow[r\to 0_+]{} 0,$$

there exists a number $\lambda > 0$ such that

(ii) $$\max\big(\inf\{a(tr)-ta(r);\ (t,r)\in\langle\sigma,\eta\rangle\times(0,\lambda)\},\ \inf\{ta(r)-a(tr);\ (t,r)\in\langle\sigma,\eta\rangle\times(0,\lambda)\}\big)\geqslant 0,$$

(iii) $$\inf\{a(t_2r)-a(t_1r)-(t_2r-t_1r);\ t_2\geqslant t_1 \text{ and } (t_1,r),(t_2,r)\in\langle\sigma,\eta\rangle\times(0,\lambda)\}\geqslant 0.$$

LEMMA 1. *If functions a, b belong to $F_{m,M}$, condition (2) is fulfilled and the space (E,ϱ') satisfies the condition of rings at the point $p\in E$, then for any sets $A\subset E$, $B\subset E$, (a,b)-clustered at the point p, the relation $(A,B)\in T\varrho_6(a,b,k,p)$ implies $(A,B)\in T\varrho_6'(a,b,k,p)$.*

Proof. Let $(A,B)\in T\varrho_6(a,b,k,p)$. Then

(3) $$\frac{1}{r^k}\inf\{\varrho(x,y);\ x\in(A\cap S_\varrho(p,r)_{a(r)}),\ y\in(B\cap S_\varrho(p,r)_{b(r)})\}\xrightarrow[r\to 0_+]{} 0.$$

Let r' be a number which satisfies the inequalities

$$mr\leqslant r'\leqslant Mr.$$

From (3) we have

$$\frac{1}{\left(\frac{r'}{M}\right)^k}\inf\left\{\varrho(x,y);\ x\in\left(A\cap S_\varrho\left(p,\frac{r'}{M}\right)_{a\left(\frac{r'}{M}\right)}\right),\ y\in\left(B\cap S_\varrho\left(p,\frac{r'}{M}\right)_{b\left(\frac{r'}{M}\right)}\right)\right\}\xrightarrow[\frac{r'}{M}\to 0_+]{} 0,$$

i.e.,

(4) $$\frac{1}{(r')^k}\inf\left\{\varrho(x,y);\ x\in\left(A\cap S_\varrho\left(p,\frac{r'}{M}\right)_{a\left(\frac{r'}{M}\right)}\right),\ y\in\left(B\cap S_\varrho\left(p,\frac{r'}{M}\right)_{b\left(\frac{r'}{M}\right)}\right)\right\}\xrightarrow[r'\to 0_+]{} 0.$$

Let us put $\delta=\min(\mu,\lambda,M\lambda)$, where μ is a number such that (1) is satisfied for $r,u\in(0,\mu)$, λ is a number such that (ii) and (iii) are fulfilled. We shall prove that

(5) $$S_\varrho\left(p,\frac{r'}{M}\right)_{a\left(\frac{r'}{M}\right)}\subset\left(S_{\varrho'}(p,r')_{a(r')}\right)\quad\text{for } r'\in(0,\delta).$$

Let $x \in S_\varrho(p, r'/M)_{a(r'/M)}$. It is easy to prove that

$$\frac{r'}{M} - a\left(\frac{r'}{M}\right) < \varrho(p, x) < \frac{r'}{M} + a\left(\frac{r'}{M}\right). \tag{6}$$

Let

$$\inf\{a(tr) - ta(r);\ (t, r) \in \langle\sigma, \eta\rangle \times (0, \lambda)\} \leqslant \inf\{ta(r) - a(tr);\ (t, r) \in \langle\sigma, \eta\rangle \times (0, \lambda)\}. \tag{7}$$

From (iii) it results that

$$a\left(\frac{r'}{m}\right) - a\left(\frac{r'}{M}\right) \geqslant \left(\frac{1}{m} - \frac{1}{M}\right) r' \quad \text{for } r' \in (0, \lambda).$$

Hence

$$\frac{r'}{M} - a\left(\frac{r'}{M}\right) \geqslant \frac{r'}{m} - a\left(\frac{r'}{m}\right). \tag{8}$$

From (6) and (8) we get

$$\frac{r'}{m} - a\left(\frac{r'}{m}\right) < \varrho(p, x) < \frac{r'}{M} + a\left(\frac{r'}{M}\right).$$

