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Some theorems on the estimate and existence of solutions of integro-differential equations of parabolic type

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In papers [7], [8] some theorems were proved concerning the "local" existence of solutions of the first Fourier problem in a bounded domain ([1]) for a system of parabolic equations with a linear main part and with a non-linear operator depending on unknown functions. These theorems involve a system of integro-differential equations as a particular case.

The above results are generalized in the present paper. Here we remove the restrictions, made in [7], [8], concerning the quantity of some constants related to the equations and the domain, and prove the existence theorems for the whole domain. At first we extend some a priori estimates of Friedman's type. These estimates enable us to apply the Leray-Schauder fixed point theorem and to prove the existence mentioned.

1. A priori estimates. Let G be a bounded domain of the Euclidean space of the variables $(x, t) = (x_1, \ldots, x_n, t)$ whose boundary consists of the domains R_0 and R_T of hyperplanes $t = 0$, $t = T = \text{const} > 0$, and of a side surface S situated in the strip $\{(x, t): 0 < t \leqslant T\}$. By $C(Q)$ we shall denote the set of all functions $v(x, t)$ which are continuous in a set $Q \subset E_{n+1}$, and by $C_{p+1,p}(Q)$ $(p = 0, 1)$ the set of all functions $v \in C(Q)$ possessing in Q the continuous derivatives $D_x^l D_t^m v$, where $l + 2m \leqslant p+1$. The symbols $C^N(Q)$, $C^N_{p+1,p}(Q)$ will denote the sets of all vector-functions $u(x, t) = \big(u^1(x, t), \ldots, u^N(x, t)\big)$ such that all u^k $(k = 1, \ldots, N)$ belong to $C(Q)$ and $C_{p+1,p}(Q)$, respectively.

In this section, applying a method similar to Friedman's ([3], p. 200), we deduce an a priori estimate for the norm $|u|^G_{1+\beta}$ ([2]), where $u = (u^1, \ldots, \ldots, u^N)$ is a solution of the problem:

$$(1.1)\qquad L^k u^k \equiv \sum_{i,j=1}^{n} a^k_{ij}(x, t)\, u^k_{x_i x_j} - u^k_t = B^k u, \qquad (x, t) \in \bar{G} \setminus \Sigma,$$

([1]) I.e., the existence of solutions in a part of the given domain contained in the strip $\{(x, t): 0 < t \leqslant \tau\}$, where $\tau > 0$ is a sufficiently small number.

([2]) In sections 1, 2 the notation of sections 1, 2 of [7] will be used.

(1.2) $$u^k(x,t) = \varphi^k(x,t), \quad (x,t)\in\Sigma \quad (k = 1,\ldots,N),$$

Σ being the union $\bar{R}_0\cup S$.

The following assumptions are introduced ($i,j = 1,\ldots,n$; $k = 1,\ldots$ $\ldots, N$):

(1.I) For any $(x,t)\in\bar{G}$ and $\xi\in E_n$ we have

$$a^k_{ij}(x,t) = a^k_{ji}(x,t), \qquad \sum_{i,j=1}^{n} a^k_{ij}(x,t)\,\xi_i\,\xi_j \geqslant A_0|\xi|^2$$

(A_0 is a certain positive constant).

(1.II) The coefficients a^k_{ij} satisfy the uniform Hölder condition with the exponent α $(0<\alpha<1)$ in $\bar{G}$ and the uniform Lipschitz condition on the surface S.

Then for some constants $A_1, A_2 > 0$

$$\sum_{i,j=1}^{n} |a^k_{ij}|^{G}_{\alpha} \leqslant A_1, \qquad \sum_{i,j=1}^{n} |a^k_{ij}|^{S}_{1-0} \leqslant A_2.$$

(1.III) The surface S belongs both to $\bar{\bar{C}}_{2+\alpha}$ and to C_{2-0}.

(1.IV) For any ν, τ $(0\leqslant\nu<\tau\leqslant T)$ B^k is an operator defined on the set $C^N_{1,0}(\overline{G^{\nu,\tau}})$ with values belonging to the set $C(\overline{G^{\nu,\tau}})$, where

$$G^{\nu,\tau} = G\cap\{(x,t):\ \nu<t<\tau\}.$$

Moreover, there are constants $A_3, A_4 > 0$ (independent of ν and τ) such that for any $u\in C^N_{1,0}(\overline{G^{\nu,\tau}})$

(1.3) $$|B^k u|^{G^{\nu,\tau}}_0 \leqslant A_3 + A_4|u|^{G^{\nu,\tau}}_{1,0},$$

where

$$|u|^{Q}_{1,0} = \sum_{k=1}^{N}|u^k|^{Q}_0 + \sum_{i=1}^{n}\sum_{k=1}^{N}|u^k_{x_i}|^{Q}_0.$$

(1.V) The vector-function $\varphi = (\varphi^1,\ldots,\varphi^N)$ defined on Σ possesses an extension $\Phi\in C^N_{1+\beta}(\bar{G})\cap C^N_{2,1}(\bar{G})$ $(0<\beta<1)$.

