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Exceptional values of a meromorphic function and its derivatives

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Abstract. Let f be a transcendental meromorphic function and k a positive integer. If there exist complex numbers $a, b, b \neq 0$ such that a is an evB (*exceptional value in the sense of Borel*) for f for distinct zeros of order $\leqslant p$ and b is an evB for $f^{(k)}$ for distinct zeros of order $\leqslant q$ and if ∞ is an evB for f for distinct zeros of order $\leqslant l$, where p, q, l are positive integers, then it is shown that

$$\frac{q+1+k}{(q+1)(l+1)}+\frac{k+1}{p+1}+\frac{1}{q+1} > 1.$$

Several consequences are deduced which extend and improve earlier results of Hiong and Singh and Gopalakrishna. It is shown, for instance, that if a $(\neq \infty)$ and ∞ are evB for f for distinct zeros then $f^{(k)}$ has no finite evB for simple zeros except possibly 0.

We denote by C the set of all finite complex numbers and by $\bar{C}$ the extended complex plane consisting of all (finite) complex numbers and ∞. By a meromorphic function we shall always mean a transcendental meromorphic function in the plane. We use the usual notations of the Nevanlinna theory of meromorphic functions as explained in [1] and [4].

If f is a meromorphic function we denote by $S(r,f)$ any quantity satisfying

$$\int_{r_0}^{r} \frac{S(x,f)}{x^{1+\lambda}}\,dx = O\left(\int_{r_0}^{r} \frac{\log T(x,f)}{x^{1+\lambda}}\,dx\right) \tag{1}$$

as $r \to \infty$, whenever $\lambda > 0$ and

$$S(r,f) = o\big(T(r,f)\big) \tag{2}$$

as $r \to \infty$, through all values if f is of finite order and outside a set of finite linear measure if f is of infinite order.

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If f is a meromorphic function, then we have the following fundamental results of Nevanlinna [3], p. 63,

$$m(r, f'/f) = S(r, f)$$

and

$$(q-2)T(r,f) \leqslant \sum_{i=1}^{q} N(r, a_i, f) - N_1(r) + S(r,f)$$

whenever $a_1, \ldots, a_q$ are distinct elements of $\bar{C}$, where

$$N_1(r) = 2N(r,f) - N(r,f') + N(r, 1/f').$$

Generalizations and extensions of these results have been obtained by Milloux, Hayman, and others and most of them are found in [1]. In [1], Hayman denotes by $S(r,f)$ any quantity satisfying (2) above. However, since all the results are obtained from the fundamental results of Nevanlinna it is easy to see that the theorems in [1] are valid with $S(r,f)$ satisfying (1) and (2) also.

In particular, we have [1], Theorem 3.1, for a meromorphic function f,

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f) \tag{3}$$

for $k \geqslant 1$.

If f is a meromorphic function of order ϱ, $0 \leqslant \varrho \leqslant \infty$, $a \in \bar{C}$ and k is a positive integer, we denote by $\bar{n}_k(r, a, f)$ the number of distinct zeros of order $\leqslant k$ of $f-a$ in $|z| \leqslant r$ (each zero is counted only once irrespective of its multiplicity) and by $n_k(r, a, f)$ we denote the number of zeros of $f-a$ in $|z| \leqslant r$, where a zero of order $< k$ is counted according to its multiplicity and a zero of order $\geqslant k$ is counted exactly k times. $\bar{N}_k(r, a, f)$ and $N_k(r, a, f)$ are defined in terms of $\bar{n}_k(r, a, f)$ and $n_k(r, a, f)$ respectively in the usual way. We further define

$$\bar{\varrho}_k(a,f) = \limsup_{r\to\infty} \frac{\log^+ \bar{n}_k(r, a, f)}{\log r} = \limsup_{r\to\infty} \frac{\log^+ \bar{N}_k(r, a, f)}{\log r},$$

$$\bar{\varrho}(a,f) = \limsup_{r\to\infty} \frac{\log^+ \bar{n}(r, a, f)}{\log r} = \limsup_{r\to\infty} \frac{\log^+ \bar{N}(r, a, f)}{\log r},$$

and

$$\varrho(a,f) = \limsup_{r\to\infty} \frac{\log^+ n(r, a, f)}{\log r} = \limsup_{r\to\infty} \frac{\log^+ N(r, a, f)}{\log r}.$$

We call a

(i) an evB (*exceptional value in the sense of Borel*) for f for distinct zeros of order $\leqslant k$ if $\bar{\varrho}_k(a,f) < \varrho$,

(ii) an evB for f for distinct zeros if $\bar{\varrho}(a,f) < \varrho$

and

(iii) an evB for f (for the whole aggregate of zeros) if $\varrho(a,f) < \varrho$.