Hence and from (7) we have

$$\frac{r'}{m} - \frac{a(r')}{m} < \varrho(p, x) < \frac{r'}{M} + \frac{a(r')}{M} \quad \text{for } r' \in (0, \lambda).$$

Therefore

$$M\varrho(p, x) < r' + a(r') \quad \text{and} \quad m\varrho(p, x) > r' - a(r'). \tag{9}$$

From (2) and (9) we have

$$r' - a(r') < \varrho'(p, x) < r' + a(r') \quad \text{for } r' \in (0, \lambda). \tag{10}$$

Hence

$$x \in S_{\varrho'}(p, r')_{a(r')} \quad \text{for } r' \in (0, \delta).$$

Let us now suppose that

$$\inf\{a(tr) - ta(r);\ (t, r) \in \langle\sigma, \eta\rangle \times (0, \lambda)\} \geqslant \inf\{ta(r) - a(tr);\ (t, r) \in \langle\sigma, \eta\rangle \times (0, \lambda)\}. \tag{11}$$

From (6) and (2) we have

$$\frac{m}{M} r' - ma\left(\frac{r'}{M}\right) < \varrho'(p, x) < r' + Ma\left(\frac{r'}{M}\right). \tag{12}$$

From (11) and (12) we obtain

$$\frac{m}{M} r' - a\left(\frac{mr'}{M}\right) < \varrho'(p, x) < r' + a(r') \quad \text{for } r' \in (0, M\lambda). \tag{13}$$

From (iii) we get

$$\frac{m}{M}r'-a\left(\frac{m}{M}r'\right)\geqslant r'-a(r')\qquad\text{for } r'\in(0,\lambda).$$

Hence, and from (13) it follows that

$$r'-a(r')<\varrho'(p,x)<r'+a(r')\qquad\text{for } r'\in\big(0,\min(\lambda, M\lambda)\big).$$

Consequently,

$$x\in S_{\varrho'}(p,r')_{a(r')}\qquad\text{for } r'\in(0,\delta).$$

Similarly we can prove that

$$(14)\qquad S_{\varrho}\left(p,\frac{r'}{M}\right)_{b\left(\frac{r'}{M}\right)}\subset S_{\varrho'}(p,r')_{b(r')}\qquad\text{for } r'\in(0,\delta).$$

From (5) and (14) we obtain

$$\left(A\cap S_{\varrho}\left(p,\frac{r'}{M}\right)_{a\left(\frac{r'}{M}\right)}\right)\times\left(B\cap S_{\varrho}\left(p,\frac{r'}{M}\right)_{b\left(\frac{r'}{M}\right)}\right)$$
$$\subset\big(A\cap S_{\varrho'}(p,r')_{a(r')}\big)\times\big(B\cap S_{\varrho'}(p,r')_{b(r')}\big).$$

Hence

$$(15)\qquad 0\leqslant\inf\{\varrho(x,y);\,x\in\big(A\cap S_{\varrho'}(p,r')_{a(r')}\big),\ y\in\big(B\cap S_{\varrho'}(p,r')_{b(r')}\big)\}$$
$$\leqslant\inf\left\{\varrho(x,y);\,x\in\left(A\cap S_{\varrho}\left(p,\frac{r'}{M}\right)_{a\left(\frac{r'}{M}\right)}\right),\ y\in\left(B\cap S_{\varrho}\left(p,\frac{r'}{M}\right)_{b\left(\frac{r'}{M}\right)}\right)\right\}.$$

From (2) and (15) we get

$$0\leqslant\frac{1}{M}\inf\{\varrho'(x,y);\,x\in\big(A\cap s_{\varrho'}(p,r')_{a(r')}\big),\ y\in\big(B\cap S_{\varrho'}(p,r')_{b(r')}\big)\}$$
$$\leqslant\inf\left\{\varrho(x,y);\,x\in\left(A\cap S_{\varrho}\left(p,\frac{r'}{M}\right)_{a\left(\frac{r'}{M}\right)}\right),\ y\in\left(B\cap S_{\varrho}\left(p,\frac{r'}{M}\right)_{b\left(\frac{r'}{M}\right)}\right)\right\}.$$

Hence and from (4) it results that

$$\frac{1}{(r')^k}\inf\{\varrho'(x,y);\,x\in\big(A\cap S_{\varrho'}(p,r')_{a(r')}\big),\ y\in\big(B\cap S_{\varrho'}(p,r')_{b(r')}\big)\}\xrightarrow[r'\to 0_+]{}0.$$

Therefore $(A,B)\in T\varrho'_6(a,b,k,p)$. This ends the proof.