(1.VI) If a vector-function $\Phi\in C^N_{2,1}(\bar{G})$ is an extension of φ, then

$$B^k\Phi = L^k\Phi^k, \quad (x,0)\in\partial R_0.$$

Theorem 1. *Let assumptions* (1.I)-(1.VI) *be satisfied and let the function* $u(x,t)\in C^N_{2,1}(\bar{G})$ *be a solution of problem* (1.1), (1.2). *Then* $u\in C^N_{1+\beta}(G)$ *and* $|u|^{G}_{1+\beta}\leqslant K$, *$K$ being a constant depending only on* β, A_i $(i = 0,1,\ldots,6)$ *and* G, *where*

$$A_5 > |\varphi|^G_{1+\beta}, \quad A_6 > |\varphi|^G_{2,1}\ (^3).$$

Proof. Let the function $\Phi(x, t)$ be such an extension of φ that

$$|\Phi|^G_{1+\beta} \leqslant A_5, \quad |\Phi|^G_{2,1} \leqslant A_6.$$

Then the function $v(x,t) = u(x,t) - \Phi(x,t)$ is the solution of the problem

$$L^k v^k = B^k u - L^k \Phi^k, \quad (x,t) \in \bar{G} \setminus \Sigma, \tag{1.4}$$

$$v^k(x,t) = 0, \quad (x,t) \in \Sigma \ (k = 1, \ldots, N). \tag{1.5}$$

The functions $B^k u - L^k \Phi^k$ are continuous in $\bar{G}$ and vanish on ∂R^0 (by assumption (1.VI)). Hence, by Lemma 2 of [7] and by the remark to that lemma, we have

$$|v^k|^{G^\tau}_{1+\beta} \leqslant \bar{K}(\beta)\tau^{(1-\beta)/2}|B^k(v+\Phi) - L^k\Phi^k|^{G^\tau}_0, \tag{1.6}$$

where $G^\tau = G^{0,\tau}$ $(0 < \tau \leqslant T)$, $\bar{K}(\beta)$ is a constant depending only on β, A_0, A_1, A_2 and on the domain G. Inequality (1.3) applied to (1.6) yields

$$|v|^{G^\tau}_{1+\beta} \leqslant \bar{K}(\beta)\tau^{(1-\beta)/2}\Big(NA_4|v|^{G^\tau}_{1+\beta} + NA_3 + NA_4|\Phi|^{G^\tau}_{1,0} + \sum_{k=1}^{N}|L^k\Phi^k|^{G^\tau}_0\Big). \tag{1.7}$$

If

$$NA_4\bar{K}(\beta)T^{(1-\beta)/2} < 1,$$

then from (1.7) follows the estimate

$$|u|^G_{1+\beta} \leqslant K,$$

which completes the proof in this case.

In the case where

$$NA_4\bar{K}(\beta)T^{(1-\beta)/2} \geqslant 1,$$

we proceed step by step to estimate the norm $|v|^G_{1+\beta}$. Let us put

$$\tau = \frac{1}{p}[2NA_4\bar{K}(\beta)]^{2/(\beta-1)},$$

where $p \geqslant 1$ is such a number that T is an integer multiple of τ. We then have from (1.7)

$$|v|^{G^\tau}_{1+\beta} \leqslant K_1\ (^4). \tag{1.8}$$

(3) These norms are defined as

$$|\varphi|^G_{1+\beta} = \inf_\Phi |\Phi|^G_{1+\beta}, \quad |\varphi|^G_{2,1} = \inf_\Phi |\Phi|^G_{2,1},$$

where the inf is taken with respect to all Φ (of the indicated classes) which coincide with φ on Σ, whereas

$$|\Phi|^G_{1+\beta} = \sum_{k=1}^{N}|\Phi^k|^G_{1+\beta}, \quad |\Phi|^G_{2,1} = \sum_{k=1}^{N}|\Phi^k|^G_0 + \sum_{k=1}^{N}\sum_{i=1}^{n}|\Phi^k_{x_i}|^G_0 + \sum_{k=1}^{N}\sum_{i,j=1}^{n}|\Phi^k_{x_ix_j}|^G_0 + \sum_{k=1}^{N}|\Phi^k_t|^G_0.$$

(4) K_i, K'_i will denote constants depending on the same parameters as K.

Now we are going to estimate the norm $|v|_{1+\beta}^{G^{3\nu,5\nu}}$, where $\nu = \tau/4$. For this purpose let us consider the function $w(x,t) = \xi(t)v(x,t)$, where

$$\xi(t) = \begin{cases} -4\tau^{-2}(t-\nu)(t-5\nu), & \nu \leqslant t < 3\nu, \\ 1, & 3\nu \leqslant t \leqslant 5\nu. \end{cases}$$

It is easy to see that

$$L^k w^k = \xi(t)[B^k(v+\Phi) - L^k\Phi^k] - \xi'(t)v^k \equiv g^k(x,t), \quad (x,t) \in \overline{G^{\nu,5\nu}} \setminus \Sigma^{\nu,5\nu},$$
$$w^k(x,t) = 0, \quad (x,t) \in \Sigma^{\nu,5\nu} \quad (k = 1, \ldots, N).$$