Thus we call a an evB for f for simple zeros if $\bar{\varrho}_1(a,f) < \varrho$ and an evB for f for distinct simple and double zeros if $\bar{\varrho}_2(a,f) < \varrho$.

Also we call a

(i) an evP (*exceptional value in the sense of Picard*) for f for zeros of order $\leqslant k$ if $\bar{n}_k(r, a, f) = O(1)$, that is, if $f-a$ has only a finite number of zeros of order $\leqslant k$ in C

and

(ii) an evP for f if $\bar{n}(r, a, f) = O(1)$, that is, if $f-a$ has only a finite number of zeros in C.

Clearly,

$$\bar{n}(r, a, f) \leqslant \frac{1}{k+1}\{k\bar{n}_k(r, a, f)+n(r, a, f)\}$$

so that

$$\bar{N}(r, a, f) \leqslant \frac{1}{k+1}\{k\bar{N}_k(r, a, f)+N(r, a, f)\}. \tag{4}$$

In this paper we derive certain conclusions involving Borel exceptional values of f and those of $f^{(k)}$. Our conclusions are valid for meromorphic functions of all orders (finite or infinite) and improve and extend certain results obtained by K. L. Hoing and He Yu-Zan [2] for meromorphic functions of finite order. Our techniques are different from those used in [2] and appear to be more elementary.

We first prove

LEMMA 1. *If f is a meromorphic function, $a, b \in C$, $b \neq 0$ and k is a positive integer, then*

$$T(r,f) \leqslant \bar{N}(r,f)+N_{k+1}(r, a, f)+\bar{N}(r, b, f^{(k)})+S(r,f). \tag{5}$$

Proof. We have

$$\begin{aligned} T\left(r, \frac{1}{f-a}\right) &= N\left(r, \frac{1}{f-a}\right)+m\left(r, \frac{1}{f-a}\right) \\ &\leqslant N\left(r, \frac{1}{f-a}\right)+m\left(r, \frac{f^{(k)}}{f-a}\right)+m\left(r, \frac{1}{f^{(k)}}\right) \\ &= N\left(r, \frac{1}{f-a}\right)+T\left(r, \frac{1}{f^{(k)}}\right)-N\left(r, \frac{1}{f^{(k)}}\right)+S(r,f), \end{aligned}$$

by (3)

since $T\left(r, \frac{1}{f-a}\right) = T(r,f)+O(1)$. This yields

$$T(r,f) \leqslant N\left(r, \frac{1}{f-a}\right)+T(r, f^{(k)})-N\left(r, \frac{1}{f^{(k)}}\right)+S(r,f). \tag{6}$$

Applying Nevanlinna's second fundamental theorem to $f^{(k)}$, we obtain

$$(7)\qquad T(r,f^{(k)}) \leqslant N(r,f^{(k)}) + N\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) - \\ -\left\{2N(r,f^{(k)}) - N(r,f^{(k+1)}) + N\left(r,\frac{1}{f^{(k+1)}}\right)\right\} + S(r,f^{(k)}) \\ = \bar{N}(r,f) + N\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) - \\ -N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f^{(k)}),$$

since $N(r,f^{(k+1)}) - N(r,f^{(k)}) = \bar{N}(r,f^{(k)}) = \bar{N}(r,f)$.

Now

$$(8)\qquad T(r,f^{(k)}) = m(r,f^{(k)}) + N(r,f^{(k)}) \\ \leqslant m(r,f) + m\left(r,\frac{f^{(k)}}{f}\right) + \{N(r,f) + k\bar{N}(r,f)\} \\ = T(r,f) + k\bar{N}(r,f) + S(r,f) \leqslant (k+1)T(r,f) + S(r,f).$$

Hence

$$(9)\qquad S(r,f^{(k)}) = S(r,f).$$

A zero of $f-a$ of order $j>k$ is a zero of $f^{(k+1)}$ of order $j-(k+1)$ and a zero of $f^{(k)}-b$ of order m is a zero of $f^{(k+1)}$ of order $m-1$. Moreover, zeros of $f-a$ of order $>k$ are zeros of $f^{(k)}$ and so are not zeros of $f^{(k)}-b$ since $b\neq 0$.