LEMMA 2. *If functions a, b belong to $F_{m,M}$, condition (2) is fulfilled and the space (E,ϱ) fulfils the condition of rings at the point $p\in E$, then for any sets $A\subset E$, $B\subset E$, (a,b)-clustered at the point p, the relation $(A,B)\in T\varrho_7(a,b,k,p)$ implies $(A,B)\in T\varrho'_7(a,b,k,p)$.*

Proof. Let $(A, B) \in T\varrho_7(a, b, k, p)$. Then

(16) $$\frac{1}{r^k}\operatorname{diam}_\varrho\big((A\cap S_\varrho(p, r)_{a(r)})\cup(B\cap S_\varrho(p, r)_{b(r)})\big) \xrightarrow[r\to 0_+]{} 0.$$

Let r' be a number satisfying the inequalities

$$mr \leqslant r' \leqslant Mr.$$

From (16) we have

$$\frac{1}{\left(\frac{r'}{m}\right)^k}\operatorname{diam}_\varrho\left(\left(A\cap S_\varrho\left(p, \frac{r'}{m}\right)_{a\left(\frac{r'}{m}\right)}\right)\cup\left(B\cap S_\varrho\left(p, \frac{r'}{m}\right)_{b\left(\frac{r'}{m}\right)}\right)\right)\xrightarrow[\frac{r'}{m}\to 0+]{} 0,$$

i.e.,

(17) $$\frac{1}{(r')^k}\operatorname{diam}_\varrho\left(\left(A\cap S_\varrho\left(p, \frac{r'}{m}\right)_{a\left(\frac{r'}{M}\right)}\right)\cup\left(B\cap S_\varrho\left(p, \frac{r'}{m}\right)_{b\left(\frac{r'}{M}\right)}\right)\right)\xrightarrow[r'\to 0_+]{} 0.$$

Let us put $\delta = \min(\mu, \lambda, m\lambda)$, where μ is a number such that (1) is fulfilled for $r, u \in (0, \mu)$ and λ is a number such that (ii) and (iii) are satisfied. We shall prove that

(18) $$S_{\varrho'}(p, r')_{a(r')} \subset S_\varrho\left(p, \frac{r'}{m}\right)_{a\left(\frac{r'}{m}\right)} \quad \text{for } r' \in (0, \delta).$$

Let $x \in S_{\varrho'}(p, r')_{a(r')}$. It is easy to show that

(19) $$r' - a(r') < \varrho'(p, x) < r' + a(r') \quad \text{for } r' \in (0, \lambda).$$

Let

(20) $$\inf\{a(tr) - ta(r); (t, r) \in \langle\delta, \eta\rangle \times (0, \lambda)\} \geqslant \inf\{ta(r) - a(tr); (t, r) \in \langle\delta, \eta\rangle \times (0, \lambda)\}.$$

From (2) and (19) we obtain

$$\frac{r'}{M} - \frac{1}{M}a(r') < \varrho(p, x) < \frac{r'}{m} + \frac{1}{m}a(r').$$

Hence and from (20) we have

(21) $$\frac{r'}{M} - a\left(\frac{r'}{M}\right) < \varrho(p, x) < \frac{r'}{m} + a\left(\frac{r'}{m}\right) \quad \text{for } r' \in (0, \lambda).$$

From (21) and condition (iii) we get

$$\frac{r'}{m} - a\left(\frac{r'}{m}\right) < \varrho(p, x) < \frac{r'}{m} + a\left(\frac{r'}{m}\right) \quad \text{for } r' \in (0, \lambda).$$

Hence and from the fact that the space (E, ϱ) satisfies the condition of rings at $p \in E$, it follows that $x \in S_\varrho(p, r'/m)_{a(r'/m)}$, which yields inclusion (18).