Therefore, as before, we have

$$|w^k|_{1+\beta}^{G^{\nu,5\nu}} \leqslant \overline{K}(\beta)\tau^{(1-\beta)/2}|g^k|_0^{G^{\nu,5\nu}}$$

Hence, recalling that

$$w^k(x,t) = v^k(x,t) \quad \text{for } (x,t) \in \overline{G^{3\nu,5\nu}}$$

and using relations (1.3), (1.8), we obtain the inqualities

$$|v^k|_{1+\beta}^{G^{3\nu,5\nu}} \leqslant K_2 + \overline{K}(\beta)\tau^{(1-\beta)/2}|B^k(v+\Phi) - L^k\Phi^k|_0^{G^{3\nu,5\nu}},$$

which easily imply the estimate

$$|v|_{1+\beta}^{G^{3\nu,5\nu}} \leqslant K_3.$$

The next step is to estimate the norm $|v|_{1+\beta}^{G^{4\nu,6\nu}}$ We use the previous method with $G^{\nu,5\nu}$ and $\xi(t)$ replaced by $G^{2\nu,6\nu}$ and $\xi(t-\nu)$, respectively.

Proceeding in the above manner, step by step, we obtain the estimates

$$|v|_{1+\beta}^{G^{i\nu,(i+2)\nu}} \leqslant K_i', \quad i = 0, 1, \ldots, i_0,$$

where $i_0 = \dfrac{T}{\nu} - 2$. Hence it follows that

$$|v|_{1+\beta}^{G^{\mu,\mu+\nu}} \leqslant \max K_i', \quad 0 \leqslant \mu \leqslant T-\nu,$$

which implies

$$|v|_{1+\beta}^{G} \leqslant K.$$

Thus

$$|u|_{1+\beta}^{G} \leqslant K + A_5$$

and the proof of Theorem 1 is completed.

Remark 1. Theorem 1 holds true for problem (1.1), (1.2) with $L^k u$ replaced by

$$\sum_{i,j=1}^{n} a_{ij}^k(x,t)u_{x_ix_j}^k + \sum_{i=1}^{n}\sum_{j=1}^{N} b_{ij}^k(x,t)u_{x_i}^j + \sum_{j=1}^{N} c_j^k(x,t)u^j - u_t^k,$$

if we additionally assume that the coefficients $b_{ij}^k(x,t)$, $c_j^k(x,t)$ are continuous in $\overline{G}$.

Remark 2. Let all the assumptions of Theorem 1 be fulfilled and suppose that

(1.VII) the operators B^k map the space $C^N_{1+\alpha}(G)$ into the set $\bigcup\limits_{0<\varepsilon<1} C_\varepsilon(G)$.

Then it follows from Lemmas 1 and 2 of [7] that any solution $u \in C^N_{2,1}(\bar{G})$ of problem (1.1), (1.2) is of class $C^N_{2+\varepsilon}(G)$, $0<\varepsilon<1$, being a constant.

Remark 3. If the assumptions of Theorem 1 are satisfied and the operators B^k map the space $C^N_{1+\alpha}(G)$ into $C_\alpha(G)$, then the assertion of Remark 2 holds with $\varepsilon = \alpha$.

Now corollaries to Theorem 1 concerning special cases of operators B^k, which were considered in paper [7], are deduced. We introduce the following assumptions:

(1.VIII) For any ν, τ $(0 \leqslant \nu < \tau \leqslant T)$ $\Psi^k\big(x, t; z(\cdot, t)\big)$ $((x,t) \in \bar{G}$, $1 \leqslant k \leqslant N)$ is a functional defined for functions $z \in C_{1,0}(\overline{G^{\nu,\tau}})$ and continuous with respect to the parameter (x, t), i.e., for any $z \in C_{1,0}(\overline{G^{\nu,\tau}})$, $\varepsilon > 0$ and $(x_0, t_0) \in \overline{G^{\nu,\tau}}$ there is such a number $\delta > 0$ that if

$$|x-x_0|^2 + |t-t_0| < \delta,$$

then

$$\big|\Psi^k\big(x, t; z(\cdot, t)\big) - \Psi^k\big(x_0, t_0; z(\cdot, t_0)\big)\big| < \varepsilon.$$

Moreover, there are constants $A_7, A_8 > 0$ (independent of ν, τ) such that for any $z \in C_{1,0}(\overline{G^{\nu,\tau}})$

$$\big|\Psi^k\big(x, t; z(\cdot, t)\big)\big|_0^{G^{\nu,\tau}} \leqslant A_7 + A_8 |z|_{1,0}^{G^{\nu,\tau}}$$

(1.IX) The functions $f^k(x, t, p, q, r)$ are defined and continuous in the set $\bar{G} \times E_{N+nN+N}$ and there exist constants $A_9, A_{10} > 0$ such that

$$|f^k(x, t, p, q, r)| \leqslant A_9 + A_{10}\Big(\sum_{i=1}^{N} |p_i| + \sum_{i=1}^{N}\sum_{j=1}^{n} |q_{ij}| + \sum_{i=1}^{N} |r_i|\Big).$$

