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Special solutions of a functional equation

by S. Czerwik (Katowice)

Abstract. In this paper we consider the problem of the existence and uniqueness of solutions of the functional equation (1), where φ is an unknown function belonging to a certain function class G, fulfilling condition (5).

We shall prove that its solution is continuous with respect to the parameter u.

1. In the present paper we consider the problem of the existence and uniqueness of solutions of the functional equation

$$\varphi(x) = h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big), \tag{1}$$

where φ is an unknown function of a real variable belonging to a certain function class G, which is defined below.

It is known that in general the solution of equation (1) depends on an arbitrary function, therefore conditions ensuring the uniqueness of a solution are of particular importance in the theory of functional equations in a single variable (cf. [2], p. 44–45, and [3]).

We assume the hypotheses:

Hypothesis 1.

(i) *Let $(Y, \|\ldots\|)$ be a Banach space. The function $h(x, y_1, \ldots, y_n, u)$, $h\colon I \times Y^n \times R \to Y$, $I = \langle 0, \infty)$, $R = (-\infty, \infty)$ is continuous in $I \times Y^n \times R$.*

(ii) *There exist constants $L_1, \ldots, L_n$ such that for every $x \in I$, $u \in R$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n) \in Y^n$ we have*

$$\|h(x, y_1, \ldots, y_n, u) - h(x, z_1, \ldots, z_n, u)\| \leqslant \sum_{i=1}^{n} L_i \|y_i - z_i\|, \tag{2}$$

$$\sum_{i=1}^{n} L_i = c. \tag{3}$$

(iii) *The real functions $f_i\colon I \to I$, $i = 1, \ldots, n$ are continuous in I.*

(iv) *There exist the locally bounded function $L\colon I \to I$ and constant $0 < q < 1$ such that the inequalities*

$$\sum_{i=1}^{n} L_i \exp\big(L[f_i(x)]\big) \leqslant q \exp\big(L(x)\big), \qquad x \in I \tag{4}$$

hold.

(v) *For every fixed* $u \in R$ *there exist* $K \geqslant 0$ *and* $(r_1, \ldots, r_n) \in Y^n$ *such that*

$$\|h(x, r_1, \ldots, r_n, u)\| \leqslant K \exp\big(L(x)\big), \quad x \in I.$$

HYPOTHESIS 2. *There exist the constant* M *and function* $A\colon I \to I$ *such that for every* $x \in I$, $u_1, u_2 \in R$, $(y_1, \ldots, y_n) \in Y^n$,

$$\|h(x, y_1, \ldots, y_n, u_1) - h(x, y_1, \ldots, y_n, u_2)\| \leqslant A(x)|u_1 - u_2|$$

and

$$\sup_{x \in I}\big[A(x)\exp\big(-L(x)\big)\big] \leqslant M.$$

Remark. Condition (4) is fulfilled if

1° $f_i(x) \leqslant x$, $i = 1, \ldots, n$, L is increasing in I and $c < 1$,

or

2° $L(x) = sQ(x)$, $Q[f_i(x)] - Q(x) \leqslant s^{-1}\ln s$, $x \in I$, $i = 1, \ldots, n$, $0 < s < (nc)^{-1}$, $g = nsc$.

2. Let G be a space of those functions $\varphi\colon I \to Y$ which are continuous in I and fulfil the condition

$$\|\varphi(x)\| = 0\big[\exp\big(L(x)\big)\big], \quad x \in I. \tag{5}$$

We define the norm (cf. also [1])

$$|\varphi| = \sup_{x \in I}\big[\|\varphi(x)\|\exp\big(-L(x)\big)\big]. \tag{6}$$

We shall verify that G with norm (6) is a Banach space. Let $\varepsilon > 0$ be arbitrary, and let $\{\varphi_k\}$ be a sequence of elements belonging to G. Let there exist a positive integer N such that for $k, m \geqslant N$,

$$|\varphi_k - \varphi_m| < \varepsilon. \tag{7}$$

We have for $x \in \langle 0, d\rangle$, $d > 0$,

$$\|\varphi_k(x) - \varphi_m(x)\| \leqslant \sup_{x \in \langle 0, d\rangle}\big[\exp\big(L(x)\big)\|\varphi_k(x) - \varphi_m(x)\|\exp\big(-L(x)\big)\big]$$
$$\leqslant \bar{K}|\varphi_k - \varphi_m| < \bar{K}\varepsilon, \quad k, m \geqslant N,$$

where

$$\bar{K} = \sup_{x \in \langle 0, d\rangle}\big[\exp\big(L(x)\big)\big],$$

and consequently $\{\varphi_k\}$ converges uniformly in $\langle 0, d\rangle$ to a function φ. The arbitrariness of d shows that φ is continuous in I. Also $\varphi(x) \in Y$ for $x \in I$. Letting $k \to \infty$ in (7) we see that $|\varphi - \varphi_m| \leqslant \varepsilon$, and next, from the inequality

$$\big||\varphi| - |\varphi_m|\big| \leqslant |\varphi - \varphi_m| \leqslant \varepsilon$$

we have

$$|\varphi| \leqslant |\varphi_m| + \varepsilon < \infty.$$

Therefore $\varphi \in G$ and G is a Banach space.