Let us now suppose that

$$(22)\qquad \inf\{a(tr)-ta(r);\ (t,r)\in\langle\sigma,\eta\rangle\times(0,\lambda)\} \leqslant \inf\{ta(r)-a(tr);\ (t,r)\in\langle\delta,\eta\rangle\times(0,\lambda)\}.$$

From condition (iii) it results that

$$\frac{M}{m}r'-a\left(\frac{M}{m}r'\right)\leqslant r'-a(r')\qquad\text{for } r'\in(0,\lambda).$$

Hence and (19) we have

$$(23)\qquad \frac{M}{m}r'-a\left(\frac{M}{m}r'\right)<\varrho'(p,x)<\frac{m}{m}r'+a\left(\frac{m}{m}r'\right).$$

From (22) and (23) we get

$$M\frac{r'}{m}-Ma\left(\frac{r'}{m}\right)<\varrho'(p,x)<m\frac{r'}{m}+ma\left(\frac{r'}{m}\right)\qquad\text{for } r'\in(0,m\lambda).$$

Hence and from (2) we obtain

$$(24)\qquad \frac{r'}{m}-a\left(\frac{r'}{m}\right)<\varrho(p,x)<\frac{r'}{m}+a\left(\frac{r'}{m}\right)\qquad\text{for } r'\in\big(0,\min(\lambda,m\lambda)\big).$$

From (24) and from the fact that the space (E,ϱ) satisfies the condition of rings we obtain inclusion (18).

Similarly we prove that

$$(25)\qquad S_{\varrho'}(p,r')_{b(r')}\subset S_{\varrho}\left(p,\frac{r'}{m}\right)_{b\left(\frac{r'}{m}\right)}.$$

From (18) and (25) we get

$$(26)\qquad 0\leqslant \operatorname{diam}_{\varrho}\big((A\cap S_{\varrho'}(p,r')_{a(r')})\cup(B\cap S_{\varrho'}(p,r')_{b(r')})\big) \leqslant \operatorname{diam}_{\varrho}\left(\left(A\cap S_{\varrho}\left(p,\frac{r'}{m}\right)_{a\left(\frac{r'}{m}\right)}\right)\cup\left(B\cap S_{\varrho}\left(p,\frac{r'}{m}\right)_{b\left(\frac{r'}{m}\right)}\right)\right).$$

Hence and from (2) it follows that

$$(27)\qquad 0\leqslant\frac{1}{M}\operatorname{diam}_{\varrho'}\big((A\cap S_{\varrho'}(p,r')_{a(r')})\cup(B\cap S_{\varrho'}(p,r')_{b(r')})\big) \leqslant \operatorname{diam}_{\varrho}\left(\left(A\cap S_{\varrho}\left(p,\frac{r'}{m}\right)_{a\left(\frac{r'}{m}\right)}\right)\cup\left(B\cap S_{\varrho}\left(p,\frac{r'}{m}\right)_{b\left(\frac{r'}{m}\right)}\right)\right).$$

From (17) and (27) we obtain

$$\frac{1}{(r')^k}\operatorname{diam}_{\varrho'}\big((A\cap S_{\varrho'}(p,r')_{a(r')})\cup(B\cap S_{\varrho'}(p,r')_{b(r')})\big)\xrightarrow[r'\to 0_+]{}0.$$

Therefore $(A,B)\in T\varrho'_1(a,b,k,p)$. This ends the proof.

From Lemma 1 and Lemma 2 results the following

THEOREM 1. *If functions a, b belong to $F_{m,M}$, condition* (2) *is satisfied and the spaces* (E, ϱ) *and* (E, ϱ') *satisfy the condition of rings at the point $p \in E$, then for any sets $A \subset E, B \subset E$, (a, b)-clustered at the point p, $(A, B) \in T\varrho_i(a, b, k, p)$ if and only if $(A, B) \in T\varrho'_i(a, b, k, p)$ for $i = 6, 7$.*

It follows from the above considerations that if $m = M$ in inequality (2), i.e., if

$$(2') \qquad \varrho'(x, y) = M\varrho(x, y) \quad \text{for } x, y \in E,$$

then we have the following

THEOREM 2. *If functions a, b belong to F^*_M and condition* (2′) *is satisfied, then for any sets $A \subset E, B \subset E$, (a, b)-clustered at the point $p \in E$, $(A, B) \in T\varrho_i(a, b, k, p)$ if and only if $(A, B) \in T\varrho'_i(a, b, k, p)$ for $i = 0, 1, \ldots, 7$; here F^*_M is the class of real non-negative increasing functions which satisfy conditions* (i), (ii).