(1.X) Assumption (1.VI) for operators B^k of the form:

$$B^k u = f^k\Big(x, t, u^1, \ldots, u^N, u^1_{x_1}, \ldots, u^1_{x_n}, \ldots, u^N_{x_1}, \ldots \\ \ldots, u^N_{x_n}, \Psi^1\big(x, t; u^1(\cdot, t)\big), \ldots, \Psi^N\big(x, t; u^N(\cdot, t)\big)\Big). \tag{1.9}$$

Theorem 2. *Let assumptions* (1.I)-(1.III), (1.V), (1.VIII)-(1.X) *be satisfied. Suppose that* $u \in C^N_{2,1}(\bar{G})$ *is a solution of problem* (1.1), (1.2) *in the case* (1.9). *Then* $u \in C^N_{1+\beta}(G)$ *and the norm* $|u|^G_{1+\beta}$ *is bounded by a constant depending only on* β, A_i $(i = 0, 1, 2, 5, \ldots, 10)$ *and on the domain* G.

In order to prove this theorem it suffices to observe that assumptions (1.VIII), (1.IX) imply (1.IV) and then to apply Theorem 1.

For operators B^k given by the formulas

$$(1.10)\qquad B^k u = f^k\Big(x, t, u^1, \ldots, u^N, u^1_{x_1}, \ldots, u^1_{x_n}, \ldots, u^N_{x_1}, \ldots, u^N_{x_n}, \int\limits_{G_t} u^1(y,t)\mu^1(x,t;dy), \ldots, \int\limits_{G_t} u^N(y,t)\mu^N(x,t;dy)\Big)$$

the following assumptions are introduced concerning the measure

$$\mu(x,t;D) = \big(\mu^1(x,t;D), \ldots, \mu^N(x,t;D)\big)$$

(cf. section 1 of [7]):

(1.XI) There is a constant $A_{11} > 0$ such that for any $(x,t) \in \bar{G}$

$$\mu^k(x,t;D_0) \leqslant A_{11}.$$

(1.XII) There exists a finite non-negative measure $\bar{\mu}$ (defined on $\mathfrak{M}$) with the following property: for any $\varepsilon > 0$ and $P_0(x_0, t_0) \in \bar{G}$ there is a number $\delta > 0$ such that if $P(x,t) \in \bar{G}$ and $d(P, P_0) < \delta$, then for any $D \in \mathfrak{M}$

$$|\mu^k(x,t;D) - \mu^k(x_0,t_0;D)| \leqslant \varepsilon\bar{\mu}(D).$$

(1.XIII) There is a constant $A_{12} > 0$ such that for any $D \in \mathfrak{M}$

$$\mu^k(x,t;D) \leqslant A_{12} m(D),$$

$m(D)$ being the Lebesgue measure of D ([5]).

Moreover, we shall use the following condition:

(1.XIV) Assumption (1.VI) for the case (1.10).

Remark 4. A simple example of measures satisfying conditions (1.XI) and (1.XII) are the measures given by the formulas

$$\mu^k(x,t;D) = \int\limits_D \varrho^k(x,t,y)\bar{\mu}(dy),$$

$\varrho^k(x,t,y)$ being functions non-negative and continuous in the domain $\bar{G} \times D_0$. Then

$$\int\limits_{G_t} z(y,t)\mu^k(x,t;dy) = \int\limits_{G_t} z(y,t)\varrho^k(x,t,y)\bar{\mu}(dy).$$

Remark 5. According to the Radon-Nikodym theorem (see, for example, [6], p. 299) assumption (1.XIII) implies the existence of functions $\varrho^k(x,t,y) \geqslant 0$ such that

$$\mu^k(x,t;D) = \int\limits_D \varrho^k(x,t,y)\,dy.$$

([5]) If G is a cylindrical domain, then condition (1.XIII) is superfluous in all the theorems of this paper concerning integro-differential equations.

THEOREM 3. *Let assumptions* (1.I)-(1.III), (1.V), (1.IX), (1.XI)-(1.XIV) *hold true and let* $u \in C^N_{2,1}(\bar{G})$ *be a solution of problem* (1.1), (1.2) *in case* (1.10). *Then the assertion of Theorem* 1 *remains true with* K *depending only on* β, A_i $(i = 0, 1, 2, 5, 6, 9, \ldots, 12)$ *and* G.

For the proof we need the following

LEMMA 1. *We assume that the measures* μ^k *satisfy conditions* (1.XI) *and* (1.XII), *or, if* G *is not a cylindrical domain, we assume that* μ^k *satisfy conditions* (1.XI)-(1.XIII) *and that* $S \in C_{2+\alpha}$. *Under these assumptions, if the function* $z(x, t)$ *is continuous in* $\bar{G}$, *then the functions*

$$u^k(x, t) = \int_{G_t} z(y, t)\mu^k(x, t; dy)$$

are continuous in $\bar{G}$ *as well.*

The method of proving this lemma is the same as for Lemma 4 of [7].