Hence

$$(10)\qquad N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) \\ \leqslant N_{k+1}\left(r,\frac{1}{f-a}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-b}\right).$$

Substituting from (7) and (9) in (6) and using (10), we obtain (5).

THEOREM 1. *Let f be a meromorphic function and k be a positive integer. Suppose that ∞ is an* evB *for f for distinct zeros of order $\leqslant l$, where l is an integer $\geqslant 1$. If there exist $a, b \in C$, $b \neq 0$ such that a is an* evB *for f for distinct zeros of order $\leqslant p$ and b is an* evB *for $f^{(k)}$ for distinct zeros of order $\leqslant q$, where p, q are positive integers, then*

$$(11)\qquad \frac{q+1+k}{(q+1)(l+1)} + \frac{k+1}{p+1} + \frac{1}{q+1} \geqslant 1.$$

Proof. We have

$$(12)\quad N_{k+1}\left(r,\frac{1}{f-a}\right)\leqslant(k+1)\bar{N}\left(r,\frac{1}{f-a}\right)$$
$$\leqslant\frac{k+1}{p+1}\left\{p\bar{N}_p\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-a}\right)\right\}\quad\text{by (4)}$$
$$\leqslant\frac{k+1}{p+1}\left\{p\bar{N}_p\left(r,\frac{1}{f-a}\right)+T(r,f)\right\}+O(1).$$

Also, by (4),

$$(13)\quad \bar{N}\left(r,\frac{1}{f^{(k)}-b}\right)\leqslant\frac{1}{q+1}\left\{q\bar{N}_q\left(r,\frac{1}{f^{(k)}-b}\right)+T(r,f^{(k)})\right\}+O(1)$$

and

$$(14)\quad \bar{N}(r,f)\leqslant\frac{1}{l+1}\{l\bar{N}_l(r,f)+T(r,f)\}.$$

Using (12) and (13), we obtain, from (5),

$$T(r,f)\leqslant\bar{N}(r,f)+\frac{p(k+1)}{p+1}\bar{N}_p\left(r,\frac{1}{f-a}\right)+\frac{q}{q+1}\bar{N}_q\left(r,\frac{1}{f^{(k)}-b}\right)+$$
$$+\frac{k+1}{p+1}T(r,f)+\frac{1}{q+1}T(r,f^{(k)})+S(r,f)$$
$$\leqslant\left(1+\frac{k}{q+1}\right)\bar{N}(r,f)+\frac{p(k+1)}{p+1}\bar{N}_p\left(r,\frac{1}{f-a}\right)+$$
$$+\frac{q}{q+1}\bar{N}_q\left(r,\frac{1}{f^{(k)}-b}\right)+\left(\frac{k+1}{p+1}+\frac{1}{q+1}\right)T(r,f)+S(r,f),\quad\text{by (8)}.$$

Hence, by (14),

$$(15)\quad \left\{1-\frac{q+l+k}{(q+1)(l+1)}-\frac{k+1}{p+1}-\frac{1}{q+1}\right\}T(r,f)$$
$$\leqslant\left(1+\frac{k}{q+1}\right)\frac{1}{l+1}\bar{N}_l(r,f)+\frac{p(k+1)}{p+1}\bar{N}_p\left(r,\frac{1}{f-a}\right)+$$
$$+\frac{q}{q+1}\bar{N}_q\left(r,\frac{1}{f^{(k)}-b}\right)+S(r,f).$$

Let the order of f be ϱ, $0\leqslant\varrho\leqslant\infty$. Then the order of $f^{(k)}$ is also ϱ and so, by the hypothesis, we can choose a positive number $\mu<\varrho$ such that

$$\bar{N}_l(r,f)=O(r^\mu),\qquad \bar{N}_p\left(r,\frac{1}{f-a}\right)=O(r^\mu)$$
$$\text{and}\qquad \bar{N}_q\left(r,\frac{1}{f^{(k)}-b}\right)=O(r^\mu).$$

Then, choosing λ such that $\mu < \lambda < \varrho$, we obtain

$$\int_{r_0}^{\infty} \frac{\bar{N}_l(x,f)}{x^{1+\lambda}}\,dx < \infty, \qquad \int_{r_0}^{\infty} \frac{\bar{N}_p(x,a,f)}{x^{1+\lambda}}\,dx < \infty$$

and

$$\int_{r_0}^{\infty} \frac{\bar{N}_q(x,b,f^{(k)})}{x^{1+\lambda}}\,dx < \infty.$$