2. Let a, b be non-negative, real functions defined in a right-hand side neighbourhood of the point 0, such that

$$(28) \qquad a(r) \xrightarrow[r\to 0_+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0_+]{} 0.$$

Let us consider metric spaces (E, ϱ) and (E, ϱ'). Assume that the metrics ϱ and ϱ' satisfy the condition

$$(29) \qquad \varrho'(x, y) = f(\varrho(x, y)) \quad \text{for } x, y \in E,$$

where f is an increasing real function such that

$$(30) \qquad f(r) \xrightarrow[r\to 0_+]{} 0.$$

LEMMA 3. *If condition* (29) *is fulfilled and the function f satisfies condition* (30) *and conditions*

$$(31) \qquad a(f(r)) \leqslant f(a(r)) \quad \text{and} \quad b(f(r)) \leqslant f(b(r)) \quad \text{for } r > 0,$$

$$(32) \qquad f(r_1 \cdot r_2) \leqslant f(r_1) \cdot f(r_2) \quad \text{for } r_1, r_2 > 0,$$

then for any sets $A \subset E, B \subset E$, (a, b)-clustered at the point $p \in E$, the relation $(A, B) \in T\varrho_7(a, b, k, p)$ implies $(A, B) \in T\varrho'_7(a, b, k, p)$.

Proof. Let $(A, B) \in T\varrho_7(a, b, k, p)$. Then

$$(33) \qquad \frac{1}{r^k} \operatorname{diam}_\varrho\left(\left(A \cap S_\varrho(p, r)_{a(r)}\right) \cup \left(B \cap S_\varrho(p, r)_{b(r)}\right)\right) \xrightarrow[r\to 0_+]{} 0.$$

Now we shall prove that

$$(34) \qquad S_{\varrho'}(p, r')_{a(r')} \subset S_\varrho(p, r)_{a(r)},$$

where $r' = f(r)$. (This inclusion follows from condition (29).) Let $x \in$

$S_{\varrho'}(p, r')_{a(r')}$. Hence and from the definition of the set $S_{\varrho'}(p, r')_{a(r')}$ it results that $x \in \bigcup_{q \in S_{\varrho'}(p,r')} K_{\varrho'}(q, a(r'))$.

Therefore there exists $q \in E$ such that

$$\varrho'(q, x) < a(r') \quad \text{and} \quad \varrho'(p, q) = r'. \tag{35}$$

From (29) and (35) it follows that

$$f(\varrho(q, x)) < a(f(r)) \quad \text{and} \quad f(\varrho(p, q)) = f(r).$$

Hence and from condition (31) we obtain

$$f(\varrho(q, x)) < f(a(r)) \quad \text{and} \quad f(\varrho(p, q)) = f(r). \tag{36}$$

From (36) and the definition of function f we have

$$\varrho(q, x) < a(r) \quad \text{and} \quad \varrho(p, q) = r. \tag{37}$$

Hence and from the definition of the set $S_\varrho(p, r)_{a(r)}$ it results that $x \in S_\varrho(p, r)_{a(r)}$. Therefore inclusion (34) is fulfilled.

Similarly one can prove that

$$S_{\varrho'}(p, r')_{b(r')} \subset S_\varrho(p, r)_{b(r)}. \tag{38}$$

From (34) and (38) it results that

$$\begin{aligned}(A \cap S_{\varrho'}(p, r')_{a(r')}) \cup (B \cap S_{\varrho'}(p, r')_{b(r')}) \\ \subset (A \cap S_\varrho(p, r)_{a(r)}) \cup (B \cap S_\varrho(p, r)_{b(r)}).\end{aligned} \tag{39}$$

Hence

$$\begin{aligned}\operatorname{diam}_\varrho\big((A \cap S_{\varrho'}(p, r')_{a(r')}) \cup (B \cap S_{\varrho'}(p, r')_{b(r')})\big) \\ \leqslant \operatorname{diam}_\varrho\big((A \cap S_\varrho(p, r)_{a(r)}) \cup (B \cap S_\varrho(p, r)_{b(r)})\big).\end{aligned} \tag{40}$$

Therefore

$$\begin{aligned}f\Big(\sup\big\{\varrho(x, y);\, x, y \in \big((A \cap S_{\varrho'}(p, r')_{a(r')}) \cup (B \cap S_{\varrho'}(p, r')_{b(r')})\big)\big\}\Big) \\ \leqslant f\Big(\sup\big\{\varrho(x, y);\, x, y \in \big((A \cap S_\varrho(p, r)_{a(r)}) \cup (B \cap S_{\varrho'}(p, r)_{b(r)})\big)\big\}\Big).\end{aligned} \tag{41}$$