Now, by Lemma 1, Theorem 3 follows immediately from Theorem 2.

2. General existence theorems. In this section we shall prove some existence theorems for problem (1.1), (1.2) which constitute generalization of theorems obtained in section 2 of paper [7]. Besides the assumptions of section 1, we shall use the following ones:

(2.I) The vector-function $\varphi = (\varphi^1, \ldots, \varphi^N)$ is of class $C^N_{1+\beta}(G) \cap C^N_{2+\alpha}(G)$, where $0 < \alpha < \beta < 1$.

(2.II) The operators B^k $(k = 1, \ldots, N)$ are continuous in the space $C^N_{1+\alpha}(G)$ in the following sense: if $u, u_m \in C^N_{1+\alpha}(G)$ and

$$\lim_{m\to\infty} |u_m - u|^G_{1+\alpha} = 0,$$

then

$$\lim_{m\to\infty} |B^k u_m - B^k u|^G_0 = 0.$$

THEOREM 4. *If assumptions* (1.I)-(1.IV), (1.VI), (1.VII), (2.I) *and* (2.II) *are satisfied, then there exists a solution* $u(x, t) = \big(u^1(x, t), \ldots$ $\ldots, u^N(x, t)\big)$ *of problem* (1.1), (1.2); *furthermore,* $u \in C^N_{1+\beta}(G) \cap C^N_{2+\varepsilon}(G)$ *for some* ε, $0 < \varepsilon < 1$.

Proof. We apply the method of Leray-Schauder. Let us denote by Ω the set of all functions $u \in C^N_{1+\alpha}(G)$ such that $u(x, t) = \varphi(x, t)$ on Σ. For $u \in \Omega$ and $\lambda \in [0, 1]$ let us consider the problem:

$$L^k v^k = \lambda(B^k u - L^k \Phi^k) + L^k \Phi^k, \qquad (x, t) \in \bar{G} \setminus \Sigma, \tag{2.1}$$

$$v^k(x, t) = \varphi^k(x, t), \qquad (x, t) \in \Sigma \ (k = 1, \ldots, N), \tag{2.2}$$

where $\Phi \in C^N_{1+\beta}(G) \cap C^N_{2+\alpha}(G)$ is an extension of φ. It follows from Lemmas 1 and 2 of [7] that this problem possesses a unique solution $v(x, t) = \big(v^1(x, t), \ldots$ $\ldots, v^N(x, t)\big)$ and, moreover, $v \in C^N_{1+\beta}(G) \cap C^N_{2+\varepsilon}(G)$ for some $0 < \varepsilon < 1$.

Now we define on the set $\Omega \times [0, 1]$ a transformation Z setting $Z(u, \lambda) = v$. We have to prove that Z fulfils all the assumptions of the Leray-Schauder theorem (see, for example, [3], p. 189).

The transformation $Z(u, \lambda)$ is continuous with respect to u, i.e. the condition

$$\lim_{m\to\infty} |u_m - u|^G_{1+\alpha} = 0$$

implies

$$\lim_{m\to\infty} |Z(u_m, \lambda) - Z(u, \lambda)|^G_{1+\alpha} = 0.$$

Indeed, by the definition of Z, we have $Z(u_m, \lambda) = v_m$, where

$$L^k v^k_m = \lambda(B^k u_m - L^k \Phi^k) + L^k \Phi^k, \quad (x, t) \in \bar{G} \setminus \Sigma,$$

$$v^k_m(x, t) = \varphi^k(x, t), \quad (x, t) \in \Sigma \ (k = 1, \ldots, N).$$

Hence and from (2.1), (2.2) it follows that

$$L^k(v^k_m - v^k) = \lambda(B^k u_m - B^k u), \quad (x, t) \in \bar{G} \setminus \Sigma, \tag{2.3}$$

$$v^k_m(x, t) - v^k(x, t) = 0, \quad (x, t) \in \Sigma \ (k = 1, \ldots, N). \tag{2.4}$$

Applying to (2.3), (2.4) Lemma 2 of [7] and using assumption (2.II), we obtain the relations

$$\lim_{m\to\infty} |v^k_m - v^k|^G_{1+\alpha} = 0 \quad \text{i.e.} \quad \lim_{m\to\infty} |Z(u_m, \lambda) - Z(u, \lambda)|^G_{1+\alpha} = 0.$$

Now let $v_1 = Z(u, \lambda_1)$ and $v_2 = Z(u, \lambda_2)$. Then

$$L^k(v^k_1 - v^k_2) = (\lambda_1 - \lambda_2)(B^k u - L^k \Phi^k), \quad (x, t) \in \bar{G} \setminus \Sigma,$$

$$v^k_1(x, t) - v^k_2(x, t) = 0, \quad (x, t) \in \Sigma \ (k = 1, \ldots, N).$$

Hence, by Lemma 2 of [7] and by condition (1.3) we get

$$|v^k_1 - v^k_2|^G_{1+\alpha} \leqslant \bar{K}(a) |\lambda_1 - \lambda_2| (A_3 + A_4 |u|^G_{1,0} + |L^k \Phi^k|^G_0).$$