If (11) does not hold it then follows from (15) that

$$\int_{r_0}^{\infty} \frac{T(x,f)}{x^{1+\lambda}}\,dx < \infty,$$

since

$$\int_{r_0}^{r} \frac{S(x,f)}{x^{1+\lambda}}\,dx = o\left(\int_{r_0}^{r} \frac{T(x,f)}{x^{1+\lambda}}\,dx\right) \quad \text{by (1)}.$$

But this implies that $\varrho =$ the order of $f \leqslant \lambda$ which is a contradiction. This completes the proof of Theorem 1.

Some consequences of Theorem 1. Let f be a meromorphic function and k a positive integer.

(i) Letting l tend to infinity in (11) we obtain $\dfrac{k+1}{p+1} + \dfrac{1}{l+1} \geqslant 1$.

If, now, $p = 2(k+1)$, it follows that $\dfrac{k+1}{2(k+1)+1} + \dfrac{1}{q+1} \geqslant 1$ which cannot hold for $q \geqslant 1$.

Thus, if ∞ is an evB for f for distinct zeros and if there exists $a \in C$ such that a is an evB for f for distinct zeros of order $\leqslant 2(k+1)$, then $f^{(k)}$ has no evB for simple zeros in C except possibly 0.

In particular, if ∞ and $a \in C$ are evB for f for distinct zeros, then $f^{(k)}$ has no evB for simple zeros in C except possibly 0.

A slightly weaker result was obtained by Singh and Gopalakrishna [4], Theorem 6, for functions of finite order.

(ii) Letting both l and q tend to infinity in (11) we obtain $\dfrac{k+1}{p+1} \geqslant 1$ which cannot hold for $p > k$.

Thus, it follows that if ∞ is an evB for f for distinct zeros and if there exists $b \in C$, $b \neq 0$ such that b is an evB for $f^{(k)}$ for distinct zeros, then f has no evB for distinct zeros of order $\leqslant k+1$ in C.

(iii) Letting q tend to infinity in (11), we obtain $\dfrac{1}{l+1} + \dfrac{k+1}{p+1} \geqslant 1$ which cannot hold for $l \geqslant 1$ if $p \geqslant 2(k+1)$.

Hence it follows that if there exist $a, b \in C$, $b \neq 0$, such that a is an evB for f for distinct zeros of order $\leqslant 2(k+1)$ and b is an evB for $f^{(k)}$ for distinct zeros, then ∞ is not an evB for f for simple zeros.

Since ∞ is always an evB for f' for simple zeros (because f' cannot have simple poles), it follows that if there exists $a \in C$ such that a is an evB for f' for distinct zeros of order $\leqslant 2(k+1)$, then $f^{(k+1)}$ has no evB for distinct zeros in C except possibly 0.

(iv) Letting p tend to infinity in (11), we obtain $\dfrac{q+1+k}{(q+1)(l+1)}+\dfrac{1}{q+1} \geqslant 1$ which yields $l \leqslant \dfrac{k+1}{q}$. Thus, if there exist $a, b \in C$, $b \neq 0$ such that a is an evB for f for distinct zeros and b is an evB for $f^{(k)}$ for distinct zeros of order $\leqslant q$, then ∞ is not an evB for f for distinct zeros of order $\leqslant \left[\dfrac{k+1}{q}\right]+1$, where, as usual, $[x]$ denotes the greatest integer $\leqslant x$ for any real number x.

In particular, with $k = 1$, it follows that if there exist $a, b \in C$, $b \neq 0$ such that a is an evB for f for distinct zeros and b is an evB for f' for distinct zeros of order $\leqslant 3$, then ∞ is not an evB for f for simple zeros.

Remark. If f is a transcendental meromorphic function, then $\log r = o(T(r, f))$ as $r \to \infty$ and hence it follows, from (2) and (15), that Theorem 1 and its consequences remain valid if 'evB' is replaced everywhere by 'evP'. This yields stronger results than the above for functions of order 0. It follows, for instance, that if f has only a finite number of poles, then either f assumes all finite values infinitely often or $f^{(k)}-b$ has an infinity of simple zeros for all $b \in C$ with $b \neq 0$.

References

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