Hence

$$\begin{aligned}\frac{1}{(r')^k} \sup\big\{f(\varrho(x, y));\, x, y \in \big((A \cap S_{\varrho'}(p, r')_{a(r')}) \cup (B \cap S_{\varrho'}(p, r')_{b(r')})\big)\big\} \\ \leqslant \frac{1}{(r')^k} f\Big(\sup\big\{\varrho(x, y);\, x, y \in \big((A \cap S_\varrho(p, r)_{a(r)}) \cup (B \cap S_\varrho(p, r)_{b(r)})\big)\big\}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{(r')^k} \sup\big\{\varrho'(x, y);\, x, y \in \big((A \cap S_{\varrho'}(p, r')_{a(r')}) \cup (B \cap S_{\varrho'}(p, r')_{b(r')})\big)\big\} \\ \leqslant \frac{1}{(f(r))^k} f\Big(\sup\big\{\varrho(x, y);\, x, y \in \big((A \cap S_\varrho(p, r)_{a(r)}) \cup (B \cap S_\varrho(p, r)_{b(r)})\big)\big\}\Big)\end{aligned} \tag{42}$$

From conditions (32) and (42) we obtain

$$(43)\qquad \frac{1}{(r')^k}\sup\{\varrho'(x,y);\,x,y\in((A\cap S_{\varrho'}(p,r')_{a(r')})\cup(B\cap S_{\varrho'}(p,r')_{b(r')}))\}$$
$$\leqslant\frac{1}{(f(r))^k}f\Big(\sup\{\varrho(x,y);\,x,y\in((A\cap S_{\varrho}(p,r)_{a(r)})\cup(B\cap S_{\varrho}(p,r)_{b(r)}))\}\Big)$$
$$\leqslant f\Big(\frac{1}{r^k}\sup\{\varrho(x,y);\,x,y\in((A\cap S_{\varrho}(p,r)_{a(r)})\cup(B\cap S_{\varrho}(p,r)_{b(r)}))\}\Big).$$

From (30), (33) and (43) we have

$$\frac{1}{(r')^k}\sup\{\varrho'(x,y);\,x,y\in((A\cap S_{\varrho'}(p,r')_{a(r')})\cup(B\cap S_{\varrho'}(p,r')_{b(r')}))\}\xrightarrow[r'\to 0_+]{}0.$$

Therefore $(A, B)\in T\varrho_7'(a, b, k, p)$. q.e.d.

Similarly one can prove that

LEMMA 4. *If* (29) *is fulfilled and the function f satisfies condition* (30) *and the conditions*

$$(44)\qquad a(f(r))\geqslant f(a(r))\quad and\quad b(f(r))\geqslant f(b(r))\quad for\ r>0,$$

$$(45)\qquad f(r_1\cdot r_2)\geqslant f(r_1)\cdot f(r_2)\quad for\ r_1,r_2>0,$$

then for any sets $A\subset E$, $B\subset E$, (a, b)-clustered at the point $p\in E$, the relation $(A, B)\in T\varrho_7'(a, b, k, p)$ implies $(A, B)\in T\varrho_7(a, b, k, p)$.

From Lemma 3 and Lemma 4 we obtain

THEOREM 3. *If function f satisfies conditions* (29) *and* (30) *and*

$$(46)\qquad a(f(r))=f(a(r))\quad and\quad b(f(r))=f(b(r))\quad for\ r>0.$$

$$(47)\qquad f(r_1\cdot r_2)=f(r_1)\cdot f(r_2)\quad for\ r_1,r_2>0,$$

then for any sets $A\subset E$, $B\subset E$, (a, b)-clustered at the point $p\in E$, $(A, B)\in T\varrho_7(a, b, k, p)$ if and only if $(A, B)\in T\varrho_7'(a, b, k, p)$.

Remark. Similarly one can prove that if the function f satisfies conditions (29), (30), (46) and (47), then for any sets $A\subset E$, $B\subset E$, (a, b)-clustered at the point $p\in E$, $(A, B)\in T\varrho_i(a, b, k, p)$ if and only if $(A, B)\in T\varrho_i'(a, b, k, p)$, for $i = 0, 1, \ldots, 6$.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY
CZĘSTOCHOWA, POLAND

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