With the aid of these inequalities it easily follows that for u in the bounded set of Ω the transformation $Z(u, \lambda)$ is uniformly continuous in λ, i.e., that for any bounded subset $\Omega_0 \subset \Omega$ and $\eta > 0$ there exists such a number $\delta > 0$, that if

$$u \in \Omega_0 \quad \text{and} \quad |\lambda_1 - \lambda_2| < \delta,$$

then

$$|Z(u, \lambda_1) - Z(u, \lambda_2)|^G_{1+\alpha} < \eta.$$

Note further that for any fixed $\lambda \in [0, 1]$ the transformation $Z(u, \lambda)$ is compact. This fact is a consequence of Lemma 2 of [7] and of condition (1.3) applied to (2.1), (2.2), since the transformation $Z(u, \lambda)$ maps (for fixed λ) every bounded subset $\Omega_0 \subset \Omega$ into a set Ω_0' which is bounded in the space $C^N_{1+\beta}(G)$, whence, by virtue of Theorem 1 of [3], p. 188, the closure $\overline{\Omega'_0}$ is a compact set of the space $C^N_{1+\alpha}(G)$.

The uniform boundedness (in the space $C^N_{1+\alpha}(G)$) of all possible solutions of the equation $Z(u, \lambda) = u$ ($u \in \Omega$, $\lambda \in [0, 1]$) is guaranteed by Theorem 1.

Finally, it follows from Lemma 1 of [7] that the equation $Z(u, 0) = u$ has a unique solution in Ω. Therefore, according to the Leray-Schauder theorem, there exists (in the set Ω) a solution u of the equation $Z(u, 1) = u$. Observe that u satisfies (1.1), (1.2) and $u \in C^N_{1+\beta}(G) \cap C^N_{2+\varepsilon}(G)$ for some $0 < \varepsilon < 1$, which was to be proved.

Now, under stronger assumptions than those of Theorem 4, we shall prove the existence and uniqueness of solutions of problem (1.1), (1.2). We retain assumptions (1.I)-(1.IV), (1.VI), (1.VII) and (2.I), whereas (2.II) is replaced by the following one:

(2.III) For every bounded set Ω in $C^N_{1+\alpha}(G)$ there exists a constant $A_{13} > 0$ such that for any ν, τ $(0 \leqslant \nu < \tau \leqslant T)$

$$|B^k u - B^k v|_0^{G^{\nu,\tau}} \leqslant A_{13} |u - v|_{1+\alpha}^{G^{\nu,\tau}},$$

if $u, v \in \Omega$.

THEOREM 5. *If assumptions* (1.I)-(1.IV), (1.VI), (1.VII), (2.I) *and* (2.III) *are fulfilled, then problem* (1.1), (1.2) *has a unique solution* $u = \{u^k\}$ *in the space* $C^N_{2,1}(\overline{G})$. *Moreover,* $u \in C^N_{1+\beta}(G) \cap C^N_{2+\varepsilon}(G)$ *for some* ε $(0 < \varepsilon < 1)$.

Proof. The existence of a solution is an immediate consequence of Theorem 4. We shall show the uniqueness step by step in a standard manner.

Let $u, v \in C^N_{2,1}(\overline{G})$ be the two solutions of problem (1.1), (1.2). Then for any τ $(0 < \tau \leqslant T)$

$$L^k(u^k - v^k) = B^k u - B^k v, \quad (x, t) \in \overline{G^\tau} \setminus \Sigma^\tau, \tag{2.5}$$

$$u^k - v^k = 0 \text{ on } \Sigma^\tau \quad (k = 1, \ldots, N). \tag{2.6}$$

Since, by Theorem 1, we have $u, v \in C^N_{1+\alpha}(G^\tau)$, applying the remark of [7] (p. 258) to (2.5), (2.6) and using assumption (2.III) we get

$$|u - v|_{1+\alpha}^{G^\tau} \leqslant N A_{13} \overline{K}(\alpha) \tau^{(1-\alpha)/2} |u - v|_{1+\alpha}^{G^\tau}. \tag{2.7}$$

If

$$N A_{13} \overline{K}(\alpha) T^{(1-\alpha)/2} < 1,$$

then inequality (2.7) implies the identity $u \equiv v$ in $\overline{G}$.

In the case where

$$N A_{13} \overline{K}(\alpha) T^{(1-\alpha)/2} \geqslant 1$$

set

$$\tau = [2 N A_{13} \overline{K}(\alpha)]^{2/(\alpha-1)}.$$

Consequently, from (2.7) it follows that $u \equiv v$ in $\overline{G^\tau}$.

Next we consider the domain $G^{\tau,2\tau}$. Since

$$L^k(u^k - v^k) = B^k u - B^k v, \qquad (x, t) \in \overline{G^{\tau,2\tau}} \setminus \Sigma^{\tau,2\tau}$$

$$u^k - v^k = 0 \text{ on } \Sigma^{\tau,2\tau} \qquad (k = 1, \ldots, N),$$

we have, as before,

$$|u - v|_{1+a}^{G^{\tau,2\tau}} \leqslant N A_{13} \overline{K}(a) \tau^{(1-a)/2} |u - v|_{1+a}^{G^{\tau,2\tau}}$$

Hence, by definition of τ, $u \equiv v$ in $\overline{G^{\tau,2\tau}}$.