Now we shall prove:

THEOREM 1. *Let Hypothesis 1 be fulfilled. Then for every $u \in R$ there exists exactly one solution $\varphi \in G$ of equation* (1), *given as the limit of successive approximations.*

Proof. Let $u \in R$ be fixed. For $\varphi \in G$ we define the transform $\Phi = h(\varphi)$ by

$$\Phi(x) = h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big). \tag{8}$$

Now we shall prove that (8) maps G into itself. Evidently by (i) and (iii) Φ is continuous in I and $\Phi(x) \in Y$ for $x \in I$. We verify that (5) is fulfilled. By (ii) and (v) we obtain

$$\begin{aligned}
&\|\Phi(x)\| \\
&\leqslant \big\|h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big) - h(x, r_1, \ldots, r_n, u)\big\| + \|h(x, r_1, \ldots, r_n, u)\| \\
&\leqslant \sum_{i=1}^{n} L_i \|\varphi[f_i(x)] - r_i\| + \|h(x, r_1, \ldots, r_n, u)\|.
\end{aligned}$$

Next, from (iv) we have

$$\begin{aligned}
\|\Phi(x)\| &\leqslant \max(|\varphi - r_1|, \ldots, |\varphi - r_n|) \sum_{i=1}^{n} L_i \exp\big(L[f_i(x)]\big) + \|h(x, r_1, \ldots, r_n, u)\| \\
&\leqslant q \max(|\varphi - r_1|, \ldots, |\varphi - r_n|) \exp\big(L(x)\big) + \|h(x, r_1, \ldots, r_n, u)\| \\
&\leqslant [q \max(|\varphi - r_1|, \ldots, |\varphi - r_n|) + K] \exp\big(L(x)\big).
\end{aligned}$$

Consequently we see that if $\varphi \in G$, then also $\Phi \in G$. Now we verify that transform (8) is a contraction map. Actually for $\Phi = h(\varphi)$ and $\Psi = h(\psi)$ we have

$$\begin{aligned}
\|\Phi(x) - \Psi(x)\| &\leqslant \big\|h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big) - \\
&\qquad - h\big(x, \psi[f_1(x)], \ldots, \psi[f_n(x)], u\big)\big\| \\
&\leqslant \sum_{i=1}^{n} L_i \|\varphi[f_i(x)] - \psi[f_i(x)]\| \leqslant |\varphi - \psi| \sum_{i=1}^{n} L_i \exp\big(L[f_i(x)]\big) \\
&\leqslant q |\varphi - \psi| \exp\big(L(x)\big).
\end{aligned}$$

Finally we have

$$|\Phi - \Psi| \leqslant q |\varphi - \psi|. \tag{9}$$

From Banach's fixed point principle for contraction maps, for every fixed $u \in R$ there exists a unique fixed point of transform (8), i.e., unique solution $\varphi \in G$ of equation (1), and it is given as the limit of successive approximations. This completes the proof.

We also have:

THEOREM 2. *If Hypotheses 1 and 2 are fulfilled, then the solution $\varphi(x, u)$ of equation* (1) *belonging to G is continuous with respect to the variables (x, u) in $I \times R$.*

Proof. For $\varphi \epsilon G$ we define the transform $T_u(\varphi) = h(\varphi)$ by

$$T_u(\varphi)(x) = h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big).$$

From (9) we have

$$|T_u(\varphi) - T_u(\psi)| \leqslant q\,|\varphi - \psi|. \tag{10}$$

Next, by Hypothesis 2, we obtain relation

$$|T_u(\varphi) - T_{u_0}(\varphi)| \leqslant \sup_{x \epsilon I} \big[A(x)\,|u - u_0| \exp\big(-L(x)\big)\big] \leqslant M\,|u - u_0|. \tag{11}$$

From Theorem 1 there exists the unique function $\varphi(x, u)$ such that

$$T_u[\varphi(x, u)] = \varphi(x, u) \quad \text{and} \quad T_{u_0}[\varphi(x, u_0)] = \varphi(x, u_0) \quad \text{for } x \epsilon I.$$

Then, by (10) we can write

$$\begin{aligned} &|\varphi(x, u) - \varphi(x, u_0)| \\ &\quad \leqslant |T_u[\varphi(x, u)] - T_u[\varphi(x, u_0)]| + |T_u[\varphi(x, u_0) - T_{u_0}[\varphi(x, u_0)]| \\ &\quad \leqslant q\,|\varphi(x, u) - \varphi(x, u_0)| + |T_u[\varphi(x, u_0)] - T_{u_0}[\varphi(x, u_0)]|. \end{aligned}$$

According to (11) we have finally

$$\begin{aligned} |\varphi(x, u) - \varphi(x, u_0)| &\leqslant (1-q)^{-1} |T_u[\varphi(x, u_0)] - T_{u_0}[\varphi(x, u_0)]| \\ &\leqslant M(1-q)^{-1} |u - u_0|, \end{aligned}$$

whence it follows that the function $\varphi(x, u)$ is continuous with respect to the variable u in R uniformly with respect to the variable x for $x \epsilon I$, and consequently $\varphi(x, u)$ is also continuous with respect to the pair of variables (x, u) in $I \times R$, which was to be proved.

References

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