The finite number of iterations of the above argument suffices to prove the uniqueness of solutions of the problem in question.

Further, let us consider the special cases of operators B^k given by formulas (1.9) and (1.10), for which Theorem 4 as well as Theorem 5 hold true. The following assumptions will be needed:

(2.IV) The functions $f^k(x, t, p, q, r)$ $(k = 1, \ldots, N)$ satisfy a uniform Hölder condition in every bounded set $\overline{G} \times H$ $(H \subset E_{N+nN+N})$.

(2.V) For any $z \in C_{1+a}(G)$ the functions $g^k(x, t) = \Psi^k(x, t; z(\cdot, t))$ satisfy a uniform Hölder condition in G.

(2.VI) There exists a finite measure $\overline{\mu}$ (defined on $\mathfrak{M}$) such that for any $D \in \mathfrak{M}$ and any points $P(x, t)$, $P'(x', t')$ of the domain $\overline{G}$ we have

$$|\mu^k(x, t; D) - \mu^k(x', t'; D)| \leqslant A_{14} \overline{\mu}(D) [d(P, P')]^\gamma,$$

$A_{14} > 0$, $0 < \gamma < 1$ being some constants.

THEOREM 6. *If assumptions* (1.I)-(1.III), (1.VIII)-(1.X), (2.I), (2.IV) *and* (2.V) *are fulfilled, then the assertion of Theorem* 4 *holds true in case* (1.9).

Since assumptions (1.VIII), (1.IX), (2.IV), (2.V) imply (1.IV), (1.VII), (2.II), this theorem follows from Theorem 4.

THEOREM 7. *Under assumptions* (1.I)-(1.III), (1.IX), (1.XI), (1.XIII), (1.XIV), (2.I), (2.IV), (2.VI) *the conclusion of Theorem* 4 *is true in case* (1.10).

The proof consists in applying Lemma 4 of [7] as well as Theorem 6.

Note that Theorem 7 includes Chabrowski's result [2].

To end this section, we formulate Theorem 5 for cases (1.9) and (1.10). For this purpose, instead of assumptions (2.IV), (2.V), we make the following ones:

(2.VII) For every bounded set $H \subset E_{N+nN+N}$ there exists such a constant $A_{15} > 0$ that for any $(x, t) \in \overline{G}$, (p, q, r), $(\overline{p}, \overline{q}, \overline{r}) \in H$

$$|f^k(x, t, p, q, r) - f^k(x, t, \overline{p}, \overline{q}, \overline{r})|$$
$$\leqslant A_{15} \Big(\sum_{i=1}^{N} |p_i - \overline{p}_i| + \sum_{i=1}^{N} \sum_{j=1}^{n} |q_{ij} - \overline{q}_{ij}| + \sum_{i=1}^{N} |r_i - \overline{r}_i| \Big) \qquad (k = 1, \ldots, N).$$

(2.VIII) For every bounded set $\Omega \subset C_{1+\alpha}(G)$ there is a constant $A_{16} > 0$ such that for any ν, τ $(0 \leqslant \nu < \tau \leqslant T)$

$$\left|\Psi^k(x, t; z(\cdot, t)) - \Psi^k(x, t; \bar{z}(\cdot, t))\right|_0^{G^{\nu,\tau}} \leqslant A_{16}|z-\bar{z}|_{1+\alpha}^{G^{\nu,\tau}} \quad (k = 1, \ldots, N),$$

if $z, \bar{z} \in \Omega$.

THEOREM 8. *Let assumptions* (1.I)-(1.III), (1.VIII)-(1.X), (2.I), (2.VII) *and* (2.VIII) *be satisfied. Then Theorem* 5 *holds true in case* (1.9).

THEOREM 9. *If assumptions* (1.I)-(1.III), (1.IX), (1.XI), (1.XIII), (1.XIV), (2.I), (2.VI) *and* (2.VII) *are satisfied, then the assertion of Theorem* 5 *remains valid in case* (1.10).

This theorem constitutes a generalization of Kusano's [4] and Bodanko's [1] results.

3. Existence theorems for the maximum solution and the minimum solution. In this section we generalize the "local" existence theorems of paper [8] obtaining the existence theorems for the whole domain.

THEOREM 9. *Let assumptions* (1.I)-(1.IV), (1.VII), (2.II) *be satisfied and suppose that*

(3.I) *the vector-function* φ *is of class* $C^N_{1+\beta}(G) \cap C^N_{2,1}(\bar{G})$ $(0 < \alpha < \beta < 1)$.

Then problem (1.1), (1.2) *possesses a solution* $u \in C^N_{1+\beta}(G) \cap W^N_{2+\varepsilon}(G)$ ([6]), *where* $0 < \varepsilon < 1$ *is some constant.*

Proof. Proceeding in the same manner as in the proof of Theorem 1 and using, instead of Lemma 2 of [7], Lemma 2 of [8], one can derive an a priori estimate of the norm $|u|^G_{1+\beta}$ for a solution u of problem (1.1), (1.2). The further argumentation is similar to that which was used in the proof of Theorem 4; namely, we apply the method of Leray-Schauder, making use of the above estimate and of Lemmas 1 and 2 of [8].

In order to formulate a theorem on the existence of the maximum and minimum solutions we make the following assumption:

(3.II) If the functions $u = (u^1, \ldots, u^N)$ and $v = (v^1, \ldots, v^N)$ of class $C^N_{2,1}(\bar{G})$ fulfil the inequalities

$$L^k u^k - B^k u > L^k v^k - B^k v, \quad (x, t) \in \bar{G} \setminus \Sigma \ (k = 1, \ldots, N),$$

$$u(x, t) < v(x, t), \quad (x, t) \in \Sigma,$$

then $u(x, t) < v(x, t)$ in $\bar{G}$.

THEOREM 10. *If the assumptions of Theorem* 9 *and* (3.II) *are satisfied, then there exist a maximum solution* $v = \{v^k\}$ *and a minimum solution* $u = \{u^k\}$ *of problem* (1.1), (1.2); *moreover,* $v, u \in C^N_{1+\beta}(G) \cap W^N_{2+\varepsilon}(G)$ *for some* $0 < \varepsilon < 1$.

([6]) The notation of the previous sections and of paper [8] will be used.

The above theorem can be proved by the same considerations as those for Theorem 2 of [8], by making use of Theorem 9.

For the special cases of operators B^k given by the formulas

$$(3.1)\qquad B^k u = f^k\big(x, t, u, u_x^k, \Psi(x, t; u(\cdot, t))\big)$$

$$(k = 1, \ldots, N),$$

$$(3.2)\qquad B^k u = f^k\Big(x, t, u, u_x^k, \int_{G_t} u(y, t)\mu(x, t; dy)\Big)$$

where $u_x^k = (u_{x_1}^k, \ldots, u_{x_n}^k)$, we introduce the following assumptions:

(3.III) The functions $f^k(x, t, p, q, r)$ $(k = 1, \ldots, N)$ defined on $\bar{G}\times E_{N+n+N}$, satisfy a uniform Hölder condition in every bounded set $\bar{G}\times H$ $(H \subset E_{N+n+N})$ and are non-increasing with respect to the variables $p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_N, r_1, \ldots, r_N$. Moreover, there are constants $A_{16}, A_{17} > 0$ such that for any $(x, t, p, q, r) \in \bar{G}\times E_{N+n+N}$

$$|f^k(x, t, p, q, r)| \leqslant A_{16} + A_{17}\Big(\sum_{i=1}^{N}|p_i| + \sum_{j=1}^{n}|q_j| + \sum_{i=1}^{N}|r_i|\Big).$$

(3.IV) The functionals $\Psi^k\big(x, t; z(\cdot, t)\big)$ $(k = 1, \ldots, N)$ are non-decreasing with respect to the functions $z(x, t) \in C_{2,1}(\bar{G})$.

THEOREM 11. *Let assumptions* (1.I)-(1.III), (1.VIII), (2.V), (3.I), (3.III) *and* (3.IV) *be satisfied. Then the assertion of Theorem* 10 *holds true in case* (3.1).

THEOREM 12. *If assumptions* (1.I)-(1.III), (1.XI), (1.XIII), (2.VI), (3.I) *and* (3.III) *are fulfilled, then the conclusion of Theorem* 10 *remains valid in case* (3.2).

The method of proving Theorems 11 and 12 is the same as the method used to prove Theorems 4 and 8 in [8].

As in paper [8], one can obtain theorems on weak inequalities for the whole domain G which are the counterparts of Theorems 3, 5 and 10 of [8].

References

[1] W. Bodanko, *Sur le premier problème de Fourier relatif à l'équation intégro-différentielle du processus stochastique markovien mixte*, Ann. Polon. Math. 21 (1968), p. 21-27.

[2] J. Chabrowski, *Le premier problème de Fourier relatif à un système parabolique d'équations non linéaires*, ibidem 22 (1969), p. 19-15.

[3] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs 1964.

[4] T. Kusano, *On the first boundary problem for quasi-linear systems of parabolic differential equations in non-cylindrical domains*, Funkcjalaj Ekvacioj (Serio Internacia), volumo 7 (1965), p. 103-118.

[5] О. А. Ладыженская, В. А. Солонников и Н. Н. Иральцева, *Линейные и квазилинейные уравнения параболического типа*, Москва 1967.

[6] R. Sikorski, *Funkcje rzeczywiste*, t. I, Warszawa 1958.

[7] H. Ugowski, *On integro-differential equations of parabolic and elliptic type*, Ann. Polon. Math. 22 (1970), p. 255-275.

[8] — *On integro-differential equations of parabolic type*, ibidem 25 (1971), p. 9-22.

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